FUNCTIONAL ANALYSIS

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CHAPTER 1

Hilbert Spaces

ELEMENTARY PROPERTIES AND EXAMPLES.

Vector Space:

Let $(V,+)$ is abelian group and let F be a field. Suppose there is mapping $\sigma : F\times V \to V$, with $\sigma(\alpha,x)$ written αx satisfying the following axioms. For all $\alpha,\beta \in F, x,y \in V$

(a) $\alpha(x+y) = \alpha x + \alpha y$
(b) $(\alpha+\beta)x = \alpha x + \beta x$
(c) $(\alpha\beta)x = \alpha(\beta x)$
(d) $1x = x$

Then V is called *vector space* over the field F.

Inner Product Space:

If V is a vector space over F, a semi-inner product on V is a function $u : V\times V \to F$ such that for all $\alpha,\beta \in F$ and $x,y,z \in V$, the following are satisfied:

(a) $u(\alpha x+\beta y,z) = \alpha u(x,z) + \beta u(y,z)$
(b) $u(x,\alpha y+\beta z) = \bar{\alpha} u(x,y) + \bar{\beta} u(x,z)$
(c) $u(x,x) \geq 0$
(d) $u(x,y) = \overline{u(y,x)}$.

Here, $\bar{\alpha}$ is complex conjugate of α.

An inner product on V is a semi-inner product that also satisfied the following:

(e) if $u(x,x) = 0$, then $x = 0$.

Note: Inner product can be denoted by $u(x,y) = \langle x,y\rangle$.

A vector space V together with some inner product $\langle\cdot,\cdot\rangle$ is called as *inner product space.*

Example 1:

Let X be collection of all sequences $\{\alpha_n | n \geq 1\}$ of scalars α_n from F such that $\alpha_n = 0$ for all but a finite number of values of n. Then X is vector space with respect to following addition and scalar multiplication.

Addition :

$$\{\alpha_n\} + \{\beta_n\} = \{\alpha_n + \beta_n\}$$

Scalar multiplication:

$$\alpha\{\alpha_n\} = \{\alpha\alpha_n\}.$$

(i) Let $u : X \times X \to F$ be a mapping defined by $u\left(\{\alpha_n\}, \{\beta_n\}\right) = \sum_{n=1}^{\infty} \alpha_n \bar{\beta_n}$ on X.

For any sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in X$ and $\alpha, \beta, \gamma \in F$.
(a) Consider,

$$\begin{aligned}
u\left(\alpha\{\alpha_n\} + \beta\{\beta_n\}, \{\gamma_n\}\right) &= u\left(\{\alpha\alpha_n + \beta\beta_n\}, \{\gamma_n\}\right) \\
&= \sum_{n=1}^{\infty} \left(\alpha\alpha_n + \beta\beta_n\right)\bar{\gamma_n} \\
&= \sum_{n=1}^{\infty} \left(\alpha\alpha_n\bar{\gamma_n} + \beta\beta_n\bar{\gamma_n}\right) \\
&= \sum_{n=1}^{\infty} \alpha\alpha_n\bar{\gamma_n} + \sum_{n=1}^{\infty} \beta\beta_n\bar{\gamma_n} \\
&= \alpha\sum_{n=1}^{\infty} \alpha_n\bar{\gamma_n} + \beta\sum_{n=1}^{\infty} \beta_n\bar{\gamma_n} \\
&= \alpha u\left(\{\alpha_n\}, \{\gamma_n\}\right) + \beta u\left(\{\beta_n\}, \{\gamma_n\}\right)
\end{aligned}$$

$\Longrightarrow u\left(\alpha\{\alpha_n\} + \beta\{\beta_n\}, \{\gamma_n\}\right) = \alpha u\left(\{\alpha_n\}, \{\gamma_n\}\right) + \beta u\left(\{\beta_n\}, \{\gamma_n\}\right)$
(b) Consider,

$$\begin{aligned}
u\left(\{\alpha_n\}, \beta\{\beta_n\} + \gamma\{\gamma_n\}\right) &= u\left(\{\alpha_n\}, \{\beta\beta_n + \gamma\gamma_n\}\right) \\
&= \sum_{n=1}^{\infty} \alpha_n\overline{(\beta\beta_n + \gamma\gamma_n)} \\
&= \sum_{n=1}^{\infty} \alpha_n\left(\bar{\beta}\bar{\beta_n} + \bar{\gamma}\bar{\gamma_n}\right) \\
&= \sum_{n=1}^{\infty} \left(\bar{\beta}\alpha_n\bar{\beta_n} + \bar{\gamma}\alpha_n\bar{\gamma_n}\right) \\
&= \bar{\beta}\sum_{n=1}^{\infty} \alpha_n\bar{\beta_n} + \bar{\gamma}\sum_{n=1}^{\infty} \alpha_n\bar{\gamma_n} \\
&= \bar{\beta}u\left(\{\alpha_n\}, \{\beta_n\}\right) + \bar{\gamma}u\left(\{\alpha_n\}, \{\gamma_n\}\right)
\end{aligned}$$

$\Longrightarrow u\left(\{\alpha_n\}, \beta\{\beta_n\} + \gamma\{\gamma_n\}\right) = \bar{\beta}u\left(\{\alpha_n\}, \{\beta_n\}\right) + \bar{\gamma}u\left(\{\alpha_n\}, \{\gamma_n\}\right)$
(c) Consider,

$$\begin{aligned}
u\left(\{\alpha_n\}, \{\alpha_n\}\right) &= \sum_{n=1}^{\infty} \alpha_n\bar{\alpha_n} \\
&= \sum_{n=1}^{\infty} |\alpha_n|^2 \geq 0
\end{aligned}$$

$\Longrightarrow u\left(\{\alpha_n\}, \{\alpha_n\}\right) \geq 0$

(d) Consider,

$$\begin{aligned} u(\{\alpha_n\},\{\beta_n\}) &= \sum_{n=1}^{\infty} \alpha_n \bar{\beta}_n \\ &= \overline{\sum_{n=1}^{\infty} \bar{\alpha_n} \beta_n} \\ &= \overline{\sum_{n=1}^{\infty} \beta_n \bar{\alpha_n}} \\ &= \overline{u(\{\beta_n\},\{\alpha_n\})} \end{aligned}$$

$\Longrightarrow u(\{\alpha_n\},\{\beta_n\}) = \overline{u(\{\beta_n\},\{\alpha_n\})}$
(e) Consider,

$$\begin{aligned} u(\{\alpha_n\},\{\alpha_n\}) &= 0 \\ \Longleftrightarrow \sum_{n=1}^{\infty} \alpha_n \bar{\alpha_n} &= 0 \\ \Longleftrightarrow \sum_{n=1}^{\infty} |\alpha_n|^2 &= 0 \\ \Longleftrightarrow |\alpha_1|^2 + |\alpha_2|^2 + ... &= 0 \\ \Longleftrightarrow \alpha_1 = 0 &= \alpha_2 = ... \\ \Longleftrightarrow \{\alpha_n\} &= 0 \end{aligned}$$

$u(\{\alpha_n\},\{\alpha_n\}) = 0 \Longleftrightarrow \{\alpha_n\} = 0$
Therefore, (X, u) is inner product space.

(ii) Suppose u is defined as $u(\{\alpha_n\},\{\beta_n\}) = \sum_{n=1}^{\infty} \alpha_{2n} \bar{\beta}_{2n}$

For any sequences $\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \in X$ and $\alpha, \beta, \gamma \in F$.
(a) Consider,

$$\begin{aligned} u(\alpha\{\alpha_n\} + \beta\{\beta_n\},\{\gamma_n\}) &= u(\{\alpha\alpha_n + \beta\beta_n\},\{\gamma_n\}) \\ &= \sum_{n=1}^{\infty} (\alpha\alpha_{2n} + \beta\beta_{2n})\bar{\gamma}_{2n} \\ &= \sum_{n=1}^{\infty} (\alpha\alpha_{2n}\bar{\gamma}_{2n} + \beta\beta_{2n}\bar{\gamma}_{2n}) \\ &= \sum_{n=1}^{\infty} \alpha\alpha_{2n}\bar{\gamma}_{2n} + \sum_{n=1}^{\infty} \beta\beta_{2n}\bar{\gamma}_{2n} \\ &= \alpha\sum_{n=1}^{\infty} \alpha_{2n}\bar{\gamma}_{2n} + \beta\sum_{n=1}^{\infty} \beta_{2n}\bar{\gamma}_{2n} \\ &= \alpha u(\{\alpha_n\},\{\gamma_n\}) + \beta u(\{\beta_n\},\{\gamma_n\}) \end{aligned}$$

$\Longrightarrow u(\alpha\{\alpha_n\} + \beta\{\beta_n\},\{\gamma_n\}) = \alpha u(\{\alpha_n\},\{\gamma_n\}) + \beta u(\{\beta_n\},\{\gamma_n\})$

(b) Consider,

$$\begin{aligned} u\left(\{\alpha_n\},\beta\{\beta_n\}+\gamma\{\gamma_n\}\right) &= u\left(\{\alpha_n\},\{\beta\beta_n+\gamma\gamma_n\}\right) \\ &= \sum_{n=1}^{\infty}\alpha_{2n}\overline{(\beta\beta_{2n}+\gamma\gamma_{2n})} \\ &= \sum_{n=1}^{\infty}\alpha_{2n}\left(\bar{\beta}\bar{\beta}_{2n}+\bar{\gamma}\bar{\gamma}_{2n}\right) \\ &= \sum_{n=1}^{\infty}\left(\bar{\beta}\alpha_{2n}\bar{\beta}_{2n}+\bar{\gamma}\alpha_{2n}\bar{\gamma}_{2n}\right) \\ &= \bar{\beta}\sum_{n=1}^{\infty}\alpha_{2n}\bar{\beta}_{2n}+\bar{\gamma}\sum_{n=1}^{\infty}\alpha_{2n}\bar{\gamma}_{2n} \\ &= \bar{\beta}u\left(\{\alpha_n\},\{\beta_n\}\right)+\bar{\gamma}u\left(\{\alpha_n\},\{\gamma_n\}\right) \end{aligned}$$

$\implies u\left(\{\alpha_n\},\beta\{\beta_n\}+\gamma\{\gamma_n\}\right)=\bar{\beta}u\left(\{\alpha_n\},\{\beta_n\}\right)+\bar{\gamma}u\left(\{\alpha_n\},\{\gamma_n\}\right)$

(c) Consider,

$$\begin{aligned} u\left(\{\alpha_n\},\{\alpha_n\}\right) &= \sum_{n=1}^{\infty}\alpha_{2n}\bar{\alpha}_{2n} \\ &= \sum_{n=1}^{\infty}|\alpha_{2n}|^2 \geq 0 \end{aligned}$$

$\implies u\left(\{\alpha_n\},\{\alpha_n\}\right)\geq 0$

(d) Consider,

$$\begin{aligned} u\left(\{\alpha_n\},\{\beta_n\}\right) &= \sum_{n=1}^{\infty}\alpha_{2n}\bar{\beta}_{2n} \\ &= \overline{\sum_{n=1}^{\infty}\bar{\alpha}_{2n}\beta_{2n}} \\ &= \overline{\sum_{n=1}^{\infty}\beta_{2n}\bar{\alpha}_{2n}} \\ &= \overline{u\left(\{\beta_{2n}\},\{\alpha_{2n}\}\right)} \end{aligned}$$

$\implies u\left(\{\alpha_n\},\{\beta_n\}\right)=\overline{u\left(\{\beta_n\},\{\alpha_n\}\right)}$

$\implies u$ is semi-inner product on X.

(e) Consider,

$$\begin{aligned} u\left(\{\alpha_n\},\{\alpha_n\}\right) &= 0 \\ \implies \sum_{n=1}^{\infty}\alpha_{2n}\bar{\alpha}_{2n} &= 0 \\ \implies \sum_{n=1}^{\infty}|\alpha_{2n}|^2 &= 0 \\ \implies |\alpha_2|^2+|\alpha_4|^2+... &= 0 \\ \implies \alpha_2 = 0 &= \alpha_4 = ... \\ \not\Rightarrow \{\alpha_n\} &= 0 \end{aligned}$$

Because, if we take $\{\alpha_n\} = \{1, 0, 1, 0, ...\} \neq 0$, then $u(\{\alpha_n\}, \{\alpha_n\}) = \sum_{n=1}^{\infty} \alpha_{2n}\bar{\alpha}_{2n} = 0$

Therefore, u is not inner product on X.

Example 2:

Let (X, Ω, μ) be a measure space consisting of a set X a $\sigma-$algebra Ω of subsets of X, and a countably additive measure μ defined on Ω as $\mu : \Omega \to [0, \infty)$.

$\sigma-$algebra

Ω is a collection of subsets of X then Ω is called $\sigma-$algebra on X if

(i) $X \in \Omega$.

(ii) If $A \in \Omega$ then $A^c \in \Omega$.

(iii) If $A_1, A_2, ... \in \Omega$ then $\cup_{i=1}^{\infty} A_i \in \Omega$.

Measure

A function $\mu : \Omega \to [0, \infty)$ is called a measure if $\mu(\cup_{i=0}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

Measurable function

A function $f : X \to [0, \infty)$ is measurable if $f^{-1}((\alpha, \infty))$ is measurable.

$L^2(\mu) = L^2(X, \Omega, \mu) = \left\{ f / \left(\int |f(t)|^2 d\mu \right)^{\frac{1}{2}} < \infty \right\}$, where f is measurable function and $\int |f(t)|^2 d\mu < \infty$ is square integrable function.

Define a mapping $\langle \cdot, \cdot \rangle$ on $L^2(\mu)$ by $\langle f, g \rangle = \int f(t) \overline{g(t)} d\mu$.

For $f, g \in L^2(\mu) \implies \left(\int |f(t)|^2 d\mu \right)^{\frac{1}{2}} < \infty$ and $\left(\int |g(t)|^2 d\mu \right)^{\frac{1}{2}} < \infty$.

$\therefore \int |f(t) \overline{g(t)}| d\mu \leq \left(\int |f(t)|^2 d\mu \right)^{\frac{1}{2}} \left(\int |g(t)|^2 d\mu \right)^{\frac{1}{2}} < \infty$.

Therefore, our definition of function is well defined.

Now for any $f, g, h \in L^2(\mu)$ and $\alpha, \beta \in F$,

(a) Consider,

$$\begin{aligned}
\langle \alpha f + \beta g, h \rangle &= \int (\alpha f(t) + \beta g(t)) \overline{h(t)} d\mu \\
&= \int \left(\alpha f(t) \overline{h(t)} + \beta g(t) \overline{h(t)} \right) d\mu \\
&= \int \alpha f(t) \overline{h(t)} d\mu + \int \beta g(t) \overline{h(t)} d\mu \\
&= \alpha \int f(t) \overline{h(t)} d\mu + \beta \int g(t) \overline{h(t)} d\mu \\
&= \alpha \langle f, h \rangle + \beta \langle g, h \rangle
\end{aligned}$$

$\implies \langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$.

(b) Consider,

$$\begin{aligned}
\langle f, \alpha g + \beta h \rangle &= \int f(t) \overline{(\alpha g(t) + \beta h(t))} d\mu \\
&= \int f(t) \left(\bar{\alpha} \overline{g(t)} + \bar{\beta} \overline{h(t)} \right) d\mu \\
&= \int f(t) \bar{\alpha} \overline{g(t)} d\mu + \int f(t) \bar{\beta} \overline{h(t)} d\mu \\
&= \bar{\alpha} \int f(t) \overline{g(t)} d\mu + \bar{\beta} \int f(t) \overline{h(t)} d\mu \\
&= \bar{\alpha} \langle f, g \rangle + \bar{\beta} \langle f, h \rangle
\end{aligned}$$

$\implies \langle f, \alpha g + \beta h \rangle = \bar{\alpha} \langle f, g \rangle + \bar{\beta} \langle f, h \rangle$.

(c) Consider,

$$\begin{aligned}\langle f,f\rangle &= \int f(t)\overline{f(t)}d\mu\\ &= \int |f(t)|^2 d\mu\\ &\geq 0\end{aligned}$$

$\langle f,f\rangle \geq 0$. (d) Consider,

$$\begin{aligned}\langle f,g\rangle &= \int f(t)\overline{g(t)}d\mu\\ &= \int \overline{g(t)\overline{f(t)}}d\mu\\ &= \overline{\int g(t)\overline{f(t)}d\mu}\\ &= \overline{\langle g,f\rangle}\end{aligned}$$

$\langle f,g\rangle = \overline{\langle g,f\rangle}$. (e) Consider,

$$\begin{aligned}\langle f,f\rangle &= 0\\ \Longrightarrow \int f(t)\overline{f(t)}d\mu &= 0\\ \Longrightarrow \int |f(t)|^2 d\mu &= 0\\ \Longrightarrow |f(t)|^2 &= 0\\ \Longrightarrow f=0 &\end{aligned}$$

$\langle f,f\rangle = 0 \Longrightarrow f=0$.
Therefore, $\langle\cdot,\cdot\rangle$ is inner product on $L^2(\mu)$.
Therefore,$(L^2(\mu), \langle\cdot,\cdot\rangle)$ is inner product space.
The Cauchy-Bunyakowski-Schwarz Inequality: *If $\langle\cdot,\cdot\rangle$ is a semi-inner product on X, then*

$$|\langle x,y\rangle|^2 \leq \langle x,x\rangle\langle y,y\rangle$$

for all $x,y \in X$. Moreover, equality occurs if and only if there are scalars α and β, both not 0, such that $\langle \alpha x+\beta y, \alpha x+\beta y\rangle = 0$.
PROOF. For any $x,y \in X$ and $\alpha \in F$

$$\begin{aligned}0 &\leq \langle x-\alpha y, x-\alpha y\rangle\\ &= \langle x, x-\alpha y\rangle + \langle -\alpha y, x-\alpha y\rangle\\ &= \langle x,x\rangle + \langle x,-\alpha y\rangle - [\langle \alpha y,x\rangle - \langle \alpha y,\alpha y\rangle]\\ &= \langle x,x\rangle - \bar{\alpha}\langle x,y\rangle - \alpha\langle y,x\rangle + \alpha\bar{\alpha}\langle y,y\rangle\\ &= \langle x,x\rangle - \bar{\alpha}\langle x,y\rangle - \alpha\langle y,x\rangle + |\alpha|^2\langle y,y\rangle.\end{aligned}$$

Suppose $\langle y,x\rangle = be^{i\theta}, b \geq 0, \Longrightarrow \langle x,y\rangle = be^{-i\theta}, \langle x,x\rangle = c, \langle y,y\rangle = a$, and let $\alpha = te^{-i\theta}, t \in R$.
Then above inequality become

$$\begin{aligned}0 &\leq c - e^{-i\theta}tbe^{i\theta} - e^{i\theta}tbe^{-i\theta} + at^2\\ &= c - 2bt + at^2\\ &= at^2 - 2bt + c \equiv q(t).\end{aligned}$$

Thus $q(t)$ is quadratic polynomial in real variable t and $q(t) \geq 0$ for all $t \in R$.
$\Longrightarrow$ The equation $q(t) = 0$ has at most one real solution t. But quadratic equation must

have two roots and complex roots appear in pair. Therefore, $q(t)$ must has no real root. $\Longrightarrow$ discriminant of $q(t)$ is not positive.

$$\begin{aligned}
\Longrightarrow 4b^2 - 4ac &\leq 0 \\
\Longrightarrow b^2 - ac &\leq 0 \\
\Longrightarrow b^2 &\leq ac \\
\Longrightarrow |\langle x, y\rangle|^2 &\leq \langle x, x\rangle\langle y, y\rangle.
\end{aligned}$$

Now we have to show equality holds if and only if there are scalars $\alpha, \beta \in F$ both not zero, such that $\langle \alpha x + \beta y, \alpha x + \beta y\rangle = 0$

Suppose there are scalars $\alpha, \beta \in F$ both not zero, such that $\langle \alpha x + \beta y, \alpha x + \beta y\rangle = 0$.

Take $\beta = \langle y, y\rangle, \alpha = -\langle x, y\rangle$ then, $z = \langle y, y\rangle x - \langle x, y\rangle y$

Consider,

$$\begin{aligned}
\langle z, z\rangle &= \langle \langle y, y\rangle x - \langle x, y\rangle y, \langle y, y\rangle x - \langle x, y\rangle y\rangle \\
&= \langle \langle y, y\rangle x, \langle y, y\rangle x - \langle x, y\rangle y\rangle - \langle \langle x, y\rangle y, \langle y, y\rangle x - \langle x, y\rangle y\rangle \\
&= \langle \langle y, y\rangle x, \langle y, y\rangle x\rangle - \langle \langle y, y\rangle x, \langle x, y\rangle y\rangle - \langle \langle x, y\rangle y, \langle y, y\rangle x\rangle + \langle \langle x, y\rangle y, \langle x, y\rangle y\rangle \\
&= \langle y, y\rangle\overline{\langle y, y\rangle}\langle x, x\rangle - \langle y, y\rangle\overline{\langle x, y\rangle}\langle x, y\rangle - \langle x, y\rangle\overline{\langle y, y\rangle}\langle y, x\rangle + \langle x, y\rangle\overline{\langle x, y\rangle}\langle y, y\rangle \\
&= \langle y, y\rangle\overline{\langle y, y\rangle}\langle x, x\rangle - \langle x, y\rangle\overline{\langle y, y\rangle}\langle y, x\rangle. \\
&= \overline{\langle y, y\rangle}\left[\langle y, y\rangle\langle x, x\rangle - \langle x, y\rangle\overline{\langle x, y\rangle}\right] \\
&= \overline{\langle y, y\rangle}\left[\langle y, y\rangle\langle x, x\rangle - |\langle x, y\rangle|^2\right].
\end{aligned}$$

Now, if

$$\begin{aligned}
\langle \beta x + \alpha y, \beta x + \alpha y\rangle &= 0 \\
\Rightarrow \langle z, z\rangle &= 0 \\
\Rightarrow \overline{\langle y, y\rangle}\left[\langle y, y\rangle\langle x, x\rangle - |\langle x, y\rangle|^2\right] &= 0 \\
\Rightarrow \langle y, y\rangle\langle x, x\rangle - |\langle x, y\rangle|^2 &= 0 \\
\Rightarrow |\langle x, y\rangle|^2 &= \langle x, x\rangle\langle y, y\rangle.
\end{aligned}$$

Conversely, Suppose $|\langle x, y\rangle|^2 = \langle x, x\rangle\langle y, y\rangle$

$$\begin{aligned}
\Rightarrow \langle y, y\rangle\langle x, x\rangle - |\langle x, y\rangle|^2 &= 0 \\
\Rightarrow \overline{\langle y, y\rangle}\left[\langle y, y\rangle\langle x, x\rangle - |\langle x, y\rangle|^2\right] &= 0 \\
\Rightarrow \langle z, z\rangle &= 0 \\
\Rightarrow \langle \beta x + \alpha y, \beta x + \alpha y\rangle &= 0
\end{aligned}$$

where $\beta = \langle y, y\rangle$ and $\alpha = -\langle x, y\rangle$. ■

Corollary. *If $\langle\cdot, \cdot\rangle$ is a semi-inner product on X and $\|x\| \equiv \langle x, x\rangle^{1/2}$ for all $x \in X$, then*
(a) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$.
(b) $\|\alpha x\| = |\alpha|\|x\|$ *for all* $x \in X$ *and* $\alpha \in F$.
If $\langle\cdot, \cdot\rangle$ is inner product, then
(c) $\|x\| = 0$ *implies* $x = 0$.

PROOF.
(a) For $x, y \in X$

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2 \\
&= \|x\|^2 + 2Re\langle x, y\rangle + \|y\|^2.
\end{aligned}$$

We know that $\text{Re}\langle x, y\rangle \leq |\langle x, y\rangle|$ and by CBS inequality we have $|\langle x, y\rangle| \leq \|x\|\|y\|$. Hence

$$\begin{aligned}\|x+y\|^2 &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2 .\end{aligned}$$

$$\implies \|x+y\| \leq \|x\| + \|y\|$$

(b) For $\alpha \in F$ and $x \in X$

$$\begin{aligned}\|\alpha x\|^2 &= \langle \alpha x, \alpha x\rangle \\ &= \alpha\overline{\alpha}\langle x, x\rangle \\ &= |\alpha|^2\|x\|^2\end{aligned}$$

$$\implies \|\alpha x\| = |\alpha|\|x\|$$

(c) Suppose $\langle\cdot,\cdot\rangle$ is inner product on X.

$$\begin{aligned}\|x\|^2 &= 0 \\ \implies \langle x, x\rangle &= 0 \\ \implies x &= 0\end{aligned}$$

■

If $\langle\cdot,\cdot\rangle$ is a semi-inner product on X and if $x, y \in X$, then $\|x+y\|^2 = \|x\|^2 + 2$ Re $\langle x, y\rangle + \|y\|^2$ is called as *polar identity*.
Let X be a vector space over F. Then the function $\|\cdot\| : X \to F$ from X to F is called norm if it satisfies following conditions.
(i) For all $x \in X$, $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$.
(ii) For all $x \in X$ and $\alpha \in F$, $\|\alpha x\| = |\alpha|\|x\|$.
(iii) For any $x, y \in X$, $\|x+y\| \leq \|x\| + \|y\|$. The quantity $\|x\|^2 = \langle x, x\rangle^{1/2}$ is called the norm of x induced by inner product.
If X is vector space together with $\|\cdot\|$ is called normed linear space.

Note

(1) A sequence $\{x_n\}$ in X converges to x in normed linear space means
$\|x_n - x\| \to 0$ as $n \to \infty$ or $\lim\limits_{n\to\infty} \|x_n - x\| = 0$.
(2) A sequence $\{x_n\}$ is said to be a Cauchy sequence in X if for given $\epsilon > 0, \exists N \in \mathbb{N}$ such that
$\|x_n - x_m\| \leq 0, \forall m, n \geq N$.
Given a normed linear space X, define metric on X as $d(x, y) = \|x - y\|$.
(i) $d(x, y) = \|x - y\| \geq 0$
$\Rightarrow d(x, y) \geq 0$.
(ii) $d(x, y) = \|x - y\| = \|y - x\| = d(y, x)$
$\Rightarrow d(x, y) = d(y, x)$
(iii) $d(x, y) = 0$
$\Leftrightarrow \|x - y\| = 0$
$\Leftrightarrow x - y = 0$
$\Leftrightarrow x = y$
(iv) $d(x, y) = \|x - y\| = \|x - z - (y - z)\| \leq \|x - z\| + \|z - y\| = d(x, z) + d(z, y)$
$\Rightarrow d(x, y) \leq d(x, z) + d(z, y)$.

Definition. A Hilbert space is a vector space H over F together with an inner product $\langle\cdot,\cdot\rangle$ such that relative to the metric $d(x,y)=\|x-y\|$ induced by the norm, H is complete metric space.
Fatou's Lemma: If $f_n : X \to [0,\infty)$ is measurable for each positive integer n. Then $\int\left(\lim\limits_{n\to\infty}\inf f_n\right)d\mu \le \lim\limits_{n\to\infty}\inf\displaystyle\int f_n d\mu$
Example 1. The measure space $H=L^2(\mu)=L^2(X,\Omega,\mu)$ is Hilbert space.
PROOF. We have $L^2(\mu)=\left\{f/\left(\int|f(t)|^2d\mu\right)^{1/2}<\infty\right\}$ where f is measurable function.
Then for $f,g\in L^2(\mu)$ define a metric on $L^2(\mu)$ by $d(f,g)=\|f-g\|_2$.
Suppose $\{f_n\}$ be Cauchy sequence in $L^2(\mu)$.
It is sufficient to prove that one of the subsequence of $\{f_n\}$ is convergent.
Let $\{f_{n_i}\}$ be a subsequence of given sequence such that

$$\|f_{n_{i+1}}-f_{n_i}\|_2<\tfrac{1}{2^i},\quad i=1,2,3,...$$

Define $g_k=\displaystyle\sum_{i=1}^{k}|f_{n_{i+1}}-f_{n_i}|$ and $g=\displaystyle\sum_{i=1}^{\infty}|f_{n_{i+1}}-f_{n_i}|$, then

$$\begin{aligned}
\|g_k\|_2 &= \left(\int|g_k|^2d\mu\right)^{1/2}\\
&= \left(\int\Big|\sum_{i=1}^{k}|f_{n_{i+1}}-f_{n_i}|^2\Big|d\mu\right)^{1/2}\\
&\le \left(\int\sum_{i=1}^{k}|f_{n_{i+1}}-f_{n_i}|^2d\mu\right)^{1/2}\\
&\le \sum_{i=1}^{k}\left(\int|f_{n_{i+1}}-f_{n_i}|^2d\mu\right)^{1/2}\\
&= \sum_{i=1}^{k}\|f_{n_{i+1}}-f_{n_i}\|_2\\
&< \sum_{i=1}^{k}\frac{1}{2^i}\\
&= \tfrac{1}{2}+\tfrac{1}{2^2}+\tfrac{1}{2^3}+...+\tfrac{1}{2^k}\\
&= \tfrac{1}{2}\left(\tfrac{1}{2}+\tfrac{1}{2^2}+\tfrac{1}{2^3}+...+\tfrac{1}{2^{k-1}}\right)\\
&< \tfrac{1}{2}\left(\tfrac{1}{2}+\tfrac{1}{2^2}+\tfrac{1}{2^3}+...+\tfrac{1}{2^{k-1}}+\tfrac{1}{2^k}+...\right)\\
&= \tfrac{1}{2}\left(\tfrac{1}{1-\frac{1}{2}}\right)\\
&= 1
\end{aligned}$$

$\therefore \|g_k\|_2<1$
$\Rightarrow \left(\int|g_k|^2\right)^{1/2}<1$
$\Rightarrow \int g_k^2\le\int|g_k|^2<1$
Now applying Fatou's lemma to g_k^2 we get:
$\int\left(\lim\limits_{k\to\infty}\inf g_k^2\right)d\mu\le\lim\limits_{k\to\infty}\inf\displaystyle\int g_k^2d\mu<1$
$\Rightarrow \int g^2d\mu<1$

$\Rightarrow \left(\int g^2 d\mu\right)^{1/2} < 1$
$\Rightarrow \|g\|_2 < 1 < \infty$
$\Rightarrow g$ must converges a. e.
$\Rightarrow \sum_{i=1}^{\infty} \left(f_{n_{i+1}} - f_{n_i}\right)$ converges absolutely a. e.
$\Rightarrow f_{n_1} + \sum_{i=1}^{\infty} \left(f_{n_{i+1}} - f_{n_i}\right)$ converges absolutely a. e.
Let $f(x) = f_{n_1} + \sum_{i=1}^{\infty} \left(f_{n_{i+1}} - f_{n_i}\right)$ for those $x \in X$ for which above series converges absolutely and denote $f(x) = 0$ for those $x \in X$ for which above series is not absolutely converging.
Suppose $f_{n_1} + \sum_{i=1}^{k} \left(f_{n_{i+1}} - f_{n_i}\right) = f_{n_k}$
$\therefore \lim_{i\to\infty} f_{n_i}(x) = f(x)$ a.e.
$\therefore \{f_{n_i}\}$ converges pointwise to f
Given a Cauchy sequence $\{f_n\}$ and $\epsilon > 0$, there exist $N \in \mathbb{N}$ such that
$\|f_n - f_m\| < \epsilon \quad \forall n, m > N$.
For $m > N$
$\int \lim_{i\to\infty} \inf |f_{n_i} - f_m| d\mu = \int |f - f_m|^2 d\mu \leq \lim_{i\to\infty} \inf \int |f_{n_i} - f_m|^2 d\mu < \epsilon^2$

$$\begin{aligned} \therefore \left(\int |f - f_m|^2 d\mu\right)^{1/2} &< \epsilon \\ \Rightarrow \|f - f_m\|_2 &< \epsilon < \infty \end{aligned}$$

$\therefore f - f_m \in L^2(\mu)$
$\|f\|_2 = \|f - f_m + f_m\|_2 \leq \|f - f_m\|_2 + \|f_m\|_2 < \infty$
$\Rightarrow f \in L^2(\mu)$
$\therefore \|f - f_m\|_2 \to 0$ as $m \to \infty$
$\Rightarrow \{f_n\}$ converges to f and $f \in L^2(\mu)$.
Therefore, $L^2(\mu)$ is complete with respect to metric induced by inner product.
Therefore, $L^2(\mu)$ is Hilbert space. ■

Note
(1) Every Cauchy sequence is bounded.(Exercise)
(2) **Minkowski Inequality:** For $1 \leq p < \infty$ and for any complex numbers $x_k, y_k \in \mathbb{C}$,

$$\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p}$$

Example 2. Let I be any set and let $l^2(I)$ denote the set of all functions $x : I \to F$ such that $x(i) = 0$ for all but a countable number of i and $\sum_{i\in I} |x(i)|^2 < \infty$.
For x and y in $l^2(I)$ define

$$\langle x, y\rangle = \sum_{i} x(i)\overline{y(i)}.$$

Then $l^2(I)$ is Hilbert space.
Note: If I is countable then $l^2(I)$ is denoted by l^2.
PROOF: Clearly l^2 is inner product space(Exercise for you).
Since $\langle x, y\rangle = \sum_{i=1}^{\infty} x(i)\overline{y(i)}$ is inner product on l^2.

$$\begin{aligned}
\Rightarrow \langle x, x\rangle &= \sum_{i=1}^{\infty} x(i)\overline{x(i)}. \\
&= \sum_{i=1}^{\infty} |x(i)|^2 \\
\therefore \|x\| &= \langle x, x\rangle^{1/2} \\
\Rightarrow \|x\| &= \left(\sum_{i=1}^{\infty} |x(i)|^2\right)^{1/2}. \\
\Rightarrow d(x, y) &= \|x - y\| \\
&= \left(\sum_{i=1}^{\infty} |x(i) - y(i)|^2\right)^{1/2}
\end{aligned}$$

Let $\{x_n\}$ be a Cauchy sequence in l^2.
That is, for given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned}
d(x_n, x_m) &< \epsilon, \quad \forall n, m > N. \\
\Rightarrow \left(\sum_{i=1}^{\infty} |x_n(i) - x_m(i)|^2\right)^{1/2} &< \epsilon, \quad \forall n, m > N \\
\Rightarrow \sum_{i=1}^{\infty} |x_n(i) - x_m(i)|^2 &< \epsilon^2, \quad \forall n, m > N
\end{aligned}$$

Consider,

$$\begin{aligned}
|x_n(i) - x_m(i)| &\leq \left(\sum_{j=1}^{\infty} |x_n(j) - x_m(j)|^2\right)^{1/2} \\
&= d(x_n, x_m) < \epsilon, \quad \forall n, m > N \\
\Rightarrow |x_n(i) - x_m(i)| &< \epsilon, \quad \forall n, m > N.
\end{aligned}$$

$\therefore \{x_n(i)\}$ is Cauchy sequence in F.
$\Rightarrow x_n(i) \to x(i)$ as $n \to \infty$

Consider,

$$\begin{aligned}
x &= (x(1), x(2), x(3), ...)\\
\Rightarrow \left(\sum_{j=1}^{i} |x(j)|^2\right)^{1/2} &= \left(\sum_{j=1}^{i} |x(j) - x_n(j) + x_n(j)|^2\right)^{1/2}\\
&\leq \left(\sum_{j=1}^{i} |x(j) - x_n(j)|^2\right)^{1/2} + \left(\sum_{j=1}^{i} |x_n(j)|^2\right)^{1/2}\\
&< \epsilon + \left(\sum_{j=1}^{i} |x_n(j)|^2\right)^{1/2} < \infty\\
\Rightarrow \left(\sum_{j=1}^{i} |x(i)|^2\right)^{1/2} &< \infty
\end{aligned}$$

Now as $i \to \infty$, then we have

$\Rightarrow \left(\sum_{j=1}^{\infty} |x(i)|^2\right)^{1/2} < \infty$

$\Rightarrow x \in l^2$.

Already we have, $\left(\sum_{i=1}^{\infty} |x_n(i) - x_m(i)|^2\right)^{1/2} < \epsilon$

Fix n, taking limit as $m \to \infty$

$\lim_{m\to\infty} \left(\sum_{i=1}^{\infty} |x_n(i) - x_m(i)|^2\right)^{1/2} < \epsilon$

$\Rightarrow \left(\sum_{i=1}^{\infty} |x_n(i) - x(i)|^2\right)^{1/2} < \epsilon$

Now as $n \to \infty$, $\left(\sum_{i=1}^{\infty} |x_n(i) - x(i)|^2\right)^{1/2} \to 0$

$\Rightarrow d(x_n, x) \to 0$ and $n \to \infty$

$\Rightarrow \{x_n\} \to x$ as $n \to \infty$ and $x \in l^2$.

l^2 is complete metric space with respect to above defined metric.

$\therefore l^2$ is Hilbert space. ■

Example 3. Show that $H = \mathbb{R}^k = \{(x(1), x(2), ..., x(k)) / x(i) \in \mathbb{R}\}$ is Hilbert space.

PROOF. Define an inner product on $\mathbb{R}^k$ as,

$$\begin{aligned}
\langle x, y\rangle &= \sum_{i=1}^{k} x\,(i)\,y\,(i) \\
\Rightarrow \|x\| &= \langle x, x\rangle^{1/2} \\
&= (\sum_{i=1}^{k} x\,(i)\,x\,(i))^{1/2} \\
&= (\sum_{i=1}^{k} [x\,(i)]^2)^{1/2}
\end{aligned}$$

The metric induced by above inner product is given by,

$$\begin{aligned}
d(x, y) &= \|x - y\| \\
&= (\sum_{i=1}^{k} [x(i) - y(i)]^2)^{1/2}
\end{aligned}$$

Suppose $\{x_n\}$ be Cauchy sequence in $\mathbb{R}^k$.
Therefore, for given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that
$d(x_n, x_m) < \epsilon \quad \forall n, m > N$
$\Rightarrow (\sum_{i=1}^{k} [x_n(i) - x_m(i)]^2)^{1/2} < \epsilon$
Consider,

$$\begin{aligned}
|x_n(i) - x_m(i)| &\leq (\sum_{i=1}^{k} |x_n(i) - x_m(i)|^2)^{1/2} \\
&= d(x_m, x_n) \\
&< \epsilon \quad \forall n, m > N
\end{aligned}$$

$\Rightarrow \{x_n(i)\}$ is Cauchy sequence in $\mathbb{R}$.
We know that $\mathbb{R}$ is complete.
$x_n(i) \to x(i)$ in $\mathbb{R}$ as $n \to \infty$.
Consider, $x = (x(1), x(2), ..., x(k))$ then $x \in \mathbb{R}^k$ because each $x(i) \in \mathbb{R}$.
Since,

$$(\sum_{i=1}^{k} |x_n(i) - x_m(i)|^2)^{1/2} < \epsilon$$

For fix n and taking $m \to \infty$.

$$\begin{aligned}
(\sum_{i=1}^{k} |x_n(i) - x(i)|^2)^{1/2} &< \epsilon \\
\Rightarrow d(x_n, x) &< \epsilon
\end{aligned}$$

That is $d(x_n, x) \to 0$ as $n \to \infty$.
$\Rightarrow \{x_n\} \to x$ as $n \to \infty$ and $x \in \mathbb{R}^k$.
$\therefore \mathbb{R}^k$ is complete.

$\therefore \mathbb{R}^k$ is Hilbert space. ∎

ORTHOGONALITY

Definition. If H is Hilbert space and $f, g \in H$, then f and g are orthogonal if $\langle f, g\rangle = 0$. In symbols, $f \perp g$.

If $A, B \subseteq H$, then $A \perp B$ if $f \perp g$ for every $f \in A$ and $g \in B$.

If $H = \mathbb{R}^2$ then two non-zero vectors in $\mathbb{R}^2$ are orthogonal precisely when the angle between them is $\frac{\pi}{2}$.

Pythagorean Theorem. *If $f_1, f_2, ..., f_n$ are pairwise orthogonal vectors in H, then*

$$\|f_1 + f_2 + ... + f_n\|^2 = \|f_1\|^2 + \|f_2\|^2 + ... + \|f_n\|^2.$$

PROOF. We will prove this theorem by induction.

If $f_1 \perp f_2$, then

$$\begin{aligned} \|f_1 + f_2\|^2 &= \langle f_1 + f_2, f_1 + f_2\rangle \\ &= \langle f_1, f_1\rangle + \langle f_1, f_2\rangle + \langle f_2, f_1\rangle + \langle f_2, f_2\rangle \\ &= \|f_1\|^2 + \|f_2\|^2 \end{aligned}$$

Therefore, result is true for $n = 2$.

Now assume that the result is true for $n = k$. That is if $f_1, f_2, ..., f_k$ are pairwise orthogonal then $\|f_1 + f_2 + ... + f_k\|^2 = \|f_1\|^2 + \|f_2\|^2 + ... + \|f_k\|^2$.

Now we have to show the result is true for $n = k + 1$.

Suppose $f_1, f_2, ..., f_{k+1}$ are pairwise orthogonal vectors in $H \Rightarrow h = f_1 + f_2 + ... + f_k \in H$.

Consider,

$$\begin{aligned} \|f_1 + f_2 + ... + f_k + f_{k+1}\|^2 &= \|h + f_{k+1}\|^2 \\ &= \|h\|^2 + \|f_{k+1}\|^2 \\ &= \|f_1 + f_2 + ... + f_k\|^2 + \|f_{k+1}\|^2 \\ &= \|f_1\|^2 + \|f_2\|^2 + ... + \|f_k\|^2 + \|f_{k+1}\|^2 \end{aligned}$$

Therefore, by mathematical induction

If $f_1, f_2, ..., f_n$ are pairwise orthogonal vectors in H, then

$\|f_1 + f_2 + ... + f_n\|^2 = \|f_1\|^2 + \|f_2\|^2 + ... + \|f_n\|^2$. ∎

Parallelogram Law: *If H is Hilbert space and f and $g \in H$, then*

$$\|f + g\|^2 + \|f - g\|^2 = 2(\|f\|^2 + \|g\|^2).$$

PROOF. For any $f, g \in H$ the polar identity implies,

$$\begin{aligned} \|f + g\|^2 &= \|f\|^2 + 2Re\langle f, g\rangle + \|g\|^2, \\ \|f - g\|^2 &= \|f\|^2 - 2Re\langle f, g\rangle + \|g\|^2. \end{aligned}$$

Adding both identities we get,

$$\|f + g\|^2 + \|f - g\|^2 = 2(\|f\|^2 + \|g\|^2).$$

∎

Result 1: Suppose X is vector space and $\langle\cdot,\cdot\rangle$ is an inner product on X. If $\|x_n - x\| \to 0$

as $n \to \infty$ and $\|y_n - y\| \to 0$ as $n \to \infty$, then show that $\langle x_n, y_n\rangle \to \langle x, y\rangle$ as $n \to \infty$.
PROOF. Consider,

$$\begin{aligned} |\langle x_n, y_n\rangle - \langle x, y\rangle| &= |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle| \\ &\leq |\langle x_n, y_n\rangle - \langle x_n, y\rangle| + |\langle x_n, y\rangle - \langle x, y\rangle| \\ &= |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \\ &\leq \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\| \\ \Rightarrow |\langle x_n, y_n\rangle - \langle x, y\rangle| &\leq \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\| \end{aligned}$$

As $n \to \infty$ RHS of above inequality tend to 0.
Therefore, $\langle x_n, y_n\rangle \to \langle x, y\rangle$ as $n \to \infty$. ■

Result 2: Subspace of Hilbert space is Hilbert space iff it is closed.
PROOF. Exercise.

Result 3: Among all norms $\|\cdot\|_p, 1 \leq p < \infty$ only $\|\cdot\|_2$ is a norm coming from inner product. That is only $\|\cdot\|_2$ induced from inner product.
PROOF. For $x = (x(1), x(2), ..., x(k)) \in F^k$,

$$\|x\|_p = (\sum_{i=1}^{k} |x(i)|^p)^{1/p}$$

Choose $x = (1, 0, 0, ..., 0)$ and $y = (0, 1, 0, ..., 0)$
By Parallelogram law we have $\|x + y\|_p^2 + \|x - y\|_p^2 = 2(\|x\|_p^2 + \|y\|_p^2)$.

$$\begin{aligned} LHS &= \|x + y\|_p^2 + \|x - y\|_p^2 \\ &= (\sum_{i=1}^{k} |x(i) + y(i)|^p)^{2/p} + (\sum_{i=1}^{k} |x(i) - y(i)|^p)^{2/p} \\ &= (1 + 1 + 0 + ... + 0)^{2/p} + (1 + 1 + 0 + ... + 0)^{2/p} \\ &= 2^{2/p} + 2^{2/p} \\ &= 2^{1+2/p} \\ RHS &= 2(\|x\|_p^2 + \|y\|_p^2) \\ &= 2[(\sum_{i=1}^{k} |x(i)|^p)^{2/p} + (\sum_{i=1}^{k} |y(i)|^p)^{2/p}] \\ &= 2[1 + 1] \\ &= 2^2 \end{aligned}$$

$$\begin{aligned} &LHS = RHS \\ &\Leftrightarrow 2^{1+2/p} = 2^2 \\ &\Leftrightarrow 1 + \tfrac{2}{p} = 2 \\ &\Leftrightarrow p = 2. \end{aligned}$$

$\therefore$ Only, $\|\cdot\|_2$ induced from inner product. ■

Result 4: Show that among all $L^p([0, 1])$ only $L^2([0, 2])$ is an inner product space.
PROOF. Exercise.

Result 5: Show that $C_{00} = \{x : I \to F\}$ such that $x(i) = 0$ for all but finitely many i, is not Hilbert space.
PROOF. Clearly C_{00} is a subspace of l^2.

It is sufficient to show there exists one Cauchy sequence which is not convergent in C_{00}.
Suppose $\{x_n\} = \{1, 1/2, 1/3, ..., 1/n, 0, 0, 0, ...\}$.
Since finitely many terms of this sequence are non-zero therefore $\{x_n\} \in C_{00}$.
Then, $\{x_n\}$ is Cauchy but it is not convergent in C_{00}.
Hence, C_{00} is not Hilbert space. ■

Definition. If X is any vector space over F and $A \subseteq H$, then A is convex set if for any $x, y \in A$ and $0 \leq t \leq 1, tx + (1-t)y \in A$.

Remark 1: Any linear subspace of X is convex(Check).

Remark 2: If H is Hilbert space, then show that every open ball $B(f;r) = \{g \in H/\|f-g\| < r\}$ is convex.

PROOF. Let $g, h \in B(f;r)$.
To show: $tg + (1-t)h \in B(f;r)$.
Consider,

$$\begin{aligned}
\|f - (tg + (1-t)h)\| &= \|f - tg + tf - tf - (1-t)h\| \\
&= \|f - tf - tg + tf - (1-t)h\| \\
&= \|(1-t)f - (1-t)h + t(f-g)\| \\
&= \|(1-t)(f-h) + t(f-g)\| \\
&\leq \|(1-t)(f-h)\| + \|t(f-g)\| \\
&= (1-t)\|(f-h)\| + t\|(f-g)\| \\
&< (1-t)r + tr \\
&= r - tr + tr \\
\Rightarrow \|f - (tg + (1-t)h)\| < r &
\end{aligned}$$

Therefore, Every open ball in H is convex set. ■

Theorem. *If H is Hilbert space, K is a closed convex non-empty subset of H, and $h \in H$, then there is a unique point k_0 in K such that*

$$\|h - k_0\| = dist(h, K) \equiv \inf\{\|h-k\|/k \in K\}.$$

PROOF. Let $d = dist(h, K) \equiv \inf\{\|h-k\|/k \in K\}$.
Therefore, $\exists \{k_n\}$ from K such that $\|h - k_n\| \to d$ as $n \to \infty$.
That is, $\lim\limits_{n\to\infty} \|h - kn\| = d$.
Let k_n and k_m be two elements from the sequence $\{k_n\}$.
Then $\frac{k_n+k_m}{2} \in K$ ($\because K$ is convex)
$\therefore \|h - \frac{k_n+k_m}{2}\| \geq d$.
$\Rightarrow \|2h - (k_n + k_m)\| \geq 2d$.
Also, $\lim\limits_{m\to\infty} \|h - k_m\| = d$ and $\lim\limits_{n\to\infty} \|h - k_n\| = d$.
Now, For any $\epsilon > 0$, choose N such that for $m, n > N \in \mathbb{N}$, then

$$\begin{aligned}
\|k_n - k_m\|^2 &= \|k_n - h + h - k_m\|^2 \\
&= 2(\|h-k_n\|^2 + \|h-k_m\|^2) - \|h - k_n + h - k_m\|^2 \\
&< 2(\|h-k_n\|^2 + \|h-k_m\|^2) - \|h - k_n + h - k_m\|^2 + \epsilon^2
\end{aligned}$$

$\Rightarrow \|k_m - k_n\| < \epsilon$.
$\Rightarrow \{k_n\}$ is a Cauchy sequence.

Since K is closed subset of Hilbert space H hence it is Hilbert space.
$\Rightarrow K$ is complete.
$\Rightarrow \{k_n\}$ must converges to some point in K.
That is, $\lim_{n\to\infty} k_n = k_0 \in K$.

$$\begin{aligned}\therefore \|h-k_0\| &= \|h - \lim_{n\to\infty} k_n\| \\ &= \|\lim_{n\to\infty}(h-k_n)\| \\ &= \lim_{n\to\infty}\|h-k_n\| \\ &= d \\ &= dist(h,K).\end{aligned}$$

Uniqueness:
Let, k_1 and k_2 be two points from K such that $\|h-k_1\| = dist(h,K)$ and $\|h-k_2\| = dist(h,K)$.
$\therefore \frac{k_1+k_2}{2} \in K$.
$\therefore \|h - \frac{k_1+k_2}{2}\| \geq d$
$\therefore \|2h-(k_1+k_2)\| \geq 2d$
Now,

$$\begin{aligned}\|k_1-k_2\|^2 &= \|k_1-h+h-k_2\|^2 \\ &= \|-(h-k_1)+(h-k_2)\|^2 \\ &= 2(\|h-k_1\|^2+\|h-k_2\|^2) - \|h-k_1+h-k_2\|^2 \\ &= 2(\|h-k_1\|^2+\|h-k_2\|^2) - \|2h-(k_1+k_2)\|^2 \\ &\leq 2d^2+2d^2-4d^2 \\ &= 0\end{aligned}$$
$$\Rightarrow \|k_1-k_2\| \leq 0.$$

But we know that norm is always greater equal to 0. That is, $\|k_1-k_2\| \geq 0$
$\Rightarrow \|k_1-k_2\| = 0$
$\Rightarrow k_1-k_2 = 0$
$\Rightarrow k_1 = k_2$. ■

Note: The above theorem also holds, if we replace convex set by closed linear subspace of H.

Theorem. *If M is a closed linear subspace of H, $h \in H$, and f_0 is unique element of M such that $\|h-f_0\| = dist(h,M)$, then $h-f_0 \perp M$. Conversely, if $f_0 \in M$ such that $h-f_0 \perp M$, then $\|h-f_0\| = dist(h,M)$.*

PROOF. Given f_0 unique element from M such that $\|h-f_0\| = dist(h,M)$.
Let $f \in M \Longrightarrow f_0+f \in M$ $\qquad \because M$ is linear subspace $\Rightarrow$ closed under addition.
We know that

$$\begin{aligned}\|h-f_0\|^2 &\leq \|h-(f_0+f)\|^2 \\ &= \|h-f_0-f\|^2 \\ &= \|h-f_0\|^2 - 2Re\langle h-f_0, f\rangle + \|f\|^2\end{aligned}$$

$\Rightarrow \|h-f_0\|^2 \leq \|h-f_0\|^2 - 2Re\langle h-f_0,f\rangle + \|f\|^2$
$\Rightarrow -2Re\langle h-f_0,f\rangle + \|f\|^2 \geq 0$

$\Rightarrow 2Re\langle h - f_0, f\rangle \leq \langle f, f\rangle, \quad \forall f \in M$
Now fix f in M and substitute $te^{i\theta}f$ for f in preceding inequality,
$\Rightarrow 2Re\langle h - f_0, te^{i\theta}f\rangle \leq \langle te^{i\theta}f, te^{i\theta}f\rangle, t \in \mathbb{R}$
$\Rightarrow 2Re\left\{te^{-i\theta}\langle h - f_0, f\rangle\right\} \leq t^2\langle f, f\rangle, t \in \mathbb{R}$
Now put $\langle h - f_0, f\rangle = re^{i\theta}$ where $r \geq 0$.
$\Rightarrow 2Re\left\{te^{-i\theta}re^{-i\theta}\right\} \leq t^2\|f\|^2$
$\Rightarrow 2tr \leq t^2\|f\|^2$
For $t \neq 0$ and $t \to 0 \Longrightarrow r \leq 0$
$\Rightarrow r = 0$
$\Rightarrow \langle h - f_0, f\rangle = 0$ for arbitrary $f \in M$
$\therefore h - f_0 \perp M$.
Conversely, suppose $f_0 \in M$ such that $h - f_0 \perp M$.
Consider, $f \in M$, then $f_0 - f \in M$.
$\therefore h - f_0 \perp f_0 - f$

$$\begin{aligned} \|h - f\|^2 &= \|h - f_0 + f_0 - f\|^2 \\ &= \|h - f_0\|^2 + \|f_0 - f\|^2 \\ &\geq \|h - f_0\|^2 \end{aligned}$$

$\therefore \|h - f\|^2 \geq \|h - f_0\|^2, \forall f \in M$.
$\therefore \|h - f_0\| = \inf\{\|h - f\|/f \in M\}$. ■

Result. If $A \subseteq H$, then $A^\perp = \{f \in H/f \perp g, \quad \forall g \in A\}$ is closed linear subspace of H.
PROOF. Let $f_1, f_2 \in A^\perp \Longrightarrow \langle f_1, g\rangle = 0$ and $\langle f_2, g\rangle = 0, \quad \forall g \in A$.
$\therefore \langle f_1 + f_2, g\rangle = \langle f_1, g\rangle + \langle f_2, g\rangle = 0, \quad \forall g \in A$.
$\therefore f_1 + f_2 \in A^\perp$.
Also, for any $\alpha \in F$ and $f \in A^\perp$.
$\langle \alpha f, g\rangle = \alpha\langle f, g\rangle = 0$.
$\therefore \alpha f \in A^\perp$.
$\Rightarrow A^\perp$ is linear subspace of H.
Suppose z is limit point of $A^\perp$, then there exists a sequence $\{z_n\}$ in $A^\perp$ such that $\{z_n\} \to z$.
$\therefore \langle z_n, g\rangle = 0, \quad \forall g \in A$.
Consider,

$$\begin{aligned} \langle z, g\rangle &= \langle \lim_{n\to\infty} z_n, g\rangle \\ &= \lim_{n\to\infty}\langle z_n, g\rangle \\ &= 0, \quad \forall g \in A. \end{aligned}$$

$\Rightarrow z \in A^\perp$
Therefore, $A^\perp$ is closed linear subspace of H. ■

Projection theorem. *Let H be a Hilbert space and M be a closed linear subspace of H then $H = M \oplus M^\perp$.*
PROOF. Suppose $h \in M \cap M^\perp$.
$\Rightarrow h \in M$ and $h \in M^\perp$.
$\Rightarrow \langle h, h\rangle = 0$.
$\Rightarrow h = 0$.
$\therefore M \cap M^\perp = \{0\}$.

Since M is closed linear subspace of H and $M \perp M^{\perp}$.
$\therefore N = M + M^{\perp}$ is closed .
$M \subset N$ and $M^{\perp} \subset N$. $\because$ if $S_1 \subset S_2 \Rightarrow S_2^{\perp} \subset S_1^{\perp}$.
$\Rightarrow N^{\perp} \subset M^{\perp}$ and $N^{\perp} \subset M^{\perp\perp}$.
$\therefore N^{\perp} \subset M^{\perp} \cap M^{\perp\perp} = \{0\}$.
$\Rightarrow N^{\perp} = \{0\}$.
$\Rightarrow N^{\perp\perp} = \{0\}^{\perp}$.
$\Rightarrow N^{\perp\perp} = H$. $\because \{0\}^{\perp} = H$.
$\Rightarrow N = H$. $\because N$ is closed linear subspace of Hilbert space H.
$H = M + M^{\perp}$.
$\therefore H = M \oplus M^{\perp}$. ■

Note. In previous two theorems we have proved that, if M is closed linear subspace of H and $h \in H$, then there exist unique point f_0 in M such that $h - f_0 \in M^{\perp}$. Thus we can define a mapping $P : H \to M$ by $Ph = f_0$.

Theorem. *If M is closed linear subspace of H and $h \in H$, let Ph be the unique point in M such that $h - Ph \perp M$. Then*
(a) *P is a linear transformation on H,*
(b) *$\|Ph\| \leq \|h\|$ for every $h \in H$,*
(c) *$P^2 = P$(here P^2 is composition of P with itself),*
(d) *$\ker P = M^{\perp}$ and $ranP = M$.*

PROOF. (a) Suppose $P : H \to M$ is a function defined by $h \mapsto Ph$.
Let $h_1, h_2 \in H, f \in M$ and $\alpha, \beta \in F$.
We want to show $P(\alpha h_1 + \beta h_2) = \alpha P(h_1) + \beta P(h_2)$.
Consider,

$$\begin{aligned} \langle \alpha h_1 + \beta h_2 - (\alpha P(h_1) + \beta P(h_2)), f\rangle &= \langle \alpha h_1 - \alpha P(h_1), f\rangle + \langle \beta h_2 - \beta P(h_2), f\rangle \\ &= \alpha\langle h_1 - P(h_1), f\rangle + \beta\langle h_2 - P(h_2), f\rangle \\ &= \alpha(0) - \beta(0) \\ &= 0 \end{aligned}$$

.

$\therefore \langle \alpha h_1 + \beta h_2 - (\alpha P(h_1) + \beta P(h_2)), f\rangle = 0$.
Given that, $\forall h \in H, h - Ph \perp M$.
$\Rightarrow \alpha h_1 + \beta h_2 - P(\alpha h_1 + \beta h_2) \perp M$ and $P(\alpha h_1 + \beta h_2)$ is unique.
$\therefore P(\alpha h_1 + \beta h_2) = \alpha P(h_1) + \beta P(h_2)$.
$\therefore P$ is linear transformation.
(b) Given $h - Ph \perp M, \forall h \in H$.

$$\begin{aligned} \therefore \|h\|^2 &= \|h - Ph + Ph\|^2 \\ &= \|h - Ph\|^2 + \|Ph\|^2 \\ &\geq \|Ph\|^2 \end{aligned}$$

$\therefore \|Ph\|^2 \leq \|h\|$.
(c) Consider, $P^2(h) = P(P(h)) = Ph$. $\because Ph \in M \subset H \Rightarrow Ph \in H$.
$\therefore P^2 = P$.
(d) $\ker P = \{h \in H/Ph = 0\}$.

Let $h \in \ker P \implies Ph = 0$.
$h = h - 0 = h - Ph \in M^\perp$.
$\implies h \in M^\perp$.
$\implies \ker P \subset M^\perp$.
Now, Let $h \in M^\perp$.
$\implies h - 0 \in M^\perp$.
Also we know that $h - Ph \in M^\perp$ and Ph is unique.
$\implies Ph = 0$.
$\implies h \in \ker P$. $\implies M^\perp \subseteq \ker P$.
$\therefore \ker P = M^\perp$.
If M is closed linear subspace of H then $H = M \oplus M^\perp$ and we know that $P : H \to M$ is a linear transformation such that $P^2 = P$. Then $H = \ker P \oplus$ ran$P$.
$\implies \ker P +$ ran$P = H = M + M^\perp$.
$\implies$ ran$P = M$.
DEFINITION. If M is closed linear subspace of H and P is linear transformation from H to M, then P is called as Orthogonal projection of H onto M.
Result. If P is orthogonal projection onto M, then $I - P$ is orthogonal projection onto $M^\perp$.
PROOF. Let $f \in M^\perp$.
Consider,

$$\begin{aligned} \langle h - (I-P)h, f\rangle &= \langle h - h + Ph, f\rangle \\ &= \langle Ph, f\rangle \\ &= 0, \quad \forall f \in M \because f \in M^\perp \text{ and } Ph \in M. \implies f \perp Ph. \end{aligned}$$

$\therefore h - (I-P)h \perp M^\perp$.
Suppose $g \in M^\perp$.
$\implies h - (I-P)h \perp (I-P)h - g$. $\because h - (I-P)h \perp M^\perp$.

$$\begin{aligned} \therefore \|h - g\|^2 &= \|h - (I-P)h + (I-P)h - g\|^2 \\ &= \|h - (I-P)h\|^2 + \|(I-P)h - g\|^2 \quad \because \text{By Pythagorean theorem.} \\ &\geq \|h - (I-P)h\|^2 \end{aligned}$$

$\therefore \|h - g\| \geq \|h - (I-P)h\|, \quad \forall g \in M^\perp$.
$\implies \|h - (I-P)h\| = dist(h, M^\perp)$.
$\therefore (I-P)h$ is unique.
$\therefore I - P$ is orthogonal projection onto $M^\perp$.
Corollary. *If M is closed linear subspace of $H(M \leq H)$, then $(M^\perp)^\perp$.*
PROOF. Let P be orthogonal projection onto M, then $I - P$ is orthogonal projection onto $M^\perp$.
Also,

$$\begin{aligned} \ker(I-P) &= \{h \in H/(I-P)h = 0\} \\ &= \{h \in H/h - Ph = 0\} \\ &= \{h \in H/h = Ph\} \\ &= \text{ran}P \\ &= M \end{aligned}$$

For $I - P$ orthogonal projection of $M^\perp$.
$\ker(I-P) = (M^\perp)^\perp$.
$\implies \ker(I-P) = (M^\perp)^\perp$.
$\implies (M^\perp)^\perp = M$. ■

DEFINITION. If $A \subseteq H$, let $\vee A \equiv$ the intersection of all closed linear subspace of H that contains A. $\vee A$ is called closed linear span of A.

Corollary. *If $A \subseteq H$, then $(A^\perp)^\perp$ is closed linear span of A in H.*

PROOF. By definition of closed linear space $\vee A = \overline{[A]}$, where $[A]$ is linear span of A.
For any subset $A \subseteq H, A \subset (A^\perp)^\perp$. $\because (A^\perp)^\perp$ is closed linear subspace of H.
$\implies \overline{[A]} \subseteq (A^\perp)^\perp$. (1) $\because \overline{[A]}$ is smallest closed set which contains A.
Also, $A \subset [A] \implies A \subset \overline{[A]}$.
$\implies \overline{[A]}^\perp \subset A^\perp$.
$\implies (A^\perp)^\perp \subset \overline{[A]}^{\perp\perp} = \overline{[A]}$. (2)
From (1) and (2) we get, $\overline{[A]} = (A^\perp)^\perp$.
$\therefore$ Closed linear span of A is $(A^\perp)^\perp$. ■

Definition. Y is called linear manifold if Y is linear subspace of H and Y may not be closed.

Corollary. *If Y is linear manifold in H, then Y is dense in H iff $Y^\perp = (0)$.*

PROOF. Suppose Y is dense in H.
$\implies \overline{Y} = H$.
$\implies \overline{Y}^\perp = H^\perp = \{0\}$.
$\implies \overline{Y}^\perp = \{0\}$.
$\implies Y^\perp = \{0\}$.
Conversely, suppose $Y^\perp = \{0\}$.
We know that $(Y^\perp)^\perp = \overline{Y}$. $\because Y$ is linear space hence $(Y^\perp)^\perp = \overline{Y}$.
$\implies \overline{Y} = \{0\}^\perp$.
$\implies \overline{Y} = H$.
$\implies Y$ is dense in H. ■

Proposition. *Let H be a Hilbert space and $L : H \to F$ be a linear functional. Then following statements are equivalent.*
(a) *L is continuous.*
(b) *L is continuous at 0.*
(c) *L is continuous at some point.*
(d) *There exist a constant $c > 0$ such that $|L(h)| \le c\|h\|$ for every h in H.*

PROOF. Clearly, $(a) \implies (b) \implies (c)$.
$(d) \implies (b)$
Suppose there exists a constant $c > 0$ such that $|L(h)| \le c\|h\|$ for every h in H.
Now for given $\epsilon > 0$, choose $\delta = \epsilon/c$ such that $\|h - 0\| < \epsilon/c$.
$\implies |L(h)| \le c\|h\| < c \cdot \epsilon/c = \epsilon$.
$\implies |L(h) - L(0)| < \epsilon$.
$\implies L$ is continuous at 0.
$(b) \implies (d)$
Suppose L is continuous at 0.

$\implies$ For given $\epsilon = 1$, there exists $\delta > 0$ such that
$B(0,\delta) \subseteq L^{-1}(\{\alpha/|\alpha| < 1\})$. $\because$ Inverse image of open set is open.
$\implies$ For $h \in H$ and $\|h\| < \delta \implies |L(h)| < 1$.
Choose, $\frac{h\delta}{\|h\|+\epsilon} \in B(0,\delta)$. $\because \delta = \delta \cdot 1 > \|\frac{h\delta}{\|h\|+\epsilon}\|$.
$\therefore \|\frac{h\delta}{\|h\|+\epsilon}\| < \delta \implies |L(\frac{h\delta}{\|h\|+\epsilon})| < 1$.
$\implies \frac{\delta}{\|h\|+\epsilon}|L(h)| < 1$.
$\implies |L(h)| < \frac{1}{\delta}(\|h\| + \epsilon)$.
Let, $\epsilon \to 0$ in above inequality,
$\implies |L(h)| < \frac{1}{\delta}\|h\|$.
Choose, $c = \frac{1}{\delta}$.
$\implies |L(h)| < c\|h\|$.
$(c) \implies (a)$
Let L be continuous at some point $h_0 \in H$.
Let $h \in H$ be any arbitrary point and consider $\{h_n\} \to h$ as $n \to \infty$.
$\implies \{h_n - h + h_0\} \to h_0$ as $n \to \infty$.
$\implies \lim_{n\to\infty} L(h_n - h + h_0) = L(h_0)$. $\because L$ is continuous at h_0.
$\implies \lim_{n\to\infty} [L(h_n) - L(h) + L(h_0)] = L(h_0)$.
$\implies \lim_{n\to\infty} L(h_n) - L(h) + L(h_0) = L(h_0)$.
$\implies \lim_{n\to\infty} L(h_n) = L(h)$.
$\implies L$ is continuous at $h \in H$.
Therefore, L is continuous on H. ■

Definition. A bounded linear functional L on H is a linear functional for which there is a constant $c > 0$ such that $|L(h)| \leq c\|h\|$ for all $h \in H$.

Note. By preceding proposition, a linear functional is bounded if and only if it is continuous.

For a bounded linear functional $L : H \to F$, define $\|L\| = \sup\{|L(h)|/\|h\| < 1\}$.
By definition $\|L\| < \infty$ and $\|L\|$ is called the norm of L.

Proposition. *If L is a bounded linear functional, then*

$$\begin{aligned}\|L\| &= \sup\{|L(h)| : \|h\| = 1\} \\ &= \sup\{|L(h)|/\|h\| : h \in H, h \neq 0\} \\ &= \inf\{c > 0 : |L(h)| \leq c\|h\|, h \in H\}.\end{aligned}$$

PROOF. Let $\alpha = \inf\{c > 0 : |L(h)| < c\|h\|, h \in H\}$. (1)
By definition of norm L,
$\|L\| = \sup\{|L(h)| : \|h\| < 1\}$. For a given $\epsilon > 0, |L(\frac{h}{\|h\|+\epsilon})| \leq \|L\|$.
$\implies \frac{h}{\|h\|+\epsilon}|L(h)| \leq \|L\|$.
$\implies |L(h)| \leq (\|h\| + \epsilon)\|L\|$.
Taking $\epsilon \to 0$,
$|L(h)| \leq \|L\|\|h\|$.
$\implies \alpha \leq \|L\|$. (*)
Let $|L(h)| \leq c\|h\|$.
If $\|h\| \leq 1$.
$|L(h)| \leq c$.

$\Longrightarrow \|L\| \leq c.$ $\because$ by definition of $\|L\|$.
Taking infimum on both side $\Longrightarrow \|L\| \leq \alpha.$ (**)
$\therefore \alpha = \|L\|$ $\because$ from (*) and (**).
Let $\beta = \sup\{|L(h)| : \|h\| < 1\}$.
Clearly, $\beta \leq \|L\|$. $\because$ by definition of $\|L\|$.
Consider,

$$\begin{aligned} |L(h)| &= |\tfrac{L(h)}{\|h\|} \cdot \|h\|| \\ &= |L(\tfrac{h}{\|h\|})| \cdot \|h\| \\ &\leq \sup\{|L(z)| : \|z\| = 1, z \in H\} \cdot \|h\|. \end{aligned}$$

$\therefore |L(h)| \leq \beta\|h\|$.
$\Longrightarrow \alpha \leq \beta.$ $\because$ by definition of α
$\Longrightarrow \|L\| \leq \beta.$
$\Longrightarrow \beta = \|L\|.$
Let $\gamma = \sup\left\{\frac{|L(h)|}{\|h\|} : h \in H, h \neq 0\right\}$.
$|L(h)| = |\frac{L(h)}{\|h\|} \cdot \|h\||$.
$\Longrightarrow |L(h)| = \frac{|L(h)|}{\|h\|} \cdot \|h\|$.
$\Longrightarrow |L(h)| \leq \sup\left\{\frac{|L(h)|}{\|h\|} : h \in H, h \neq 0\right\} \cdot \|h\|$.
$\Longrightarrow |L(h)| \leq \gamma \cdot \|h\|$.
$\Longrightarrow \alpha \leq \gamma.$ $\because$ by definition of α
$\Longrightarrow \|L\| \leq \gamma.$
Clearly, $\gamma \leq \|L\|$.
$\therefore \gamma = \|L\|$. ∎
Fix an h_0 in H and define $L : H \to F$ by $L(h) = \langle h, h_0\rangle$.
For $h_1, h_2 \in H$ and $\alpha \in F$.
Consider,

$$\begin{aligned} L(h_1 + \alpha h_2) &= \langle h_1 + \alpha h_2, h_0\rangle \\ &= \langle h_1, h_0\rangle + \langle \alpha h_2, h_0\rangle \\ &= \langle h_1, h_0\rangle + \alpha\langle h_2, h_0\rangle \\ &= L(h_1) + \alpha L(h_2) \end{aligned}$$

$\Longrightarrow L$ is linear.
Also, by CBS inequality $|L(h)| = |\langle h, h_0\rangle| \leq \|h\|\|h_0\|$.
$\Longrightarrow L$ is bounded.
$\Longrightarrow \|L\| \leq \|h_0\|$.
Now for $\frac{h_0}{\|h_0\|} \in H$.
$L(\frac{h_0}{\|h_0\|}) = \langle \frac{h_0}{\|h_0\|}, h_0\rangle$.
$= \frac{1}{\|h_0\|}\langle h_0, h_0\rangle$.
$= \frac{\|h_0\|^2}{\|h_0\|}$.
$\Longrightarrow L(\frac{h_0}{\|h_0\|}) = \|h_0\|$.
$\Longrightarrow \|L\| = \|h_0\|$.
The Riesz Representation Theorem. *If $L : H \to F$ is a bounded linear functional, then there is a unique vector $h_0 \in H$ such that $L(h) = \langle h, h_0\rangle$ for every $h \in H$. Moreover,*

$\|L\| = \|h_0\|$.
PROOF. Let $M = \ker L$.
Let $h_1, h_2 \in \ker L$ and $\alpha \in F$.
Consider, $L(h_1 + \alpha h_2) = L(h_1) + \alpha L(h_2) = 0$. $\because h_1, h_2 \in \ker L \Rightarrow L(h_1) = 0 = L(h_2)$.
$\Longrightarrow \ker L$ is linear subspace of H.
Let $x \in H$ be a limit point of $\ker L$.
$\Longrightarrow \exists \{x_n\}$ of points from $\ker L$ such that $\{x_n\} \to x$.
$\Longrightarrow \lim\limits_{n\to\infty} L(\{x_n\}) = L(x)$. $\because L$ is linear and bounded$\Rightarrow L$ is continuous.
$\Longrightarrow 0 = L(x)$. $\because x_n \in \ker L$ for all n.
$\Longrightarrow x \in \ker L$.
$\Longrightarrow \ker L$ is closed linear subspace of H.
$\therefore M$ is closed linear subspace of H.
case(i) If $M = H \Longrightarrow M^{\perp} = (0)$.
Then, simply choose $L(h) = \langle h, 0\rangle \Longrightarrow \|L\| = \|h_0\|$, where $h_0 = 0$.
case(ii) If $M \neq H \Longrightarrow M^{\perp} \neq (0)$.
$\Longrightarrow \exists$ some non-zero $f_0 \in M^{\perp}$ such that $L(f_0) = 1$.
Now if $h \in H$ and $\alpha = L(h)$, then $L(h - \alpha f_0) = L(h) - \alpha L(f_0) = L(h) - L(h) = 0$.
$\Longrightarrow h - \alpha f_0 \in \ker L = M$.
$\Longrightarrow h - \alpha f_0 \in M$.
$\Longrightarrow \langle h - \alpha f_0, f_0\rangle = 0$. $\because f_0 \in M^{\perp}$.
$\Longrightarrow \langle h, f_0\rangle - \alpha\langle f_0, f_0\rangle = 0$.
$\Longrightarrow \langle h, f_0\rangle - \alpha\|f_0\|^2 = 0$.
$\Longrightarrow \alpha = \langle h, \frac{f_0}{\|f_0\|^2}\rangle$.
$\Longrightarrow L(h) = \langle h, \frac{f_0}{\|f_0\|^2}\rangle$.
Choose, $h_0 = \frac{f_0}{\|f_0\|^2}$.
$\Longrightarrow L(h) = \langle h, h_0\rangle$.
Uniqueness, Suppose there are two $h_1, h_2 \in H$ such that $L(h) = \langle h, h_1\rangle$ and $L(h) = \langle h, h_2\rangle$ for all $h \in H$.
$\Longrightarrow \langle h, h_1\rangle = \langle h, h_2\rangle$.
$\Longrightarrow \langle h, h_1\rangle - \langle h, h_2\rangle = 0$.
$\Longrightarrow \langle h, h_1 - h_2\rangle = 0, \quad \forall h \in H$.
$\Longrightarrow h_1 - h_2 = 0$.
$\Longleftarrow h_1 = h_2$.
Also, we have proved $\|L\| = \|h_0\|$. ■

Corollary. *If (X, Ω, μ) is a measure space and $F : L^2(\mu) \to \mathbb{F}$ is bounded linear functional, then there is unique h_0 in $L^2(\mu)$ such that $F(h) = \int h\bar{h}_0 d\mu$, for every h in $L^2(\mu)$.*
PROOF. Choose, $H = L^2(\mu)$ and $L = F$, then by Riesz representation theorem there exists $h_0 \in H$ such that,
$L(h) = \langle h, h_0\rangle$.
$\Longrightarrow L(h) = \int h\bar{h}_0 d\mu$.
$\Longrightarrow F(h) = \int h\bar{h}_0 d\mu$. ■

ORTHOGONAL SET OF VECTORS AND BASES

Definition. An orthonormal subset of a Hilbert space H is subset $\mathcal{E}$ having the properties:

(a) For $e \in \mathcal{E}, \|e\| = 1$.
(b) If $e_1, e_2 \in \mathcal{E}$ and $e_1 \neq e_2$, then $e_1 \perp e_2$.
Definition. A basis for H is a maximal orthonormal set. Also called as Hamal basis.
Note. The concept of basis is different from Hamal basis because for an infinite dimensional Hilbert space, basis is not Hamal basis.
Proposition. *Every Hilbert space has an orthonormal basis.*
PROOF. Let $\sum$ be the collection of all orthonormal subsets of H ordered by inclusion. We know that $0 \neq h \in H$, then the singleton set $\left\{\frac{h}{\|h\|}\right\}$ is orthonormal and hence $\sum$ is non-empty. If we take $C : E_1 \subseteq E_2 \subseteq \ldots$ be chain in $\sum$, then $\cup E_i, i = 1, 2, \ldots$ is orthonormal set which is upper bound. Therefore by Zorn's Lemma there exists a maximal element $\mathcal{E} \in \sum$ which is required maximal orthonormal set in H. ■
Example 1. Let $H = L^2_{\mathbb{C}}[0, 2\pi]$ and for n in $\mathbb{Z}$ define $e_n \in H$ by $e_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}$. Then $\{e_n : n \in \mathbb{Z}\}$ is an orthogonal set in H.
Solution. Let $e_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}$ then,

$$\begin{aligned} \|e_n(t)\| &= \langle e_n, e_n \rangle^{1/2} \\ &= (\int_0^{2\pi} e_n \overline{e_n} dt)^{1/2} \\ &= (\int_0^{2\pi} |e_n|^2 dt)^{1/2} \\ &= (\int_0^{2\pi} \frac{1}{2\pi} dt)^{1/2} \\ &= \frac{1}{\sqrt{2\pi}} \sqrt{2\pi} \\ &= 1 \end{aligned}$$

$\implies \|e_n\| = 1$.
Now if $n \neq m$, then

$$\begin{aligned} \langle e_n, e_m \rangle &= \int_0^{2\pi} e_n \overline{e_m} dt \\ &= \int_0^{2\pi} \frac{1}{\sqrt{2\pi}} e^{int} \frac{1}{\sqrt{2\pi}} e^{-imt} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)t} dt \\ &= 0 \end{aligned}$$

$\therefore \{e_n : n \in \mathbb{Z}\}$ is orthogonal.
Example 2. If $H = F^d$ and for $1 \leq k \leq d, e_k =$ the d-tuple with 1 in the k^{th} place and zeros elsewhere, then $\{e_1, e_2, \ldots, e_d\}$ is a basis for H.
Solution. Exercise.
Example 3. Let $H = l^2(I)$. For each $i \in I$ define e_i in H by $e_i(i) = 1$ and $e_i(j) = 0$ for all $i \neq j$. Then $\{e_i : i \in I\}$ is a basis.
Solution. Exercise.
The Gram-Schmidt Orthogonalization Process. If H is a Hilbert space and $\{h_n : n \in \mathbb{N}\}$ is a linearly independent subset of H, then there is an orthonormal set $\{e_n : n \in \mathbb{N}\}$ such that for every n, the linear space of $\{e_1, e_2, \ldots, e_n\}$ equals the linear span of $\{h_1, h_2, \ldots, h_n\}$.
Proposition. *Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal set in H and let $M = \vee \{e_1, e_2, \ldots, e_n\}$. If P is the orthogonal projection on H onto M, then*

$$Ph = \sum_{k=1}^{n} \langle h, e_k \rangle e_k$$

for all h in H.

PROOF. Let $Qh = \sum_{k=1}^{n} \langle h, e_k \rangle e_k$

$$\begin{aligned}
\langle Qh, e_1 \rangle &= \langle \sum_{k=1}^{n} \langle h, e_k \rangle e_k, e_1 \rangle \\
&= \langle \langle h, e_1 \rangle e_1 + \langle h, e_2 \rangle e_2 + ... + \langle h, e_n \rangle e_n, e_1 \rangle \\
&= \langle h, e_1 \rangle \langle e_1, e_1 \rangle + \langle h, e_2 \rangle \langle e_2, e_1 \rangle + ... + \langle h, e_n \rangle \langle e_n, e_1 \rangle \\
&= \langle h, e_1 \rangle
\end{aligned}$$

Consider, $\langle h - Qh, e_1 \rangle = \langle h, e_1 \rangle - \langle Qh, e_1 \rangle = 0$.
$\Longrightarrow h - Qh \perp e_1$.
Similarly, $h - Qh \perp e_j \quad \forall 1 \leq j \leq n$.
$\Longrightarrow h - Qh \perp M, \quad \forall h \in H$.
But we know that P is orthogonal projection of H on M, then $h - Ph \perp M$ and Ph is unique.
$\Longrightarrow Ph = Qh$.
$\Longrightarrow Ph = \sum_{k=1}^{n} \langle h, e_k \rangle e_k$. ■

Bessel's Inequality. *If $\{e_n : n \in \mathbb{N}\}$ is an orthonormal set and $h \in H$, then*

$$\sum_{n=1}^{\infty} |\langle h, e_n \rangle|^2 \leq \|h\|^2.$$

PROOF. For any fixed n, let $h_n = h - \sum_{k=1}^{n} \langle h, e_k \rangle e_k$.
Consider,

$$\begin{aligned}
\langle h_n, e_1 \rangle &= \langle h - \sum_{n=1}^{n} \langle h, e_k \rangle e_k, e_1 \rangle \\
&= \langle h, e_1 \rangle - \langle \sum_{n=1}^{n} \langle h, e_k \rangle e_k, e_1 \rangle \\
&= \langle h, e_1 \rangle - \langle h, e_1 \rangle \langle e_1, e_1 \rangle - \langle h, e_2 \rangle \langle e_2, e_1 \rangle - ... - \langle h, e_n \rangle \langle e_n, e_1 \rangle \\
&= \langle h, e_1 \rangle - \langle h, e_1 \rangle \\
&= 0
\end{aligned}$$

$\Longrightarrow h_n \perp e_1$.
Similarly, $h_n \perp e_j, \quad \forall 1 \leq j \leq n$.
Also,

$$\begin{aligned}
\langle h_n, \sum_{k=1}^{n} \langle h, e_k \rangle e_k \rangle &= \langle h_n, \langle h, e_1 \rangle e_1 + \langle h, e_2 \rangle e_2 + ... + \langle h, e_n \rangle e_n \rangle \\
&= \overline{\langle h, e_1 \rangle} \langle h_n, e_1 \rangle + \overline{\langle h, e_2 \rangle} \langle h_n, e_2 \rangle + ... + \overline{\langle h, e_n \rangle} \langle h_n, e_n \rangle \\
&= 0
\end{aligned}$$

Since, $h = h_n + \sum_{k=1}^{n} \langle h, e_k \rangle e_k$.

$$\begin{aligned} \implies \|h\|^2 &= \|h_n + \sum_{k=1}^{n} \langle h, e_k \rangle e_k\|^2 && \because \text{By Pythagorean theorem} \\ &= \|h_n\|^2 + \|\sum_{k=1}^{n} \langle h, e_k \rangle e_k\|^2 && \because h_n \perp \sum_{k=1}^{n} \langle h, e_k \rangle e_k \\ &\geq \|\sum_{k=1}^{n} \langle h, e_k \rangle e_k\|^2 \qquad (1) \end{aligned}$$

Now consider,

$$\begin{aligned} \|\sum_{k=1}^{n} \langle h, e_k \rangle e_k\|^2 &= \langle \sum_{k=1}^{n} \langle h, e_k \rangle e_k, \sum_{k=1}^{n} \langle h, e_k \rangle e_k \rangle \\ &= \langle \langle h, e_1 \rangle e_1 + \langle h, e_2 \rangle e_2 + ... + \langle h, e_n \rangle e_n, \langle h, e_1 \rangle e_1 + \langle h, e_2 \rangle e_2 + ... + \langle h, e_n \rangle e_n \rangle \\ &= \langle h, e_1 \rangle \overline{\langle h, e_1 \rangle} \langle e_1, e_1 \rangle + \langle h, e_2 \rangle \overline{\langle h, e_2 \rangle} \langle e_2, e_2 \rangle + ... + \langle h, e_n \rangle \overline{\langle h, e_n \rangle} \langle e_n, e_n \rangle \\ &= |\langle h, e_1 \rangle|^2 + |\langle h, e_2 \rangle|^2 + ... + |\langle h, e_n \rangle|^2 \\ &= \sum_{k=1}^{n} |\langle h, e_k \rangle|^2. \end{aligned}$$

Therefore, inequality (1)

$\implies \|h\|^2 \geq \sum_{k=1}^{n} |\langle h, e_k \rangle|^2, \quad \forall h \in H.$

$\implies \sum_{k=1}^{\infty} |\langle h, e_k \rangle|^2 \leq \|h\|^2.$ ■

Proposition. *If $\mathcal{E}$ is an orthonormal set in H and $h \in H$, then $\langle h, e \rangle \neq 0$ for at most a countable number of vectors $e \in \mathcal{E}$.*

PROOF. Let $E_n = \left\{ e \in \mathcal{E} / |\langle h, e \rangle| \geq \frac{1}{n} \right\}$.

$\therefore \sum_{n=1}^{k} \frac{1}{n^2} \leq \sum_{n=1}^{k} |\langle h, e \rangle|^2 \leq \|h\|^2.$

$\therefore E_n$ must be finite.

$\cup_{n=1}^{\infty} E_n$ is countable union of finite set and hence countable.

$\therefore \langle h, e \rangle \neq 0$ for countable number of $e \in \mathcal{E}$. ■

Corollary. *Let $\mathcal{E}$ be an orthonormal set, then* $\sum_{e \in \mathcal{E}} |\langle h, e \rangle|^2 \leq \|h\|^2$.

PROOF. Proof of this corollary follows from Bassel's inequality and above corollary. ■

Result. Show that $\sum_{n=1}^{\infty} |\langle h, e_n \rangle|^2 = \|h\|^2$ iff $h = \sum_{n=1}^{\infty} \langle h, e_n \rangle e_n$.

PROOF. Let, $h_m = \sum_{n=1}^{m} \langle h, e_n \rangle e_n$.

$$\begin{aligned}
\|h_m - h\|^2 &= \langle h_m - h, h_m - h\rangle \\
&= \|h_m\|^2 - \langle h_m, h\rangle - \langle h, h_m\rangle + \|h\|^2 \\
&= \|h_m\|^2 - \langle \sum_{n=1}^{m}\langle h, e_n\rangle e_n, h\rangle - \langle h, \sum_{n=1}^{m}\langle h, e_n\rangle e_n\rangle + \|h\|^2 \\
&= \|h_m\|^2 - \langle\langle h, e_1\rangle e_1 + \langle h, e_2\rangle e_2 + ... + \langle h, e_m\rangle e_m, h\rangle \\
&\quad -\langle h, \langle h, e_1\rangle e_1 + \langle h, e_2\rangle e_2 + ... + \langle h, e_m\rangle e_m\rangle + \|h\|^2 \\
&= \|h_m\|^2 - \langle h, e_1\rangle\langle e_1, h\rangle - \langle h, e_2\rangle\langle e_2, h\rangle - ... - \langle h, e_m\rangle\langle e_m, h\rangle \\
&\quad -\overline{\langle h, e_1\rangle}\langle h, e_1\rangle - \overline{\langle h, e_2\rangle}\langle h, e_2\rangle - ... - \overline{\langle h, e_m\rangle}\langle h, e_m\rangle + \|h\|^2 \\
&= \|h_m\|^2 - \langle h, e_1\rangle\overline{\langle h, e_1\rangle} - \langle h, e_2\rangle\overline{\langle h, e_2\rangle} - ... - \langle h, e_m\rangle\overline{\langle h, e_m\rangle} \\
&\quad -\overline{\langle h, e_1\rangle}\langle h, e_1\rangle - \overline{\langle h, e_2\rangle}\langle h, e_2\rangle - ... - \overline{\langle h, e_m\rangle}\langle h, e_m\rangle + \|h\|^2 \\
&= \|h_m\|^2 - 2\langle h, e_1\rangle\overline{\langle h, e_1\rangle} - 2\langle h, e_2\rangle\overline{\langle h, e_2\rangle} - ... - 2\langle h, e_m\rangle\overline{\langle h, e_m\rangle} + \|h\|^2 \\
&= \|h_m\|^2 - 2\sum_{n=1}^{m}|\langle h, e_n\rangle|^2 + \|h\|^2 \\
&= \|h\|^2 - \sum_{n=1}^{m}|\langle h, e_n\rangle|^2. \qquad \because \|h_m\|^2 = \sum_{n=1}^{m}|\langle h, e_n\rangle|^2
\end{aligned}$$

$\therefore \|h_m - h\|^2 = \|h\|^2 - \sum_{n=1}^{m}|\langle h, e_n\rangle|^2.$

Now taking $m \to \infty$.

$\Longrightarrow \|h - \sum_{n=1}^{\infty}\langle h, e_n\rangle e_n\|^2 = \|h\|^2 - \sum_{n=1}^{\infty}|\langle h, e_n\rangle|^2.$

$\Longrightarrow \|h - \sum_{n=1}^{\infty}\langle h, e_n\rangle e_n\|^2 = \|h\|^2 - \|h\|^2.$

$\Longrightarrow \|h - \sum_{n=1}^{\infty}\langle h, e_n\rangle e_n\|^2 = 0.$

$\Longrightarrow h - \sum_{n=1}^{\infty}\langle h, e_n\rangle e_n = 0.$

$\Longrightarrow h = \sum_{n=1}^{\infty}\langle h, e_n\rangle e_n.$

Converse is left for exercise. ■

If I is an infinite uncountable set. Then consider $\mathcal{F}$ be collection of all finite subset of I. Define order on $\mathcal{F}$ by inclusion, then define $h_F = \sum\{h_i/i \in F, F \in \mathcal{F}\}$. Then $\{h_F/F \in \mathcal{F}\}$ is called as net in H.

Definition. The sum $\sum\{h_i/i \in I\}$ is convergent if then net $\{h_F/F \in \mathcal{F}\}$ converges; the value of sum is the limit of the net.

Result. If I is countable, then $\sum\{h_i/i \in I\}$ converges implies $\sum_{n=1}^{\infty} h_n$ converges.

PROOF. Let $|F| = n$ for each $F \in \mathcal{F}$.
Then $h_F = h_1 + h_2 + ... + h_n$.
$\Longrightarrow \sum_{n=1}^{\infty} h_n$ converges.
Converse is not true.
That is, convergence of $\sum_{n=1}^{\infty} h_n$ does not implies convergence of $\sum\{h_i/i \in I\}$.

Lemma. *If $\mathcal{E}$ is an orthonormal set and $h \in H$, then $\sum\{\langle h, e\rangle e/e \in \mathcal{E}\}$ converges in H.*
PROOF. We have seen that $\{e \in \mathcal{E}/\langle h, e\rangle \neq 0\}$ is countable.
Therefore, $\{e \in \mathcal{E}/\langle h, e\rangle \neq 0\} = \{e_1, e_2, ...\}$.
From Bessel's inequality we have,
$\sum_{n=1}^{\infty} |\langle h, e_n\rangle|^2 \leq \|h\|^2$.
$\therefore \exists$ some $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} |\langle h, e_n\rangle|^2 < \epsilon$.
$F_0 = \{e_1, e_2, ..., e_{N-1}\}$.
Define, $h_F = \sum\{\langle h, e\rangle e/e \in F\}$, where $F \in \mathcal{F}$ and $\mathcal{F}$ is collection of all finite subsets of $\mathcal{E}$.
Let F and G be two members of $\mathcal{F}$ such that $F_0 \subseteq F$ and $F_0 \subseteq G$.
Consider,

$$\begin{aligned}\|h_F - h_G\|^2 &= \sum\{|\langle h, e\rangle|^2 : e \in (F - G) \cup (G - F)\} \\ &< \sum_{n=N}^{\infty} |\langle h, e_n\rangle|^2 \\ &< \epsilon\end{aligned}$$

$\therefore \{h_F : F \in \mathcal{F}\}$ is Cauchy net and hence convergent and converges to $\sum_{n=1}^{\infty}\langle h, e_n\rangle e_n$. ■

Theorem. *If $\mathcal{E}$ is an orthonormal set in H, then following statements are equivalent.*
(a) *$\mathcal{E}$ is a basis for H.*
(b) *If $h \in H$ and $h \perp \mathcal{E}$, then $h = 0$.*
(c) $\vee\mathcal{E} = H$.
(d) **If** $h \in H$**, then** $h = \sum\{\langle h, e\rangle e : e \in \mathcal{E}\}$.
(e) *If $h, g \in H$, then $\langle g, h\rangle = \sum\{\langle g, e\rangle\langle e, h\rangle : e \in \mathcal{E}\}$.*
(f) *If $h \in H$, then $\|h\|^2 = \sum\{|\langle h, e\rangle|^2 : e \in \mathcal{E}\}$ (Parseval's Inequality).*
PROOF. (a) $\Longrightarrow$ (b)
Suppose $\mathcal{E}$ is basis for H.

$\Longrightarrow \mathcal{E}$ is maximal orthonormal set in H.
If $h \in H$ and $h \perp \mathcal{E}$ then $h = 0$.
Because if $h \neq 0$ then $\mathcal{E}_1 = \mathcal{E} \cup \left\{ \frac{h}{\|h\|} \right\}$ become orthonormal set containing $\mathcal{E}$.
$\rightarrow\leftarrow$ to maximality of $\mathcal{E}$.
(b) $\Longleftrightarrow$ (c)
Suppose, if $h \in H$ and $h \perp \mathcal{E}$, then $h = 0$.
$\Longleftrightarrow \mathcal{E}^{\perp} = \{0\}$.
$\Longleftrightarrow (\mathcal{E}^{\perp})^{\perp} = \{0\}^{\perp}$.
$\Longleftrightarrow \vee\mathcal{E} = H$.
For $h \in H, h = \sum \{\langle h, e\rangle e : e \in \mathcal{E}\}$.
(b) $\Longrightarrow$ (d)
If $h \in H$ and $h \perp \mathcal{E}$ then $h = 0$.
To show: If $h \in H$ then $h = \sum \{\langle h, e\rangle e : e \in \mathcal{E}\}$.
Suppose $f = h - \sum \{\langle h, e\rangle e : e \in \mathcal{E}\}$.
$\Longrightarrow$ If $e_1 \in \mathcal{E}$, then $\langle f, e_1\rangle = \langle h, e_1\rangle - \sum \{\langle h, e\rangle\langle e, e_1\rangle : e \in \mathcal{E}\}$.
$\Longrightarrow \langle f, e_1\rangle = \langle h, e_1\rangle - \langle h, e_1\rangle = 0$.
$\Longrightarrow \langle f, e_1\rangle = 0$.
$\Longrightarrow f \in \mathcal{E}^{\perp}$ hence $f = 0$.
$\Longrightarrow h = \sum \{\langle h, e\rangle e : e \in \mathcal{E}\}$.
(d) $\Longrightarrow$ (e)
Suppose $h, g \in H$ then $h = \sum \{\langle h, e\rangle e : e \in \mathcal{E}\}$ and $g = \sum \{\langle g, e\rangle e : e \in \mathcal{E}\}$.
Consider,
$\langle g, h\rangle = \langle \sum \langle g, e\rangle e, \sum \langle h, e\rangle e\rangle$
$\Longrightarrow \langle g, h\rangle = \sum \left\{ \langle g, e\rangle \overline{\langle h, e\rangle} \langle e, e\rangle : e \in \mathcal{E} \right\}$
$\Longrightarrow \langle g, h\rangle = \sum \{\langle g, e\rangle\langle e, h\rangle : e \in \mathcal{E}\}$
(e) $\Longrightarrow$ (f)
Suppose $h, g \in H$, then $\langle g, h\rangle = \sum \{\langle g, e\rangle\langle e, h\rangle : e \in \mathcal{E}\}$.
Consider, $\|h\|^2 = \langle h, h\rangle$.
$\Longrightarrow \|h\|^2 = \sum \{\langle h, e\rangle\langle e, h\rangle : e \in \mathcal{E}\}$.
$\Longrightarrow \|h\|^2 = \sum \{|\langle h, e\rangle|^2 : e \in \mathcal{E}\}$.
(f) $\Longrightarrow$ (a)
Suppose $\mathcal{E}$ is orthonormal set in H.
To show: $\mathcal{E}$ is maximal orthonormal set in H.
On contrary assume that $\mathcal{E}$ is not maximal orthonormal set.
$\Longrightarrow \exists$ orthonormal set $\mathcal{E}_1$ such that $\mathcal{E} \subseteq \mathcal{E}_1$.
$\Longrightarrow$ There is unit vector $e_0 \in \mathcal{E}_1$ such that $\|e_0\| = 1, e_0 \perp \mathcal{E}$.
But by our assumption, $\|e_0\|^2 = \sum \{|\langle e_0, e\rangle|^2 : e \in \mathcal{E}\} = 0$.
$\rightarrow\leftarrow$.
Therefore, $\mathcal{E}$ is maximal orthonormal set in H and hence basis. ∎

Proposition. *If H is a Hilbert space, any two bases have same cardinality.*
PROOF. Let H be Hilbert space.
Consider, $\mathcal{E}$ and $\mathcal{F}$ be two bases for H.
case(i) If $|\mathcal{E}| = \eta_1$ and $|\mathcal{F}| = \eta_2$, where η_1, η_2 both are infinite.
For each $e \in \mathcal{E}$, consider, $\mathcal{F}_e = \{f \in \mathcal{F} / \langle f, e\rangle \neq 0\}$.

$\implies \mathcal{F}_e$ is countable.
Each $f \in \mathcal{F}$ must belongs to one of the $\mathcal{F}_e$.
If $f \in \mathcal{F}$ and $f \notin \mathcal{F}_e$ for all $e \in \mathcal{E}$.
$\implies \langle f, e\rangle = 0, \quad \forall e \in \mathcal{E}$.
$\implies f \perp \mathcal{E} \implies f = 0 \rightarrow\leftarrow$.
$\therefore \mathcal{F} = \cup_{e\in\mathcal{E}}\mathcal{F}_e$.
$\implies |\mathcal{F}| \leq \sum_{e\in\mathcal{E}} |\mathcal{F}_e|$.
$\implies \eta_2 \leq \eta_1 \cdot \aleph_0 = \eta_1$.
$\therefore \eta_2 \leq \eta_1$.
Similarly, $\eta_1 \leq \eta_2$.
$\implies \eta_1 = \eta_2$.
case(ii) If $|\mathcal{E}| = \eta_1$ and $|\mathcal{F}| = \eta_2$, where η_1, η_2 both are finite.
If $\mathcal{E}$ is basis for H and $\mathcal{F}$ is orthonormal set in H.
$\implies \eta_2 \leq \eta_1$. $\because \mathcal{E}$ is maximal orthonomal set.
Similarly, if $\mathcal{F}$ is basis of H and $\mathcal{E}$ is orthonormal set in H.
$\implies \eta_1 \leq \eta_2$. $\because \mathcal{F}$ is maximal orthonomal set.
$\therefore \eta_1 = \eta_2$. ■

Definition. The dimension of a Hilbert space is cardinality of a basis and denoted by $\dim H$.

Result. If (X, d) is a metric space that is separable and $\{B_i = B(x_i, \epsilon_i) : i \in I\}$ is a collection of pairwise disjoint open balls in X, then I must be countable.
PROOF. Let D be countable dense set in X.
Then for each $B(x_i, \epsilon_i), B(x_i, \epsilon_i) \cap D \neq \phi$.
Let $y_i \in B(x_i, \epsilon_i) \cap D$.
$\{y_i/i \in I\}$ is countable. $\because y_i \in D$ and D is countable.
$\implies I$ is countable. ■

Proposition. *If H is infinite dimensional Hilbert space, then H is separable if and only if* $\dim H = \aleph_0$.
PROOF. Suppose H is separable Hilbert space.
Let $\mathcal{E}$ be a basis for H.
For $e_1, e_2 \in \mathcal{E}, \|e_1 - e_2\|^2 = \|e_1\|^2 + \|e_2\|^2$.
$\implies \|e_1 - e_2\| = \sqrt{2}$.
Hence, $\left\{B(e, \frac{1}{\sqrt{2}}) : e \in \mathcal{E}\right\}$ is collection of pairwise disjoint open balls in H.
$\implies \mathcal{E}$ is countable.
$\therefore \dim H = \aleph_0$.
Conversely, suppose $\dim H = \aleph_0$.
That is, H has countable basis. Say $\mathcal{E} = \{e_1, e_2, ...\}$.
If H is Hilbert space over R, then $D_n = \left\{\sum_{i=1}^{n} q_i e_i : q_i \in \mathbb{Q} \text{ for } 1 \leq i \leq n\right\}$ are countable for $1 \leq n \leq \infty$.
$\implies D = \cup_{n=1}^{\infty} D_n$ is countable and dense in H.
The closure of each D_n is seen to be the linear span of $\{e_1, e_2, ..., e_n\}$ and so the closure

of D includes all finite linear combinations $\sum_{i=1}^{n} h_i e_i$. But, each $h \in H$ is a limit of such finite linear combinations. Hence the closure of D is all of H. As D is countable, this shows that H must be separable.
If H is Hilbert space over $\mathbb{C}$. Then we take $q_i \in \mathbb{Q} + i\mathbb{Q}$, so that we can get all finite linear combinations of e_i in the closure of D. ■

ISOMORPHIC HILBERT SPACES

Definition. If H and K are Hilbert spaces, an isomorphism between H and K is a linear surjection $U : H \to K$ such that
$\langle Uh, Ug\rangle = \langle h, g\rangle$ for all $h, g \in H$. In this case H and K are said to be isomorphic.

Definition. An isometry between metric spaces is a map that preserves distance.
That is, A map $T : X \to Y$ is called an isometry if $d_X(x_1, x_2) = d_Y(T(x_1), T(x_2)) \quad \forall x_1, x_2 \in X$.

Theorem. *If $V : H \to K$ is a linear map between Hilbert spaces, then V is isometry if and only if $\langle Vh, Vg\rangle = \langle h, g\rangle \quad \forall h, g \in H$.*

PROOF. Let $V : H \to K$ is an isometry.
That is, V preserves distance.
$\therefore$ For $h, g \in H$ and $\lambda \in \mathbb{F}$, $\|h + \lambda g\|^2 = \|V(h + \lambda g)\|^2$.

$$\Longrightarrow \langle h + \lambda g, h + \lambda g\rangle = \langle Vh + \lambda Vg, Vh + \lambda Vg\rangle.$$

$$\Longrightarrow \langle h, h\rangle + \langle h, \lambda g\rangle + \langle \lambda g, h\rangle + \langle \lambda g, \lambda g\rangle = \langle Vh, Vh\rangle + \langle Vh, \lambda Vg\rangle + \langle \lambda Vg, Vh\rangle + \langle \lambda Vg, \lambda Vg\rangle.$$

$$\Longrightarrow \|h\|^2 + \overline{\lambda}\langle h, g\rangle + \lambda\langle g, h\rangle + \lambda\overline{\lambda}\|g\|^2 = \|Vh\|^2 + \overline{\lambda}\langle Vh, Vg\rangle + \lambda\langle Vg, Vh\rangle + \lambda\overline{\lambda}\|Vg\|^2.$$

$$\Longrightarrow \overline{\lambda}\langle h, g\rangle + \overline{\overline{\lambda}\langle h, g\rangle} = \overline{\lambda}\langle Vh, Vg\rangle + \overline{\overline{\lambda}\langle Vh, Vg\rangle}.$$

$$\Longrightarrow 2\mathrm{Re}\overline{\lambda}\langle h, g\rangle = 2\mathrm{Re}\overline{\lambda}\langle Vh, Vg\rangle.$$

Case (i) If $\mathbb{F} = \mathbb{R}$.
$\Longrightarrow \mathrm{Re}\overline{\lambda}\langle h, g\rangle = \mathrm{Re}\overline{\lambda}\langle Vh, Vg\rangle$.
$\Longrightarrow \langle h, g\rangle = \langle Vh, Vg\rangle$.
Case (ii) If $\mathbb{F} = \mathbb{C}$.
$\lambda = 1 \Longrightarrow \mathrm{Re}\langle h, g\rangle = \mathrm{Re}\langle Vh, Vg\rangle$. (1)
$\lambda = i \Longrightarrow -\mathrm{Re}\ i\langle h, g\rangle = -\mathrm{Re}\ i\langle Vh, Vg\rangle$. (2)
Equation (1) and (2) $\Longrightarrow \langle h, g\rangle = \langle Vh, Vg\rangle$.
Conversely, suppose $\langle h, g\rangle = \langle Vh, Vg\rangle, \quad \forall h, g \in H$.
$\therefore$ In particular, $\langle h, h\rangle = \langle Vh, Vh\rangle$.
$\Longrightarrow \|h\|^2 = \|Vh\|^2$.
$\Longrightarrow \|h\| = \|Vh\|, \quad \forall h \in H$.
Consider,

$$\begin{aligned} d_H(h, g) &= \|h - g\| \\ &= \|V(h - g)\| \\ &= \|Vh - Vg\| \\ &= d_K(Vh, Vg) \end{aligned}$$

$\therefore V$ is an isometry. ■

Example. Define $S : l^2 \to l^2$ by $S(\alpha_1, \alpha_2, ...) = (0, \alpha_1, \alpha_2, ...)$. Then S is an isometry that is not surjective.

PROOF. Consider, $S : l^2 \to l^2$ defined by $S(\alpha_1, \alpha_2, ...) = (0, \alpha_1, \alpha_2, ...)$, then S is not onto because $(1, 0, 0, ...) \in l^2$ but there does not exists $(\alpha_1, \alpha_2, ...) \in S$ such that $S(\alpha_1, \alpha_2, ...) = (1, 0, 0, ...)$.

Let $\alpha = (\alpha_1, \alpha_2, ...)$ and $\beta = (\beta_1, \beta_2, ...) \in l^2$. Then

$$\begin{aligned}
\langle S(\alpha), S(\beta)\rangle &= \langle (0, \alpha_1, \alpha_2, ...), (0, \beta_1, \beta_2, ...)\rangle \\
&= \sum_{i=1}^{\infty} \alpha_i \overline{\beta_i} \\
&= \langle (\alpha_1, \alpha_2, ...), (\beta_1, \beta_2, ...)\rangle \\
&= \langle \alpha, \beta\rangle
\end{aligned}$$

$\therefore S$ is an isometry but not surjective. ■

Theorem. If H is Hilbert space and $\mathcal{E}$ is a basis for H, then H is isomorphic to $l^2(\mathcal{E})$.

PROOF. Let $U : H \to l^2(\mathcal{E})$ be a map defined by $Uh = \hat{h}$, where $\hat{h} : \mathcal{E} \to \mathbb{F}$ defined by $\hat{h}(e) = \langle h, e\rangle$ and $\sum_{e\in\mathcal{E}} |\hat{h}(e)|^2 < \infty$.

To show U is linear: Let $h_1, h_2 \in H$ and $\alpha \in F$.

Consider, $U(h_1 + \alpha h_2) = \widehat{h_1 + \alpha h_2}$.

Claim: $\widehat{h_1 + \alpha h_2} = \hat{h_1} + \alpha\hat{h_2}$.

For $e \in \mathcal{E}$,

$$\begin{aligned}
(\widehat{h_1 + \alpha h_2})(e) &= \langle h_1 + \alpha h_2, e\rangle \\
&= \langle h_1, e\rangle + \langle \alpha h_2, e\rangle \\
&= \langle h_1, e\rangle + \alpha\langle h_2, e\rangle \\
&= \hat{h_1}(e) + \alpha\hat{h_2}(e), \quad \forall e \in \mathcal{E}
\end{aligned}$$

$$\therefore \widehat{h_1 + \alpha h_2} = \hat{h_1} + \alpha\hat{h_2}$$

$\therefore U(h_1 + \alpha h_2) = \widehat{h_1 + \alpha h_2} = \hat{h_1} + \alpha\hat{h_2}$.

$\implies U(h_1 + \alpha h_2) = U(h_1) + \alpha U(h_2)$.

To show U is isometry:

Consider,

$$\begin{aligned}\|Uh\|^2 &= \|\hat{h}\|^2 \\ &= \langle \hat{h}, \hat{h}\rangle \\ &= \sum_{e\in\mathcal{E}} |\hat{h}(e)|^2 \\ &= \sum_{e\in\mathcal{E}} |\langle h, e\rangle|^2 \\ &= \|h\|^2 \qquad \because \text{By Parseval's Identity.}\end{aligned}$$

$\Longrightarrow U$ is isometry.
To show: U is surjective.
$l^2(\mathcal{E})$ contains all functions $f : \mathcal{E} \to \mathbb{F}$ such that $f(e) = 0$ for all but finitely many $e \in \mathcal{E}$.
$\therefore l^2(\mathcal{E}) \subseteq \mathrm{Im}(U)$.
$\Longrightarrow \overline{l^2(\mathcal{E})} \subseteq \overline{\mathrm{Im}(U)}$.
$\Longrightarrow \overline{\mathrm{Im}(U)} = l^2(\mathcal{E})$.
$\Longrightarrow \mathrm{Im}(U)$ is dense in $l^2(\mathcal{E})$. $\because$ It contains all indicators δ_e for $e \in \mathcal{E}$
$\therefore \mathrm{Im}(U)$ is complete. $\because \mathrm{Im}(U)$ is image of complete space under isometry.
$\Longrightarrow \mathrm{Im}(U)$ is closed.
$\Longrightarrow \mathrm{Im}(U) = l^2(\mathcal{E})$.
$\Longrightarrow U$ is surjective.
$\therefore U$ is an isomorphism.
$\therefore H$ is isomorphic to $l^2(\mathcal{E})$. ■

Theorem. *Two Hilbert spaces are isomorphic if and only if they have the same dimension.*

PROOF. Let H and K are Hilbert spaces such that H and K are isomorphic to each other.
$\therefore \exists U : H \to K$ is an isomorphism.
Let $\mathcal{E}$ is basis for H.
$\Longrightarrow U(\mathcal{E}) = \{U(e)/e \in \mathcal{E}\}$ is basis for K.
$\therefore |\mathcal{E}| = |U(\mathcal{E})|$.
$\therefore \dim H = \dim K$.
Conversely, suppose $\dim H = \dim K$.
Let $\mathcal{E}$ be basis for H and $\mathcal{F}$ be a basis for K.
$\Longrightarrow |\mathcal{E}| = |\mathcal{F}|$.
$\therefore \exists$ an isomorphism $T : l^2(\mathcal{E}) \to l^2(\mathcal{F})$.
$\Longrightarrow T(e_\alpha) = f_\alpha$, where $e_\alpha \in l^2(\mathcal{E})$ and $f_\alpha \in l^2(\mathcal{F})$. $\because |\mathcal{E}| = |\mathcal{F}|$.
$\therefore H \cong l^2(\mathcal{E}) \cong l^2(\mathcal{F}) \cong K$.
$\Longrightarrow H \cong K$.
$\therefore H$ and K are isomorphic to each other. ■

Corollary. All separable infinite dimensional Hilbert spaces are isomorphic.

PROOF. If Hilbert spaces H and K are infinite dimensional and separable, then $\dim H =$

$\aleph_0$ and $\dim K = \aleph_0$.
$\therefore \dim H = \dim K$.
$\implies H \cong K$.
Therefore, all separable infinite dimensional Hilbert spaces are isomorphic. ∎
Theorem. If $f : \partial\mathbb{D} \to \mathbb{C}$, where $\partial\mathbb{D} = \{z \in \mathbb{C}/|z| = 1\}$ is a continuous function, then there is a sequence $\{p_n(z, \bar{z})\}$ of polynomials in z and $\bar{z}$ such that $p_n(z, \bar{z}) \to f(z)$ uniformly on $\partial\mathbb{D}$.
Note that if $z \in \partial\mathbb{D}, \bar{z} = z^{-1}$. Thus a polynomial in z and z^{-1} on $\partial\mathbb{D}$ become a function of the form $\displaystyle\sum_{k=-m}^{m} \alpha_k z^k$.
If we put $z = e^{i\theta}$, this become function of the form $\displaystyle\sum_{k=-m}^{m} \alpha_k e^{ik\theta}$.
such functions are called as trigonometric polynomials.
Theorem. If for each $n \in \mathbb{Z}, e_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}$, then $\{e_n : n \in \mathbb{Z}\}$ is a basis for $L^2_{\mathbb{C}}[0, 2\pi]$.
PROOF. Already we have shown $\mathcal{E} = \{e_n : n \in \mathbb{Z}\}$ is orthonormal set.
We want to show $\vee\mathcal{E} = L^2_{\mathbb{C}}[0, 2\pi]$.
For this consider, $\mathcal{T} = \left\{ \displaystyle\sum_{k=-n}^{n} \alpha_k e_k / \alpha_k \in \mathbb{C}, n \geq 0 \right\}$, where $e_k(t) = \frac{1}{\sqrt{2\pi}} e^{ikt}$.
Also consider, $\mathcal{C} = \{f \in C_{\mathbb{C}}[0, 2\pi]/f(0) = f(2\pi)\}$.
To show uniform closure of $\mathcal{T}$ is $\mathcal{C}$.
That is to show, $\overline{\mathcal{T}} = \mathcal{E}$.
Let $f \in \mathcal{C} \implies f(0) = f(2\pi)$.
For any $h \in \mathcal{T}, h(0) = h(2\pi)$.
Now define $F : \partial\mathbb{D} \to \mathbb{C}$, where $\partial\mathbb{D} = \{z \in \mathbb{C}/|z| = 1\}$ by $F(e^{i\theta}) = f(\theta)$.
Then F is continuous function on $\partial\mathbb{D}$. $\because f$ is continuous.
$\therefore p_n(e^{i\theta}, e^{-i\theta})$ converges uniformly to $F(e^{i\theta})$.
$\implies \overline{\mathcal{T}} = \mathcal{C}$. $\because p_n(e^{i\theta}, e^{-i\theta}) \in \mathcal{T}$.
Also, we know that $\overline{\mathcal{C}} = L^2_{\mathbb{C}}[0, 2\pi]$.
$\implies \overline{\overline{\mathcal{T}}} = \overline{\mathcal{C}} = L^2_{\mathbb{C}}[0, 2\pi]$.
$\implies \overline{\mathcal{T}} = L^2_{\mathbb{C}}[0, 2\pi]$.
$\implies \vee\mathcal{E} = \overline{\mathcal{T}} = L^2_{\mathbb{C}}[0, 2\pi]$.
$\therefore \vee\mathcal{E} = L^2_{\mathbb{C}}[0, 2\pi]$.
$\therefore \mathcal{E}$ is a basis for $L^2_{\mathbb{C}}[0, 2\pi]$. ∎
Remark 1. If $f \in L^2_{\mathbb{C}}[0, 2\pi]$, then $\hat{h}(n) = \langle f, e_n \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt$ is called n^{th} Fourier coefficient of f.
Consider, $f \in L^2_{\mathbb{C}}[0, 2\pi]$ and $\mathcal{E} = \{e_n/n \in \mathbb{Z}\}$ is basis for $L^2_{\mathbb{C}}[0, 2\pi]$.
Therefore, $f = \displaystyle\sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n = \sum_{n=-\infty}^{\infty} \hat{f}(n) e_n$ is called Fourier series corresponding to f.
Remark 2. If $f \in L^2_{\mathbb{C}}[0, 2\pi]$, then

$$\begin{aligned}
\|f\|^2 &= \langle f, f\rangle \\
&= \langle \sum_{n=-\infty}^{\infty} \hat{f}(n)e_n, \sum_{n=-\infty}^{\infty} \hat{f}(n)e_n\rangle \\
&= \langle \hat{f}(1)e_1 + \hat{f}(2)e_2, \hat{f}(1)e_1 + \hat{f}(2)e_2\rangle \qquad \text{For } n = 1, 2 \\
&= \hat{f}(1)\overline{\hat{f}(1)}\langle e_1, e_1\rangle + \hat{f}(1)\overline{\hat{f}(2)}\langle e_1, e_2\rangle + \hat{f}(2)\overline{\hat{f}(1)}\langle e_2, e_1\rangle + \hat{f}(2)\overline{\hat{f}(2)}\langle e_2, e_2\rangle \\
&= |\hat{f}(1)|^2 + |\hat{f}(2)|^2
\end{aligned}$$

$\therefore$ For $f \in L^2_{\mathbb{C}}[0, 2\pi]$.

$\|f\|^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 < \infty.$

The Reimann-Lebesgue Lemma. *If $f \in L^2_{\mathbb{C}}[0, 2\pi]$, then $\int_0^{2\pi} f(t)e^{-int}dt \to 0$ as $n \to \pm\infty$.*

PROOF. From result 2. we have $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 < \infty.$

$\implies \lim_{n\to\infty} |\hat{f}(n)|^2 = 0.$

$\implies \lim_{n\to\infty} |\hat{f}(n)| = 0.$

$\implies \lim_{n\to\infty} \hat{f}(n) = 0.$

$\implies \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int} \to 0$ and $n \to \infty.$

$\implies \int_0^{2\pi} f(t)e^{-int} \to 0$ and $n \to \infty.$

Note. For f in $L^2_{\mathbb{C}}[0, 2\pi]$, the function $\hat{f} : \mathbb{Z} \to \mathbb{C}$ is called the Fourier transform of f. That is, the map $U : L^2_{\mathbb{C}}[0, 2\pi] \to l^2(\mathbb{Z})$ defined by $Uf = \hat{f}$ is the Fourier transform.

Theorem. The Fourier transform is a linear isometry from $L^2_{\mathbb{C}}[0, 2\pi]$ onto $l^2(\mathbb{Z})$.

PROOF. Let $U : L^2_{\mathbb{C}}[0, 2\pi] \to l^2(\mathbb{Z})$ is Fourier transform defined by $Uf = \hat{f}$, where $\hat{f}(n) = \langle f, e_n\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt.$

To show U is linear:

Let $f_1, f_2 \in L^2_{\mathbb{C}}[0, 2\pi]$ and $\alpha \in F$.

To show $U(f_1 + \alpha f_2) = U(f_1) + \alpha U(f_2)$.

That is, to show $\widehat{f_1 + \alpha f_2} = \hat{f}_1 + \alpha\hat{f}_2$.

For $e_n \in \mathcal{E}$ consider,

$$\begin{aligned}
\widehat{f_1 + \alpha f_2}(n) &= \langle f_1 + \alpha f_2, e_n\rangle \\
&= \langle f_1, e_n\rangle + \alpha\langle f_2, e_n\rangle \\
&= \hat{f}_1(n) + \alpha\hat{f}_2(n) \\
&= (\hat{f}_1 + \alpha\hat{f}_2)(n) \\
\implies \widehat{f_1 + \alpha f_2} &= \hat{f}_1 + \alpha\hat{f}_2
\end{aligned}$$

To show U is isometry:

Consider,

$$\begin{aligned}\|Uf\|^2 &= \|\hat{f}\|^2\\ &= \langle \hat{f}, \hat{f}\rangle\\ &= \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2\\ &= \|f\|^2 \qquad \because \text{Parsevals identity}\end{aligned}$$

$\therefore U$ is isometry.
To show U is onto:
Let $\{\{\alpha_n\} \in l^2(\mathbb{Z})/\alpha_n = 0$ for all but finitely many $n \in \mathbb{Z}\}$
Then clearly T is dense in $l^2(\mathbb{Z})$.
Now we want to show that range$(U) = T$.
Take $\{\alpha_n\} \in T$.
Consider, $f = \displaystyle\sum_{n=-\infty}^{\infty} \alpha_n e_n$.
Here,

$$\begin{aligned}\|f\|^2 &= \langle f, f\rangle\\ &= \langle \sum_{n=-\infty}^{\infty} \alpha_n e_n, \sum_{n=-\infty}^{\infty} \alpha_n e_n\rangle\\ &= \sum_{n=-\infty}^{\infty} |\alpha_n|^2\\ &< \infty\end{aligned}$$

$\therefore f \in L^2_{\mathbb{C}}[0, 2\pi]$.

$$\begin{aligned}U(f(n)) &= \hat{f}(n)\\ &= \langle f, e_n\rangle\\ &= \langle \sum_{n=-\infty}^{\infty} \alpha_n e_n, e_n\rangle\\ &= \alpha_n\end{aligned}$$

$\therefore U(f) = \{\alpha_n\}$.
$\Longrightarrow T \subseteq \text{range}(U)$
$\Longrightarrow \overline{T} \subseteq \overline{\text{range}(U)} \subseteq l^2(\mathbb{Z})$.
$\Longrightarrow l^2(\mathbb{Z}) \subseteq \overline{\text{range}(U)} \subseteq l^2(\mathbb{Z})$.
But range(U) is closed being image of complete space under isometry.
$\therefore l^2(\mathbb{Z}) \subseteq \text{range}(U) \subseteq l^2(\mathbb{Z})$.
$\Longrightarrow \text{range}(U) = l^2(\mathbb{Z})$.
Therefore, U is onto map.
Therefore, U is linear isometry map from $L^2_{\mathbb{C}}[0, 2\pi]$ to $l^2(\mathbb{Z})$.
THE DIRECT SUM OF HILBERT SPACES

Suppose H and K are Hilbert space, then $H \oplus K = \{h \oplus k : h \in H, k \in K\}$.
If $h_1 \oplus k_1, h_2 \oplus k_2 \in H \oplus K$, then $(h_1 \oplus k_1) + (h_2 \oplus k_2) = (h_1 + h_2) \oplus (k_1 + k_2)$.
Definition. If H and K are Hilbert spaces then $\langle h_1 \oplus k_1, h_2 \oplus k_2 \rangle = \langle h_1, h_2 \rangle + \langle k_1, k_2 \rangle$. $H \oplus K$ is complete inner product space with respect to above inner product hence it is Hilbert space.
Proposition. *If $H_1, H_2, ...$ are Hilbert spaces, let $H = \{(h_n)_{n=1}^{\infty} : h_n \in H_n\}$ for all n and $\sum_{n=1}^{\infty} \|h_n\|^2 < \infty$. For $h = (h_n)$ and $g = (g_n)$ in H, define $\langle h, g \rangle = \sum_{n=1}^{\infty} \langle h_n, g_n \rangle$. Then $\langle \cdot, \cdot \rangle$ is an inner product on H. With this inner product H is Hilbert space.*
PROOF. Exercise.
Definition. $H_1, H_2, ...$ are Hilbert spaces, the Hilbert space H in previous preposition is called direct sum of $H_1, H_2, ...$ and it is denoted by $H = H_1 \oplus H_2 \oplus ....$

♣♣♣

CHAPTER 2

Operators on Hilbert Spaces

ELEMENTARY PROPERTIES AND EXAMPLES

Proposition. *Let H and K be Hilbert spaces and $A : H \to K$ a linear transformation. The following statements are equivalent.*
(a) *A is continuous.*
(b) *A is continuous at 0.*
(c) *A is continuous at some point.*
(d) *There is a constant $c > 0$ such that $\|Ah\| \le c\|h\|$ for all $h \in H$.*

PROOF. Already done. ■

If

$$\|A\| = \sup\{\|Ah\| : h \in H, \|h\| \le 1\},$$

then

$$\begin{aligned} \|A\| &= \sup\{\|Ah\| : \|h\| = 1\} \\ &= \sup\{\|Ah\|/\|h\| : h \ne 0\} \\ &= \inf\{c > 0 : \|Ah\| \le c\|h\|, h \in H\}. \end{aligned}$$

Also $\|Ah\| \le \|A\|\|h\|$. $\|A\|$ is called the norm of A and a linear transformation with finite norm is called bounded. Let $\mathcal{B}(H, K)$ be the set of bounded linear transformations from H to K. For $H = K, \mathcal{B}(H, H) \equiv \mathcal{B}(H)$. Note that $\mathcal{B}(H, \mathbb{F})$ = all the bounded linear functionals on H.

Proposition. (a) *If A and $B \in \mathcal{B}(H, K)$, then $A + B \in \mathcal{B}(H, K)$, and $\|A + B\| \le \|A\| + \|B\|$.*
(b) *If $\alpha \in \mathbb{F}$ and $A \in \mathcal{B}(H, K)$, then $\alpha A \in \mathcal{B}(H, K)$ and $\|\alpha A\| = |\alpha|\|A\|$.*
(c) *If $A \in \mathcal{B}(H, K)$ and $B \in \mathcal{B}(K, L)$, then $BA \in \mathcal{B}(H, L)$ and $\|BA\| \le \|B\|\|A\|$.*

PROOF. (a) Let $A, B \in \mathcal{B}(H, K)$.
$\therefore A + B : H \to K$ is a linear transformation.

$$\begin{aligned} \|(A + B)h\| &= \|Ah + Bh\| \\ &\le \|Ah\| + \|Bh\| \\ &\le \|A\|\|h\| + \|B\|\|h\| \\ &\le (\|A\| + \|B\|)\|h\|, \quad \forall h \in H \end{aligned}$$

$\Longrightarrow A + B \in \mathcal{B}(H, K)$.
Also, $\|A + B\| = \inf\{c > 0 : \|(A + B)h\| \le c\|h\|\}$.
$\Longrightarrow \|A + B\| \le \|A\| + \|B\|$.
(b) Clearly, $\alpha A : H \to K$ is a linear transformation.
Consider,

$$\begin{aligned} \|\alpha A(h)\| &= |\alpha|\|Ah\| \\ &\le |\alpha|\|A\|\|h\|, \quad \forall h \in H. \end{aligned}$$

$\therefore \alpha A$ is bounded and it is bounded by $|\alpha|\|A\|$.
By definition of $\|\alpha A\|$,

$$\begin{aligned}\|\alpha A\| &= \sup\{\|\alpha A(h)\| : \|h\| \leq 1\} \\ &= |\alpha| \sup\{\|Ah\| : \|h\| \leq 1\} \\ &= |\alpha|\|A\|\end{aligned}$$

(c) Clearly, BA is a linear transformation from H to L.

$$\begin{aligned}\|(BA)h\| &= \|B(A(h))\| \\ &\leq \|B\|\|Ah\| \\ &\leq \|B\|\|A\|\|h\|\end{aligned}$$

$\therefore BA$ is a bounded operator with upper bound as $\|B\|\|A\|$.
$\therefore \|BA\| \leq \|B\|\|A\|$. ■

Definition. If H and K are Hilbert spaces, a function $u : H \times K \to \mathbb{F}$ is sesquilinear form if for $h, g, f \in H$ and $\alpha, \beta \in \mathbb{F}$.
(a) $u(\alpha h + \beta g, k) = \alpha u(h, k) + \beta u(g, k)$;
(b) $u(h, \alpha k + \beta f) = \bar{\alpha} u(h, k) + \bar{\beta} u(h, f)$.
A sesquilinear form is bounded if there is constant M such that $|u(h, k)| \leq M\|h\|\|k\|, \quad \forall h \in H$ and $k \in K$. The constant M is called a bound for u.

Result. Given $A \in \mathcal{B}(H, K)$ we can construct a sesquilinear form $u(h, k) = \langle Ah, k\rangle$.
Solution: For $h_1, h_2 \in H, k_1, k_2 \in K$ and $\alpha, \beta \in \mathbb{F}$. Then

$$\begin{aligned}u(\alpha h_1 + \beta h_2, k) &= \langle A(\alpha h_1 + \beta h_2), k\rangle \\ &= \langle \alpha A(h_1) + \beta A(h_2), k\rangle \\ &= \alpha\langle A(h_1), k\rangle + \beta\langle A(h_2), k\rangle \\ &= \alpha u(h_1, k) + \beta u(h_2, k)\end{aligned}$$

Similarly, $u(h, \alpha k_1 + \beta k_2) = \bar{\alpha} u(h, k_1) + \bar{\beta} u(h, k_2)$.
Now boundedness,

$$\begin{aligned}|u(h, k)| &= |\langle Ah, k\rangle| \\ &\leq \|Ah\|\|k\| \qquad \because \text{ By CBS inequality.} \\ &\leq \|A\|\|h\|\|k\|, \quad \forall h \in H, k \in K.\end{aligned}$$

Theorem. If $u : H \times K \to \mathbb{F}$ is bounded sesquilinear form with bound M, then there is a unique operator $A \in \mathcal{B}(H, K)$ and $B \in \mathcal{B}(K, H)$ such that $u(h, k) = \langle Ah, k\rangle = \langle h, Bk\rangle$ for all $h \in H$ and $k \in K$ and $\|A\|, \|B\| \leq M$.
PROOF. Define $L_h : K \to \mathbb{F}$ by,
$L_h(k) = \overline{u(h, k)}$.
Claim: L_h is linear functional.
Consider, $k_1, k_2 \in K$ and $\alpha \in \mathbb{F}$.

$$\begin{aligned}L_h(k_1 + \alpha k_2) &= \overline{u(h, k_1 + \alpha k_2)} \\ &= \overline{u(h, k_1)} + \overline{u(h, \alpha k_2)} \\ &= \overline{u(h, k_1)} + \alpha\overline{u(h, k_2)} \\ &= L_h(k_1) + \alpha L_h(k_2)\end{aligned}$$

$\therefore L_h$ is linear functional.
Consider, $|L_h(k)| = |\overline{u(h, k)}| = |u(h, k)| \leq M\|h\|\|k\|, \quad \forall k \in K$.

$\therefore L_h$ is a bounded linear functional with upper bound $M\|h\|$.
$\therefore \|L_h\| \leq M\|h\|$.
$\therefore L_h : K \to \mathbb{F}$ is a bounded linear functional, hence by Riesz Representation theorem, there is some $k_1 \in K$ such that
$L_h(k) = \langle k, k_1 \rangle, \quad \forall k \in K$ and $\|L_h\| = \|k_1\|$.
Put, $k_1 = Ah$.
Claim: A is linear.
Consider, $\langle k, k_1 \rangle = \langle k, Ah \rangle$.
Let $h_1, h_2 \in H, \alpha \in \mathbb{F}$.

$$\begin{aligned} \langle k, A(h_1 + \alpha h_2) \rangle &= \overline{u(h_1 + \alpha h_2, k)} \\ &= \overline{u(h_1, k)} + \overline{u(\alpha h_2, k)} \\ &= \langle k, Ah_1 \rangle + \langle k, \alpha A h_2 \rangle \\ &= \langle k, Ah_1 + \alpha A h_2 \rangle, \quad \forall k \in K \end{aligned}$$

$\implies A(h_1 + \alpha h_2) = Ah_1 + \alpha A h_2$.
Also, $\langle Ah, k \rangle = \overline{\langle k, Ah \rangle} = \overline{\langle k, k_1 \rangle} = u(h, k)$.
$\therefore u(h, k) = \langle Ah, k \rangle$.
We know, $\|L_h\| = \|k_1\|$
$\implies \|L_h\| \leq M\|h\|$
$\implies \|k_1\| \leq M\|h\|$
$\implies \|Ah\| \leq M\|h\|, \quad \forall h \in H$.
Uniqueness:
If possible, there exists $A_1 \in \mathcal{B}(H, K)$ such that
$u(h, k) = \langle A_1 h, k \rangle$.
$\implies \langle Ah, k \rangle = \langle A_1 h, k \rangle, \quad \forall h \in H$ and $k \in K$.
$\implies \langle Ah, k \rangle - \langle A_1 h, k \rangle = 0$.
$\implies \langle Ah - A_1 h, k \rangle = 0$.
$\implies \langle (A - A_1)h, k \rangle = 0, \quad \forall h \in H$.
$\implies A - A_1 = 0$.
$\implies A = A_1$. ∎

Definition. If $A \in \mathcal{B}(H, K)$, then the unique operator $B \in \mathcal{B}(K, H)$ such that $\langle Ah, k \rangle = \langle h, Bk \rangle$ is called adjoint of A and denoted by A^*.

Proposition. If $U \in \mathcal{B}(H)$, then U is an isomorphism if and only if U is invertible and $U^{-1} = U^*$.

PROOF. Let $U : H \to K$ is an isomorphism.
That is, U is linear, surjective isometry.
Since U is isometry,
$\implies \langle Uh, Ug \rangle = \langle h, g \rangle, \quad \forall h, g \in H$.
In particular, $\langle Uh, Uh \rangle = \langle h, h \rangle$.
If $Uh = 0 \implies \langle h, h \rangle = 0 \implies h = 0. \implies \ker U = \{0\}$.
$\implies U$ is injective.
$\implies U$ is bijective and hence invertible.
For $h_1, h_2 \in H, \langle Uh_1, Uh_2 \rangle = \langle h_1, h_2 \rangle$.
Also, $\langle Uh_1, Uh_2 \rangle = \langle h_1, U^* U h_2 \rangle$.

$\Longrightarrow \langle h_1, U^*Uh_2\rangle = \langle h_1, h_2\rangle, \quad \forall h_1, h_2 \in H.$
$\Longrightarrow \langle h_1, U^*Uh_2\rangle - \langle h_1, h_2\rangle = 0.$
$\Longrightarrow \langle h_1, U^*Uh_2 - h_2\rangle = 0.$
$\Longrightarrow \langle h_1, (U^*U - I)h_2\rangle = 0.$
$\Longrightarrow U^*U - I = 0.$
$\Longrightarrow U^*U = I.$
$\Longrightarrow U^{-1} = U^*.$
Conversely, suppose U is invertible and $U^* = U^{-1}$.
Clearly, U is linear, surjective mapping.
Consider,

$$\begin{aligned}\langle Uh_1, Uh_2\rangle &= \langle h_1, U^*Uh_2\rangle \\ &= \langle h_1, Ih_2\rangle \\ &= \langle h_1, h_2\rangle\end{aligned}$$

$\therefore U$ is an isometry.
$\therefore U$ is an isomorphism. ■

Proposition. *If $A, B \in \mathcal{B}(H, K)$ and $\alpha \in \mathbb{F}$, then:*
(a)$(\alpha A + B)^* = \bar{\alpha}A^* + B^*$
(b)$(AB)^* = B^*A^*$
(c)$A^{**} = (A^*)^* = A$
(d) If A is invertible in $\mathcal{B}(H)$ and A^{-1} is its inverse, then A^* is invertible and $(A^*)^{-1} = (A^{-1})^*$.

PROOF. (a) If $A \in \mathcal{B}(H)$ then $\alpha A \in \mathcal{B}(H)$. Also, for $\alpha A, B \in \mathcal{B}(H)$.
Consider, $\langle(\alpha A + B)h_1, h_2\rangle = \langle h_1, (\alpha A + B)^*h_2\rangle$.
Now,

$$\begin{aligned}\langle h_1, (\alpha A + B)^*h_2\rangle &= \langle(\alpha A + B)h_1, h_2\rangle \\ &= \langle \alpha Ah_1, h_2\rangle + \langle Bh_1, h_2\rangle \\ &= \alpha\langle Ah_1, h_2\rangle + \langle Bh_1, h_2\rangle \\ &= \alpha\langle h_1, A^*h_2\rangle + \langle h_1, B^*h_2\rangle \\ &= \langle h_1, \bar{\alpha}A^*h_2\rangle + \langle h_1, B^*h_2\rangle \\ &= \langle h_1, (\bar{\alpha}A^* + B^*)h_2\rangle\end{aligned}$$

$\langle h_1, (\alpha A + B)^*h_2\rangle - \langle h_1, (\bar{\alpha}A^* + B^*)h_2\rangle = 0.$
$\langle h_1, [(\alpha A + B)^* - (\bar{\alpha}A^* + B^*)]h_2\rangle = 0, \quad \forall h_1, h_2 \in H.$
$\Longrightarrow [(\alpha A + B)^* - (\bar{\alpha}A^* + B^*)]h_2 = 0, \quad \forall h_2 \in H.$
$\Longrightarrow (\alpha A + B)^* - (\bar{\alpha}A^* + B^*) = 0.$
$\Longrightarrow (\alpha A + B)^* = \bar{\alpha}A^* + B^*.$
(b) Suppose $A, B \in \mathcal{B}(H)$.
Consider,

$$\begin{aligned}\langle(AB)h_1, h_2\rangle &= \langle A(Bh_1), h_2\rangle \\ &= \langle(Bh_1), A^*h_2\rangle \\ &= \langle h_1, B^*A^*h_2\rangle\end{aligned}$$

$\Longrightarrow \langle(AB)h_1, h_2\rangle - \langle h_1, B^*A^*h_2\rangle = 0.$
$\Longrightarrow \langle h_1, (AB)^*h_2\rangle - \langle h_1, B^*A^*h_2\rangle = 0.$
$\Longrightarrow \langle h_1, (AB)^*h_2 - B^*A^*h_2\rangle = 0.$

$\implies \langle h_1, [(AB)^* - B^*A^*]h_2\rangle = 0, \quad h_1, h_2 \in H.$
$\implies [(AB)^* - B^*A^*]h_2 = 0, \quad h_2 \in H.$
$\implies (AB)^* = B^*A^*.$
(c) Let $A \in \mathcal{B}(H)$.
Consider,

$$\begin{aligned}\langle Ah_1, h_2\rangle &= \langle h_1, A^*h_2\rangle \\ &= \langle (A^*)^*h_1, h_2\rangle\end{aligned}$$

$\implies \langle Ah_1, h_2\rangle - \langle (A^*)^*h_1, h_2\rangle = 0.$
$\implies \langle (A-(A^*)^*)h_1, h_2\rangle = 0, \quad h_1, h_2 \in H.$
$\implies (A-(A^*)^*)h_1 = 0, \quad h_1 \in H.$
$\implies A - (A^*)^* = 0.$
$\implies A = (A^*)^* = A^{**}.$
(d) Suppose $A \in \mathcal{B}(H)$ and A is invertible.
Let A^{-1} is inverse of A.
Clearly, A^* is surjective mapping.
Let $A^*h = 0$.
Consider, $\langle h_1, A^*h\rangle = \langle Ah_1, h\rangle$.
$\implies \langle Ah_1, h\rangle = 0, \quad \forall h_1, h \in H.$
In particular, $\langle h, h\rangle = 0$.
$\implies h = 0.$
$\implies \ker A^* = \{0\}.$
$\implies A^*$ is injective.
$\therefore A^*$ is invertible.
Consider, $\langle h_1, h_2\rangle = \langle A^*(A^*)^{-1}h_1, h_2\rangle$.
$\implies \langle h_1, h_2\rangle = \langle (A^*)^{-1}h_1, Ah_2\rangle.$
Also, $\langle h_1, h_2\rangle = \langle h_1, A^{-1}Ah_2\rangle$.
$\implies \langle h_1, h_2\rangle = \langle (A^{-1})^*h_1, Ah_2\rangle, \quad \forall h_1, h_2 \in H.$
$\implies \langle (A^{-1})^*h_1, Ah_2\rangle = \langle (A^*)^{-1}h_1, Ah_2\rangle.$
$\implies \langle (A^{-1})^*h_1, Ah_2\rangle - \langle (A^*)^{-1}h_1, Ah_2\rangle = 0.$
$\implies \langle [(A^{-1})^* - (A^*)^{-1}]h_1, Ah_2\rangle = 0.$
In particular, $\langle [(A^{-1})^* - (A^*)^{-1}]h_1, [(A^{-1})^* - (A^*)^{-1}]h_1\rangle = 0.$
$\implies [(A^{-1})^* - (A^*)^{-1}]h_1 = 0, \quad \forall h_1 \in H.$
$\implies (A^{-1})^* = (A^*)^{-1}.$ ■

Proposition. *If* $A \in \mathcal{B}(H), \|A\| = \|A^*\| = \|A^*A\|^{1/2}$.
PROOF. Let $h \in H$ and $\|h\| \leq 1$.
Consider,

$$\begin{aligned}\|Ah\|^2 &= \langle Ah, Ah\rangle \\ &= \langle A^*Ah, h\rangle \\ &\leq \|A^*Ah\|\|h\| \\ &\leq \|A^*A\|\|h\|\|h\| \\ &\leq \|A^*A\|\end{aligned}$$

$\implies \sup\{\|Ah\|^2 : \|h\| \leq 1\} \leq \|A^*A\|.$
$\implies \|A\|^2 \leq \|A^*A\| \leq \|A^*\|\|A\|. \qquad (1)$
$\implies \|A\|^2 \leq \|A^*\|\|A\|.$

$\implies \|A^*\|\|A\| - \|A\|^2 \geq 0.$
$\implies (\|A^*\| - \|A\|)\|A\| \geq 0.$
$\implies \|A^*\| \geq \|A\|.$ (2)
For $A \in \mathcal{B}(H)$ we have $A^* \in \mathcal{B}(H)$.
Replacing A by A^* we get, $\|A^{**}\| \geq \|A^*\|$.
$\implies \|A\| \geq \|A^*\|.$ (3) $\because A^{**} = A$
$\therefore$ By (2) and (3) $\|A\| = \|A^*\|$.
Put $\|A^*\| = \|A\|$ in inequality (1).
$\|A\|^2 \leq \|A^*A\| \leq \|A\|^2$.
$\implies \|A^*A\| = \|A\|^2.$
$\implies \|A^*A\|^{1/2} = \|A\|.$
$\therefore \|A\| = \|A^*\| = \|A^*A\|^{1/2}.$ ■

Proposition. *If $S : l^2 \to l^2$ is defined by $S(\alpha_1, \alpha_2, ...) = (0, \alpha_1, \alpha_2, ...)$, then S is isometry and $S^*(\alpha_1, \alpha_2, ...) = (\alpha_2, \alpha_3, ...)$.*
PROOF. Let $\alpha = (\alpha_1, \alpha_2, ...), \beta = (\beta_1, \beta_2, ...) \in l^2$.
Consider,

$$\begin{aligned}
\langle S\alpha, S\beta \rangle &= \langle S(\alpha_1, \alpha_2, ...), S(\beta_1, \beta_2, ...)\rangle \\
&= \langle (0, \alpha_1, \alpha_2, ...), (0, \beta_1, \beta_2, ...)\rangle \\
&= 0 \cdot 0 + \alpha_1\overline{\beta_1} + \alpha_2\overline{\beta_2} + ... \\
&= \sum_{n=1}^{\infty} \alpha_n\overline{\beta_n} \\
&= \langle (\alpha_1, \alpha_2, ...), (\beta_1, \beta_2, ...)\rangle \\
&= \langle \alpha, \beta \rangle
\end{aligned}$$

$\implies S$ is isometry.
Now, to show: $S^*(\alpha_1, \alpha_2, ...) = (\alpha_2, \alpha_3, ...)$.
Consider,

$$\begin{aligned}
\langle S^*\alpha, \beta \rangle &= \langle S^*(\alpha_1, \alpha_2, ...), (\beta_1, \beta_2, ...)\rangle \\
&= \langle (\alpha_1, \alpha_2, ...), S(\beta_1, \beta_2, ...)\rangle \\
&= \langle (\alpha_1, \alpha_2, ...), (0, \beta_1, \beta_2, ...)\rangle \\
&= \alpha_1 \cdot 0 + \alpha_2\overline{\beta_1} + ... \\
&= \sum_{n=1}^{\infty} \alpha_{n+1}\overline{\beta_n} \\
&= \langle (\alpha_2, \alpha_3, ...), (\beta_1, \beta_2, ...)\rangle \\
&= \langle (\alpha_2, \alpha_3, ...), \beta \rangle
\end{aligned}$$

$\implies \langle S^*\alpha, \beta\rangle = \langle (\alpha_2, \alpha_3, ...), \beta\rangle.$
$\implies \langle S^*(\alpha_1, \alpha_2, ...), \beta\rangle - \langle (\alpha_2, \alpha_3, ...), \beta\rangle = 0.$
$\implies \langle S^*(\alpha_1, \alpha_2, ...) - (\alpha_2, \alpha_3, ...), \beta\rangle = 0, \quad \forall \beta \in l^2.$
$\implies S^*(\alpha_1, \alpha_2, ...) - (\alpha_2, \alpha_3, ...) = 0.$
$\implies S^*(\alpha_1, \alpha_2, ...) = (\alpha_2, \alpha_3, ...).$ ■

Note. The operator S is called unilateral shift and the operator S^* is called backward shift.

Definition. If $A \in \mathcal{B}(H)$, then: (a) A is hermitian or self-adjoint if $A^* = A$; (b) A is normal if $AA^* = A^*A$.

Proposition. *If H is $\mathbb{C}$-Hilbert space and $A \in \mathcal{B}(H)$, then A is hermitian if and only if $\langle Ah, h\rangle \in \mathbb{R}$ for all h in H.*
PROOF. Suppose A is hermitian.
$\Longrightarrow A^* = A$.
Consider,

$$\begin{aligned}\langle Ah, h\rangle &= \langle h, A^*h\rangle \\ &= \langle h, Ah\rangle \\ &= \overline{\langle Ah, h\rangle}\end{aligned}$$

$\Longrightarrow \langle Ah, h\rangle \in \mathbb{R}$.
Conversely, suppose $\langle Ah, h\rangle \in \mathbb{R}, \quad \forall h \in H$.
To show: A is hermitian.
That is to show: $A^* = A$.
For $h, g \in H$ and $\alpha \in \mathbb{F} \Longrightarrow h + \alpha g \in H$.
$\therefore \langle A(h+\alpha g), h+\alpha g\rangle \in \mathbb{R}$.

$\Longrightarrow \langle A(h+\alpha g), h+\alpha g\rangle = \overline{\langle A(h+\alpha g), h+\alpha g\rangle}$.

$\Longrightarrow \langle Ah, h\rangle + \bar{\alpha}\langle Ah, g\rangle + \alpha\langle Ag, h\rangle + |\alpha|^2\langle Ag, g\rangle = \overline{\langle Ah, h\rangle + \bar{\alpha}\langle Ah, g\rangle + \alpha\langle Ag, h\rangle + |\alpha|^2\langle Ag, g\rangle}$.

$\Longrightarrow$
$\langle Ah, h\rangle + \bar{\alpha}\langle Ah, g\rangle + \alpha\langle Ag, h\rangle + |\alpha|^2\langle Ag, g\rangle = \overline{\langle Ah, h\rangle} + \alpha\overline{\langle Ah, g\rangle} + \bar{\alpha}\overline{\langle Ag, h\rangle} + |\alpha|^2\overline{\langle Ag, g\rangle}$.

$\Longrightarrow \bar{\alpha}\langle Ah, g\rangle + \alpha\langle Ag, h\rangle = \alpha\overline{\langle Ah, g\rangle} + \bar{\alpha}\overline{\langle Ag, h\rangle}$.

$\Longrightarrow \bar{\alpha}\langle Ah, g\rangle + \alpha\langle Ag, h\rangle = \alpha\langle A^*g, h\rangle + \bar{\alpha}\langle A^*h, g\rangle$.

For $\alpha = 1$
$\Longrightarrow \langle Ah, g\rangle + \langle Ag, h\rangle = \langle A^*g, h\rangle + \langle A^*h, g\rangle$. (1)
For $\alpha = i$
$\Longrightarrow -i\langle Ah, g\rangle + i\langle Ag, h\rangle = i\langle A^*g, h\rangle - i\langle A^*h, g\rangle$
$\Longrightarrow \langle Ah, g\rangle - \langle Ag, h\rangle = -\langle A^*g, h\rangle + \langle A^*h, g\rangle$ (2)
Subtracting (2) from (1) we get,
$2\langle Ag, h\rangle = 2\langle A^*g, h\rangle$
$\Longrightarrow \langle Ag, h\rangle = \langle A^*g, h\rangle$
$\Longrightarrow \langle Ag, h\rangle - \langle A^*g, h\rangle = 0, \quad \forall h, g \in H$.
$\Longrightarrow \langle Ag - A^*g, h\rangle, \quad \forall h, g \in H$.
$\Longrightarrow Ag - A^*g = 0, \quad \forall g \in H$.
$\Longrightarrow A = A^*$.
$\therefore A$ is hermitian. ■
Remark. If H is $\mathbb{R}-$Hilbert space, then above proposition may not be true.
Counter example:

Consider, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$, Then

$$\begin{aligned} \langle Ah, h\rangle &= \langle \begin{bmatrix} h_2 \\ -h_1 \end{bmatrix}, \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \rangle \\ &= \langle (h_2, -h_1), (h_1, h_2)\rangle \\ &= h_2h_1 - h_1h_2 \\ &= 0 \end{aligned}$$

$\implies \langle Ah, h\rangle = 0 \in \mathbb{R}$.
Here A^* is conjugate transpose of matrix A.
$\implies A^* = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \neq A$.
$\implies A$ is not hermitian. ■

Proposition. *If $A = A^*$, then*

$$\|A\| = \sup\{|\langle Ah, h\rangle| : \|h\| = 1\}.$$

PROOF. Suppose $M = \sup\{|\langle Ah, h\rangle| : \|h\| = 1\}$.
If $\|h\| = 1$, then

$$\begin{aligned} |\langle Ah, h\rangle| &\leq \|Ah\|\|h\| && \because \text{By CBS inequality.} \\ &\leq \|A\|\|h\|\|h\| \\ &\leq \|A\| && \because \|h\| = 1. \end{aligned}$$

$\implies |\langle Ah, h\rangle| \leq \|A\|$.
$\implies \sup\{|\langle Ah, h\rangle| : \|h\| = 1\} \leq \|A\|$.
$\therefore M \leq \|A\|$. (1)
If $h, g \in H$ and $\|h\| = \|g\| = 1$, then

$$\begin{aligned} \langle A(h \pm g), h \pm g\rangle &= \langle Ah, h\rangle \pm \langle Ah, g\rangle \pm \langle Ag, h\rangle + \langle Ag, g\rangle \\ &= \langle Ah, h\rangle \pm \langle Ah, g\rangle \pm \langle g, A^*h\rangle + \langle Ag, g\rangle \\ &= \langle Ah, h\rangle \pm \langle Ah, g\rangle \pm \langle g, Ah\rangle + \langle Ag, g\rangle && \because A = A^* \\ &= \langle Ah, h\rangle \pm \langle Ah, g\rangle \pm \overline{\langle Ah, g\rangle} + \langle Ag, g\rangle \\ &= \langle Ah, h\rangle \pm 2\text{Re}\langle Ah, g\rangle + \langle Ag, g\rangle \end{aligned}$$

$\therefore \langle A(h+g), h+g\rangle = \langle Ah, h\rangle + 2\text{Re}\langle Ah, g\rangle + \langle Ag, g\rangle$ (2)
and $\langle A(h-g), h-g\rangle = \langle Ah, h\rangle - 2\text{Re}\langle Ah, g\rangle + \langle Ag, g\rangle$ (3)
Subtracting equation (3) from equation (2) we get,

$$\langle A(h+g), h+g\rangle - \langle A(h-g), h-g\rangle = 4\text{Re}\langle Ah, g\rangle.$$

$\implies 4\text{Re}\langle Ah, g\rangle \leq |\langle A(h+g), h+g\rangle| + |\langle A(h-g), h-g\rangle|$. (4)
Now for any $u = \frac{f}{\|f\|} \in H$,

$$\begin{aligned} |\langle Au, u\rangle| &= |\langle A\tfrac{f}{\|f\|}, \tfrac{f}{\|f\|}\rangle| \\ &= \tfrac{1}{\|f\|^2}|\langle Af, f\rangle| \end{aligned}$$

$\Longrightarrow \frac{1}{\|f\|^2}|\langle Af, f\rangle| = |\langle Au, u\rangle| \leq M.$
$\Longrightarrow \frac{1}{\|f\|^2}|\langle Af, f\rangle| \leq M.$
$\Longrightarrow |\langle Af, f\rangle| \leq M\|f\|^2, \quad \forall f \in H.$
$\therefore 4\text{Re}\langle Ah, g\rangle \leq |\langle A(h+g), h+g\rangle| + |\langle A(h-g), h-g\rangle|.$
$\Longrightarrow 4\text{Re}\langle Ah, g\rangle \leq M(\|h+g\|^2 + \|f-g\|^2).$
$\Longrightarrow 4\text{Re}\langle Ah, g\rangle \leq 2M(\|h\|^2 + \|g\|^2).$
$\Longrightarrow 4\text{Re}\langle Ah, g\rangle \leq 2M(1+1).$
$\Longrightarrow 4\text{Re}\langle Ah, g\rangle \leq 4M.$
$\therefore \text{Re}\langle Ah, g\rangle \leq M, \quad \forall h, g \in H$ such that $\|h\| = \|g\| = 1.$
Now suppose $\langle Ah, g\rangle = e^{i\theta}|\langle Ah, g\rangle|.$
Replacing h in above inequality by $e^{-i\theta}h$ gives,
$\text{Re}\langle Ae^{-i\theta}h, g\rangle \leq M.$
$\Longrightarrow |\langle Ah, g\rangle| \leq M, \quad \forall g \in H$ such that $\|g\| = 1.$ $\because \langle Ah, g\rangle = e^{i\theta}|\langle Ah, g\rangle|.$
Taking supremum over all $g \in H$ such that $\|g\| = 1$ on both side,
$\|Ah\| \leq M, \quad \forall h \in H$ such that $\|h\| = 1.$
$\Longrightarrow \sup\{\|Ah\| : \|h\| = 1\} \leq M.$
$\Longrightarrow \|A\| \leq M.$ (4)
$\therefore$ from equation (1) and (4).
$\|A\| = M.$
$\Longrightarrow \|A\| = \sup\{|\langle Ah, h\rangle| : \|h\| = 1\}.$ ■

Proposition. *If $A = A^*$ and $\langle Ah, h\rangle = 0$ for all h in H, then $A = 0$.*
PROOF. Suppose $A = A^*$ and $\langle Ah, h\rangle = 0, \quad \forall h \in H.$
$\therefore$ By previous proposition $\|A\| = \{|\langle Ah, h\rangle| : \|h\| = 1\} = 0.$
$\therefore \|A\| = 0.$
$\Longrightarrow A = 0.$ ■

Proposition. *If H is $\mathbb{C}-$Hilbert space and $A \in \mathcal{B}(H)$ such that $\langle Ah, h\rangle = 0, \quad \forall h \in H$, then $A = 0$.*
PROOF. If $\langle Ah, h\rangle = 0, \quad \forall h \in H.$
That is, $\langle Ah, h\rangle \in \mathbb{R} \Longrightarrow A = A^*.$
$\therefore \langle Ah, h\rangle = 0$ and $A = A^*.$
$\therefore$ By previous corollary $A = 0.$ ■

Note. If H is a $\mathbb{C}-$Hilbert space and $A \in \mathcal{B}(H)$, then $B = (A + A^*)/2$ and $C = (A - A^*)/2i$ are self-adjoint and $A = B + iC$. The operator B and C are called, respectively, the real and imaginary parts of A.

Proposition. *If $A \in \mathcal{B}(H)$, the following statement are equivalent.*
(a) A is normal.
(b) $\|Ah\| = \|A^*h\|$ for all h.
If H is a $\mathbb{C}-$Hilbert space, then these statements are also equivalent to:
(c) The real and imaginary parts of A commute.
PROOF. Consider,

$$\begin{aligned}\|Ah\|^2 - \|A^*h\|^2 &= \langle Ah, Ah\rangle - \langle A^*h, A^*h\rangle \\ &= \langle A^*Ah, h\rangle - \langle AA^*h, h\rangle \\ &= \langle (A^*A - AA^*)h, h\rangle, \quad \forall h \in H\end{aligned}$$

$\Longrightarrow \|Ah\|^2 - \|A^*h\|^2 = \langle (A^*A - AA^*)h, h\rangle.$ (1)

$(a) \Longrightarrow (b)$
Let A is normal.
$\Longrightarrow AA^* = A^*A$.
From equation (1), $\|Ah\|^2 - \|A^*h\|^2 = \langle 0, h\rangle$.
$\Longrightarrow \|Ah\|^2 - \|A^*h\|^2 = 0$.
$\Longrightarrow \|Ah\|^2 = \|A^*h\|^2$.
$\Longrightarrow \|Ah\| = \|A^*h\|$.
$(b) \Longrightarrow (a)$
Let $\|Ah\| = \|A^*h\|$.
From (1), $\Longrightarrow \langle (A^*A - AA^*)h, h\rangle = 0$.
Let $B = A^*A - AA^*$.

$$\begin{aligned} B^* &= (A^*A - AA^*)^* \\ &= (A^*A)^* - (AA^*)^* \\ &= A^*A^{**} - A^{**}A^* \\ &= A^*A - AA^* \\ &= B \end{aligned}$$

$\Longrightarrow B = B^*$.
$\therefore \langle Bh, h\rangle = 0, \quad \forall h \in H, B \in \mathcal{B}(H)$ and $B = B^*$.
$\Longrightarrow B = 0$.
$\Longrightarrow A^*A - AA^* = 0 \Longrightarrow A^*A = AA^*$.
$\Longrightarrow A$ is normal.
$(c) \Longrightarrow (a)$
$A = \frac{A+A^*}{2} + i\frac{A-A^*}{2i}$.
Let $B = \frac{A+A^*}{2}, C = \frac{A-A^*}{2i}$.
Therefore, B and C are real and imaginary parts of A.
$\therefore$ By our assumption, $BC = CB$.
$A = B + iC$.
$\Longrightarrow A^* = (B + iC)^* = B^* - iC^* = B - iC$.

$$\begin{aligned} A^*A &= (B - iC)(B + iC) \\ &= B^2 + iBC - iCB + C^2 \\ &= B^2 + C^2 && \because BC = CB \\ AA^* &= (B + iC)(B - iC) \\ &= B^2 - iBC + iCB + C^2 \\ &= B^2 + C^2 && \because BC = CB \end{aligned}$$

$\Longrightarrow AA^* = A^*A$.
$\Longrightarrow A$ is normal.
$(a) \Longrightarrow (c)$
Let A is normal.
That is, $A^*A = AA^*$
$\Longrightarrow B^2 + iBC - iCB + C^2 = B^2 - iBC + iCB + C^2$.
$\Longrightarrow iBC - iCB = -iBC + iCB$
$\Longrightarrow BC - CB = -BC + CB$
$\Longrightarrow 2(BC - CB) = 0$

$\Longrightarrow BC = CB.$ ■

Proposition. *If $A \in \mathcal{B}(H)$, the following statements are equivalent.*
(a) *A is an isometry.*
(b) *$A^*A = I$.*
(c) *$\langle Ah, Ag\rangle = \langle h, g\rangle$ for all $h, g \in H$.*

PROOF. $(a) \Longrightarrow (b)$
A is isometry.
$\Longrightarrow \|Ah\|^2 = \|h\|^2, \quad \forall h \in H.$
$\Longrightarrow \langle Ah, Ah\rangle = \langle h, h\rangle, \quad \forall h \in H.$
$\Longrightarrow \langle A^*Ah, h\rangle = \langle h, h\rangle, \quad \forall h \in H.$
$\Longrightarrow \langle A^*Ah, h\rangle - \langle h, h\rangle = 0, \quad \forall h \in H.$
$\Longrightarrow \langle A^*Ah - h, h\rangle = 0, \quad \forall h \in H.$
$\Longrightarrow \langle (A^*A - I)h, h\rangle = 0, \quad \forall h \in H.$
$\Longrightarrow (A^*A - I)h = 0, \quad \forall h \in H.$
$\Longrightarrow A^*A - I = 0.$
$\Longrightarrow A^*A = I.$
$(b) \Longrightarrow (c)$
Let $A^*A = I$

$$\begin{aligned} A^*A &= I \\ \langle h, g\rangle &= \langle A^*Ah, g\rangle \\ &= \langle Ah, Ag\rangle, \quad \forall h, g \in H \end{aligned}$$

$(c) \Longrightarrow (b)$
$\langle Ah, Ag\rangle = \langle h, g\rangle, \quad \forall h, g \in H.$
$\langle A^*Ah, g\rangle - \langle h, g\rangle = 0, \quad \forall h, g \in H.$
$\langle (A^*A - I)h, g\rangle = 0 \quad \forall h, g \in H.$
$\Longrightarrow A^*A - I = 0.$
$\Longrightarrow A^*A = I.$ ■

Proposition. *If $A \in \mathcal{B}(H)$, then the following statements are equivalent.*
(a) *$A^*A = AA^* = I$.*
(b) *A is unitary.(That is, A is a surjective isometry.)*
(c) *A is a normal isometry.*

PROOF. $(a) \Longrightarrow (b)$
Let $A^*A = AA^* = I$.
Clearly, A is linear and A is surjective.
Take, $A^*A = I$.
Then, $\langle h, g\rangle = \langle A^*Ah, g\rangle = \langle Ah, Ag\rangle, \quad \forall h, g \in H.$
$\Longrightarrow A$ is an isometry which is surjective and hence it is unitary.
$(b) \Longrightarrow (c)$
Let A is unitary.
That is, A is linear, surjective isometry.
$\Longrightarrow A^{-1}$ is also isometry.
$\Longrightarrow (A^{-1})^*A^{-1} = I. \qquad \because \langle A^{-1}h, A^{-1}g\rangle = \langle h, g\rangle \Longrightarrow \langle (A^{-1})^*A^{-1}h, g\rangle = \langle h, g\rangle$
$\Longrightarrow I = (A^{-1})^*A^{-1} = (A^*)^{-1}A^{-1} = (AA^*)^{-1}.$
$\Longrightarrow AA^* = I.$

Also, A is an isometry $\implies A^*A = I$.
$\therefore A^*A = I = AA^*$.
$\implies A^*A = AA^* = I$.
$\implies A$ is normal isometry. ∎
Theorem. *If* $A \in \mathcal{B}(H)$*, then* $\ker A = (\text{ran}A^*)^\perp$.
PROOF. We know that $(\text{ran}A^*)^\perp = \{h \in H : \langle h, y\rangle = 0, \forall y \in \text{ran}A^*\}$.
Let $y \in \text{ran}A^* \implies y = A^*g$ for some $g \in H$.
Consider, $\langle h, y\rangle = \langle h, A^*g\rangle = \langle Ah, g\rangle = \langle 0, g\rangle = 0, \quad \forall y \in \text{ran}A^*$.
$\therefore h \in (\text{ran}A^*)^\perp \implies \ker A \subseteq (\text{ran}A^*)^\perp$.
Conversely, let $h \in (\text{ran}A^*)^\perp$.
$\implies \langle h, y\rangle = 0, \quad \forall y \in \text{ran}A^*$.
$\implies \langle h, A^*g\rangle = 0, \quad \forall g \in H$.
$\implies \langle Ah, g\rangle = 0, \quad \forall g \in H$.
$\implies Ah = 0$.
$\implies h \in \ker A$.
$\implies (\text{ran}A^*)^\perp \subseteq \ker A$.
$\implies (\text{ran}A^*)^\perp = \ker A$. ∎
Observation. We know for $A \in \mathcal{B}(H), A^* \in \mathcal{B}(H)$ and by replacing A by A^* in previous theorem we get,
$\ker A^* = (\text{ran}A^{**})^\perp = (\text{ran}A)^\perp$.
Notation. Let Ball$H$ denote unit ball in H.
Definition. A linear transformation $T : H \to K$ is compact if, $\text{cl}[T(\text{Ball}H)]$ is compact in K.
Notation. $\mathcal{B}_0(H, K) =$ The set of all compact operators from H to K.
Note. Let X be a complete metric space. If A is totally bounded then cl$A$ is compact.
Proposition. (a) $\mathcal{B}_0(H, K) \subseteq \mathcal{B}(H, K)$.
(b) $\mathcal{B}_0(H, K)$ *is a linear space and if* $\{T_n\} \subseteq \mathcal{B}_0(H, K)$ *and* $T \in \mathcal{B}(H, K)$ *such that* $\|T_n - T\| \to 0$*, then* $T \in \mathcal{B}_0(H, K)$.
(c) *If* $A \in \mathcal{B}(H), B \in \mathcal{B}(K)$*, and* $T \in \mathcal{B}_0(H, K)$*, then* TA *and* $BT \in \mathcal{B}_0(H, K)$.
PROOF. (a) Let $T \in \mathcal{B}(H, K)$.
$\implies \text{cl}[T(\text{Ball}H)]$ is compact.
$\implies \text{cl}[T(\text{Ball}H)] \subseteq \{k \in K : \|k\| \le M, \text{ for some } M > 0\}$.
$\implies T(\text{Ball}H) \subseteq \{k \in K : \|k\| \le M, \text{ for some } M > 0\}$.
$\implies$ If $h \in H$, then $\|Th\| \le M$.
If $\|h\| \le 1$ and $\|Th\| \le M$.
$\implies \|T\| \le M \le \infty$.
$\implies T$ is bounded.
$\implies T \in \mathcal{B}(H, K)$.
$\therefore \mathcal{B}_0(H, K) \subseteq \mathcal{B}(H, K)$.
Clearly, $\mathcal{B}_0(H, K)$ is subspace of $\mathcal{B}(H, K)$(Exercise).
Suppose, $\{T_n\} \subseteq \mathcal{B}_0(H, K)$ and $T \in \mathcal{B}(H, K)$ such that $\|T_n - T\| \to 0$.
To show: $T \in \mathcal{B}_0(H, K)$.
That is, To show: $\text{cl}[T(\text{Ball}H)]$ is compact.
Here K is Hilbert space and hence it is complete.
Now, if some how we can show that, $T(\text{Ball}H)$ is totally bounded that will prove that,

cl$[T(\text{Ball}H)]$ is compact.
It is given that, $\|T_n - T\| \to 0$ as $n \to 0$.
That is, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that
$\|T_n - T\| < \epsilon/3, \quad \forall n \geq N$.
Also, it is given that $\{T_n\}$ is sequence of compact linear transformations from H to K.
$\Longrightarrow$ cl$[T_n(\text{Ball}H)]$ is compact for all n.
Let $\cup_\alpha B(T_n h_\alpha, \epsilon/3)$ be an open covering of $T(\text{Ball}H)$.
$\Longrightarrow$ cl$[T_n(\text{Ball}H)] \subseteq \cup_\alpha B(T_n h_\alpha, \epsilon/3)$.
Since T_n is compact for all n hence, there are vectors $h_1, h_2, ..., h_m$ in H such that
cl$[T_n(\text{Ball}H)] \subseteq \cup_{j=1}^m B(T_n h_j, \epsilon/3)$.
$\Longrightarrow T_n(\text{Ball}H) \subseteq \cup_{j=1}^m B(T_n h_j, \epsilon/3)$.
So if $\|h\| \leq 1$, then there is an h_j such that $\|Th - T_n h_j\| \leq \epsilon/3$.
Consider,

$$\begin{aligned}
\|Th - Th_j\| &= \|Th - T_n h + T_n h - T_n h_j + T_n h_j - Th_j\| \\
&\leq \|Th - T_n h\| + \|T_n h - T_n h_j\| + \|T_n h_j - Th_j\| \\
&< \|(T - T_n)h\| + \epsilon/3 + \|(T - T_n)h_j\| \\
&< \|T - T_n\| + \epsilon/3 + \|T - T_n\| \\
&< 2\|T - T_n\| + \epsilon/3 \\
&< 2 \cdot \epsilon/3 + \epsilon/3 \\
&< \epsilon
\end{aligned}$$

$\Longrightarrow \|Th - Th_j\| < \epsilon$.
$\Longrightarrow Th \in B(Th_j, \epsilon)$.
$\Longrightarrow T(\text{Ball}H) \subseteq \cup_{j=1}^m B(Th_j, \epsilon)$.
$\Longrightarrow$ cl$[T(\text{Ball}H)]$ is compact.
$\Longrightarrow T \in \mathcal{B}_0(H, K)$.
To show: $\mathcal{B}_0(H, K)$ is linear space.
Let $T_1, T_2 \in \mathcal{B}_0(H, K) \Longrightarrow$ cl$[T_1(\text{Ball}H)]$ and cl$[T_2(\text{Ball}H)]$ are compact.
We know that,

$$\begin{aligned}
\Longrightarrow \text{cl}[(T_1 + T_2)(\text{Ball}H)] &= \text{cl}[T_1(\text{Ball}H) + T_2(\text{Ball}H)] \\
&= \text{cl}[T_1(\text{Ball}H)] + \text{cl}[T_2(\text{Ball}H)]
\end{aligned}$$

$\Longrightarrow$ cl$[(T_1 + T_2)(\text{Ball}H)]$ is compact.
$\Longrightarrow T_1 + T_2 \in \mathcal{B}_0(H, K)$.
For $\alpha \in \mathbb{F}$ and $T \in \mathcal{B}_0(H, K)$.
$\Longrightarrow$ cl$[\alpha T(\text{Ball}H)]$ is compact.
$\Longrightarrow \alpha T \in \mathcal{B}_0(H, K)$.
(c) Let $A \in \mathcal{B}(H), B \in \mathcal{B}(K)$.
To show: $TA \in \mathcal{B}_0(H, K)$.
Since, $A \in \mathcal{B}(H) \Longrightarrow A(\text{Ball}H)$ is compact. $\because$ Ball$H$ is closed and bounded subset of H.
$\Longrightarrow TA(\text{Ball}H)$ is compact. $\because T \in \mathcal{B}_0(H, K) \subseteq \mathcal{B}(H, K)$.
$\Longrightarrow$ cl$TA(\text{Ball}H)$ is compact.
$\Longrightarrow TA \in \mathcal{B}_0(H, K)$.
Similarly, $BT \in \mathcal{B}_0(H, K)$. ■

Definition. An operator T on H has finite rank if ran$T$ is finite dimensional. The set of continuous finite rank operators is denoted by $\mathcal{B}_{00}(H,K); \mathcal{B}_{00}(H) = \mathcal{B}_{00}(H,H)$.
Exercise. Show that $\mathcal{B}_{00}(H,K) \subseteq \mathcal{B}_0(H,K)$.
Theorem. *If $T \in \mathcal{B}(H,K)$, the following statements are equivalent.*
(a) *T is compact.*
(b) *T^* is compact.*
(c) *There is sequence $\{T_n\}$ of operators of finite rank such that $\|T - T_n\| \to 0$.*
PROOF. (c) $\implies$ (a)
From part (b) of previous theorem we have $\{T_n\}$ be sequence of continuous operators with finite rank with $\|T - T_n\| \to 0$ then, $T \in \mathcal{B}_0(H,K)$.
$\implies T$ is compact.
(a) $\implies$ (c)
Let T is compact operator.
$\implies$ cl(T(Ball H)) is compact.
$\implies$ cl(ran$T$) $= \mathcal{L}$ is separable subspace of K. $\because X$ is compact metric space then, X is separable.
Assume that $\{e_1, e_2, ...\}$ is basis for $\mathcal{L}$.
Let $M = \vee \{e_i : 1 \leq i \leq n\}$ and let P_n be the projection of K onto M.
Denote $T_n = P_n T$.
Claim: For $h \in H, \|T_n h - Th\| \to 0$.
Consider, $k \in K, k = \displaystyle\sum_i \langle k, e_i \rangle e_i$.

Also, any element $P_n k \in M$ can be written as $P_n k = \displaystyle\sum_{i=1}^{n} \langle k, e_i \rangle e_i$.
$\implies \|P_n k - k\| \to 0$ as $n \to \infty$.
In particular, $\|P_n Th - Th\| \to 0$. Hence claim.
It is given that T is compact.
$\implies$ cl(T(Ball H)) is compact.
We know that every compact space is complete and totally bounded.
$\implies$ cl(T(Ball H)) is totally bounded.
$\implies \exists h_1, h_2, ..., h_m \in H$ such that cl(T(Ball H)) $\subseteq \cup_{j=1}^{m} B(Th_j, \epsilon/3)$.
$\implies \|Th - Th_j\| < \epsilon/3$.
Consider,

$$\begin{aligned}
\|Th - T_n h\| &= \|Th - Th_j + Th_j - T_n h_j + T_n h_j - T_n h\| \\
&\leq \|Th - Th_j\| + \|Th_j - T_n h_j\| + \|T_n h_j - T_n h\| \\
&\leq \|Th - Th_j\| + \|Th_j - T_n h_j\| + \|P_n Th_j - P_n Th\| \\
&\leq \|Th - Th_j\| + \|Th_j - T_n h_j\| + \|P_n(Th_j - Th)\| \\
&\leq \|Th - Th_j\| + \|Th_j - T_n h_j\| + \|Th_j - Th\| \\
&\leq 2\|Th - Th_j\| + \|Th_j - T_n h_j\| \\
&\leq 2\epsilon/3 + \epsilon/3
\end{aligned}$$

$\implies \|Th - T_n h\| < \epsilon, \quad \forall n \geq n_0$ for some $n_0 \in \mathbb{N}$. (1)
Now, $\|T_n - T\| = \sup\{\|T_n h - Th\| : \|h\| \leq 1\}$.
From (1) $\|T_n - T\| \to 0$. Also, $T_n = P_n T$.

$\Longrightarrow T_n$ has finite rank.
$\therefore \{T_n\}$ is sequence of finite rank operators such that $\|T_n - T\| \to 0$.
(c) $\Longrightarrow$ (b)
Let $\{T_n\}$ is a sequence in $\mathcal{B}_{00}(H,K)$ such that $\|T_n - T\| \to 0$.
Consider,

$$\begin{aligned} \|T_n^* - T^*\| &= \|(T_n - T)^*\| \\ &= \|T_n - T\| \end{aligned}$$

$\therefore \|T_n^* - T^*\| = \|T_n - T\| \to 0$.
But $T_n^* \in \mathcal{B}_{00}(K,H)$ and $\|T_n^* - T^*\| \to 0$.
$\Longrightarrow T^* \in \mathcal{B}_{00}(K,H) \subseteq \mathcal{B}_0(K,H)$.
$\Longrightarrow T^*$ is compact.
(b) $\Longrightarrow$ (a)
Apply (c) $\Longrightarrow$ (b) for T^*. ■

Corollary. *If $T \in \mathcal{B}_0(H,K)$, then* cl(ran$T$) *is separable and if $\{e_i\}$ is a basis for* cl(ran$T$) *and P_n is the projection of K onto $\vee\{e_i : 1 \le i \le n\}$, then $\|P_nT - T\| \to 0$.*
PROOF. Exercise. ■

Definition. If $A \in \mathcal{B}(H)$, a scalar α is an eigenvalue of A if $\ker(A - \alpha I) \neq 0$.
If h is non-zero vector in $\ker(A - \alpha I)$, h is called eigenvector for α.

Notation. $\sigma_p(A) =$ Set of eigenvalues of A.

Proposition. *If $A \in \mathcal{B}_0(H), \lambda \in \sigma_p(T)$ and $\lambda \neq 0$, then eigenspace $\ker(T - \lambda I)$ is finite dimensional.*
PROOF. Suppose there is an infinite orthonormal sequence in $\{e_n\}$ in $\ker(T - \lambda I)$.
Since T is compact operator.
$\therefore$ there is subsequence $\{e_{n_k}\}$ of $\{e_n\}$ such that $\{T(e_{n_k})\}$ is convergent.
$\Longrightarrow \{T(e_{n_k})\}$ is Cauchy sequence.
But for $n_k \neq n_j$.
Consider,

$$\begin{aligned} \|T(e_{n_k} - Te_{n_j})\|^2 &= \|Te_{n_k} - Te_{n_j}\|^2 \\ &= \|\lambda e_{n_k} - \lambda e_{n_j}\|^2 \\ &= |\lambda|^2 \|e_{n_k} - e_{n_j}\|^2 \\ &= 2|\lambda|^2 \\ &> 0 \qquad\qquad \because \lambda \neq 0. \end{aligned}$$

$\to\leftarrow$ to saying that $\ker(T - \lambda I)$ contain an infinite orthonormal sequence $\{e_n\}$.
$\therefore \ker(T - \lambda I)$ must be finite dimensional. ■

Result. If T is a compact self-adjoint operator, then there is a sequence $\{\mu_n\}$ of real numbers and an orthonormal basis $\{e_n\}$ for $(\ker T)^\perp$ such that for all h,

$$Th = \sum_{n=1}^{\infty} \mu_n \langle h, e_n \rangle e_n.$$

Proposition. *If $T \in \mathcal{B}_0(H), T = T^*$ and $\ker T = (0)$, then H is separable.*
PROOF. Suppose T is compact self adjoint operator.
$\Longrightarrow \ker T = (\text{ran}T)^\perp$.
Given that $\ker T = (0)$.
$\Longrightarrow (\ker T)^\perp = (0)^\perp = H$.

$\implies (\text{ran}T) = H.$ $\because \ker T = (\text{ran}T^*)^\perp$ and $T = T^* \implies \ker T = (\text{ran}T)^\perp$. (1)
Also, if T is compact self adjoint operator on H, then there is sequence of $\{\mu_n\}$ of real numbers and orthonormal basis $\{e_n\}$ for $(\ker T)^\perp$ such that for all h,

$$Th = \sum_{n=1}^{\infty} \mu_n \langle h, e_n \rangle e_n. \qquad (2)$$

From (1) and (2) we can say that ran$T$ is a countable dense subset of H.
$\implies H$ is separable. ■

Proposition. *If A is normal operator and $\lambda \in \mathbb{F}$, then $\ker(A-\lambda) = \ker(A-\lambda)^*$ and $\ker(A-\lambda)$ is a reducing subspace for A.*

PROOF. Suppose A is normal operator.
Consider,

$$\begin{aligned}
(A-\lambda)(A-\lambda)^* &= (A-\lambda)(A^*-\overline{\lambda}) \\
&= AA^* - A\overline{\lambda} - A^*\lambda + \lambda\overline{\lambda} \\
&= A^*A - A^*\lambda - A\overline{\lambda} + \lambda\overline{\lambda} \qquad \because A \text{ is normal operator.} \\
&= A^*(A-\lambda) - \overline{\lambda}(A-\lambda) \\
&= (A^*-\overline{\lambda})(A-\lambda) \\
&= (A-\lambda)^*(A-\lambda)
\end{aligned}$$

$\implies (A-\lambda)$ is normal operator.
$\therefore \|(A-\lambda)h\| = \|(A-\lambda)^*h\|.$ $\because$ If A is normal operator if and only if $\|Ah\| = \|A^*h\|$.
Now, $h \in \ker(A-\lambda) \iff (A-\lambda)h = 0.$
$\iff \|(A-\lambda)h\| = 0.$
$\iff \|(A-\lambda)^*h\| = 0.$
$\iff (A-\lambda)^*h = 0.$
$\iff h \in \ker(A-\lambda)^*.$
$\therefore \ker(A-\lambda) = \ker(A-\lambda)^*.$
If $h \in \ker(A-\lambda) = \ker(A-\lambda)^*.$
$\implies h \in \ker(A-\lambda)^*.$
$\implies h \in \ker(A^*-\overline{\lambda}).$
$\implies A^*h = \overline{\lambda}h.$

$$\begin{aligned}
(A^*-\overline{\lambda})(\overline{\lambda}h) &= A^*\overline{\lambda}h - (\overline{\lambda})^2h \\
&= \overline{\lambda}A^*h - (\overline{\lambda})^2h \\
&= (\overline{\lambda})^2h - (\overline{\lambda})^2h \\
&= 0
\end{aligned}$$

$\therefore$ For $h \in \ker(A-\lambda); \overline{\lambda}h \in \ker(A-\lambda)^* = \ker(A-\lambda).$
$\therefore \ker(A-\lambda)$ reduces a subspace for A. ■

Proposition. *If A is normal operator and μ, λ are distinct eigenvalues of A, then $\ker(A-\lambda) \perp \ker(A-\mu)$.*

PROOF. Let $h \in \ker(A-\lambda)$ and $g \in \ker(A-\mu) = \ker(A-\mu)^*.$
$\implies Ah = \lambda h, g \in \ker(A^*-\bar{\mu}) \implies A^*g = \bar{\mu}g.$

Consider,

$$\begin{aligned}\lambda\langle h,g\rangle &= \langle \lambda h,g\rangle\\ &= \langle Ah,g\rangle\\ &= \langle h,A^*g\rangle\\ &= \langle h,\bar{\mu}g\rangle\\ &= \mu\langle h,g\rangle\end{aligned}$$

$\Longrightarrow \lambda\langle h,g\rangle - \mu\langle h,g\rangle = 0.$
$\Longrightarrow (\lambda-\mu)\langle h,g\rangle = 0.$
$\Longrightarrow \langle h,g\rangle = 0.$ $\because \lambda \neq \mu.$
$\therefore h \perp g, \quad \forall h \in \ker(A-\lambda), \forall g \in \ker(A-\mu).$
$\Longrightarrow \ker(A-\lambda) \perp \ker(A-\mu).$ ■

Proposition. *If $A = A^*$ and $\lambda \in \sigma_p(A)$, then λ is real number.*
PROOF. Let $h \in \ker(A-\lambda)$.
$\Longrightarrow Ah = \lambda h.$ (1)
Also, $h \in \ker(A-\lambda)^*$.
$\Longrightarrow h \in \ker(A^* - \bar{\lambda}).$
$\Longrightarrow A^*h = \bar{\lambda}h.$ (2)
It is given that $A^* = A$.
From (1) and (2) we can write,
$\lambda h = \bar{\lambda}h.$
$\Longrightarrow \lambda h - \bar{\lambda}h = 0.$
$\Longrightarrow (\lambda-\bar{\lambda})h = 0, \quad \forall h \in H.$
$\Longrightarrow \lambda - \bar{\lambda} = 0.$
$\Longrightarrow \lambda = \bar{\lambda}.$
$\therefore \lambda$ is real number. ■

Result. If T is a compact operator on H, $\lambda \neq 0$, and $\inf\{\|(T-\lambda)h\| : \|h\| = 1\} = 0$, then $\lambda \in \sigma_p(T)$.

Lemma. *If T is compact self-adjoint operator, then either $\pm\|T\|$ is an eigenvalue of T.*
PROOF. We know that for self-adjoint operator T,
$\|T\| = \sup\{|\langle Th,h\rangle| : \|h\| = 1\}.$
$\Longrightarrow \exists$ a sequence $\{h_n\}$ of units vectors such that
$|\langle Th_n,h_n\rangle| \to \|T\|.$
Let $|\lambda| = \|T\|$.
So $|\langle Th_n,h_n\rangle| \to \lambda.$
Claim: $\lambda \in \sigma_p(T)$.

$$
\begin{aligned}
0 &\leq \|(T-\lambda)h_n\|^2 \\
&\leq \|Th_n - \lambda h_n\|^2 \\
&\leq \langle Th_n - \lambda h_n, Th_n - \lambda h_n\rangle \\
&\leq \langle Th_n, Th_n\rangle - \langle Th_n, \lambda h_n\rangle - \langle \lambda h_n, Th_n\rangle + \langle \lambda h_n, \lambda h_n\rangle \\
&\leq \|Th_n\|^2 - \lambda\langle Th_n, h_n\rangle - \lambda\langle h_n, Th_n\rangle + \lambda^2\langle h_n, h_n\rangle. && \because \lambda = \bar{\lambda} \text{ for self-adjoint operator.} \\
&\leq \|Th_n\|^2 - \lambda\langle Th_n, h_n\rangle - \lambda\langle T^* h_n, h_n\rangle + \lambda^2\|h_n\|^2 \\
&\leq \|Th_n\|^2 - \lambda\langle Th_n, h_n\rangle - \lambda\langle Th_n, h_n\rangle + \lambda^2 && \because \|h_n\| = 1 \\
&\leq \|Th_n\|^2 - 2\lambda\langle Th_n, h_n\rangle + \lambda^2 \\
&\leq \|T\|^2 - 2\lambda\langle Th_n, h_n\rangle + \lambda^2 \\
&\leq \lambda^2 - 2\lambda\langle Th_n, h_n\rangle + \lambda^2 \\
&\leq 2\lambda^2 - 2\lambda\langle Th_n, h_n\rangle
\end{aligned}
$$

$\Longrightarrow 0 \leq \|(T-\lambda)h_n\|^2 \leq 2\lambda^2 - 2\lambda\langle Th_n, h_n\rangle$.
But $\langle Th_n, h_n\rangle \to \lambda$.
$\Longrightarrow \|(T-\lambda)h_n\|^2 = 0$ and $n \to \infty$.
Also, we know that for compact operator if $\inf\{\|(T-\lambda)h\| : \|h\| = 1\} = 0$,
then $\lambda \in \sigma_p(T)$.
$\Longrightarrow \inf\{\|(T-\lambda)h\| : \|h\| = 1\} = 0$. $\because \|(T-\lambda)h_n\| = 0$ as $n \to \infty$.
$\Longrightarrow \lambda \in \sigma_p(T)$. ■

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CHAPTER 3

Banach Spaces

ELEMENTARY PROPERTIES AND EXAMPLES

Definition. If X is vector space over $\mathbb{F}$, a seminorm is function $p : X \to [0, \infty)$ having the properties:
(a) $p(x + y) = p(x) + p(y)$ for all $x, y \in X$.
(b) $p(\alpha x) = |\alpha| p(x)$ for all $\alpha \in \mathbb{F}$ and $x \in X$.
A norm is seminorm p such that
(c) $p(x) = 0 \Longrightarrow x = 0$.

Notation. Usually norm is denoted by $\| \cdot \|$.

Normed space. A vector space X together with some norm is called normed space. That is, A normed space is pair $(X, \| \cdot \|)$, where X is vector space and $\| \cdot \|$ is norm on X.

Banach space. Banach space is a normed space which is complete with respect to the metric defined by norm.

NORMED SPACES

Let X be metric space.
(1) The set of all bounded continuous $\mathbb{F}-$valued functions on X is denoted by $C(X)$. That is, $C(X) = \{f : X \to F : f \text{ is bounded and continuous}\}$
This is linear space w.r.t pointwise addition and scalar multiplication and

$$\|f\| = \sup \{|f(x)| : x \in X\}$$

defines norm on $C(X)$.
(2) The set of all functions $f \in C(X)$ satisfying the property: For every $\epsilon > 0$ there is compact set S of X, depending on f and ϵ such that

$$|f(x)| < \epsilon, \quad \forall x \notin S.$$

is denoted by $C_0(X)$. Then $C_0(X)$ is subspace of $C(X)$ and hence normed space.
That is, $C_0(X)$ is set consisting of functions $f \in C(X)$ such that,
for given $\epsilon > 0, \quad S = \{x \in X : |f(x)| \geq \epsilon\}$ is compact set in X.
(3) The set of all functions $f \in C(X)$ with the property: There is compact set S of X, depending on f, such that $f(x) = 0$ for all $x \notin S$ is denoted by $C_c(X)$.
That is, $C_c(X)$ is set consisting of functions $f \in C(X)$ such that, $S = \{x \in X : f(x) \neq 0\}$ is compact set in X.
(4) Let X be the set of natural numbers with the discrete metric. Then $C(X)$ is the set of all bounded sequences $x(n)$ in $\mathbb{F}$. This space is denoted by l^∞, and the norm is given by

$$\|x\| = \sup \{|x(n)| : n \in \mathbb{N}\}, \quad x \in l^\infty.$$

The subspace $C_0(X)$ becomes

$$c_0 = \left\{x \in l^\infty : \lim_{n\to\infty} x(n) = 0\right\},$$

and $C_c(X)$ becomes

$$c_{00} = \{x \in l^\infty : x(n) = 0 \text{ for all but finitely many } n \in \mathbb{N}\}.$$

(5) Let X be measure space with measure μ. If $1 \le p < \infty$, then set of all measurable functions f with $\int_X |f|^p d\mu < \infty$ is denoted by $L^p(X)$ or $L^p(\mu)$. This is linear space and

$$\|f\|_p = (\int_X |f|^p d\mu)^{\frac{1}{p}}, \quad f \in L^p(X),$$

defines a norm on $L^p(X)$.
(6) The set of all essentially bounded measurable functions is denoted by $L^\infty(X)$ or $L^\infty(\mu)$. This also is linear space, and

$$\|f\|_\infty = \text{ess.}\sup\{|f(x)| : x \in X\}, \quad x \in L^\infty(X),$$

defines a norm on $L^\infty(X)$. The spaces $L^p(X)$, with $1 \le p \le \infty$, are called Lebesgue spaces.
(7) Let X be the set of all natural numbers with counting measure. If $1 \le p < \infty$, then $L^p(X)$ becomes the set l^p of all scalar sequences $\{x(n)\}$ with

$$\|x\|_p = (\sum_{n=1}^{\infty} |x(n)|^p)^{\frac{1}{p}} < \infty.$$

(8) The space $L^\infty(X)$ become the sequence space l^∞.
(9) If $X = \{1, 2, ..., n\}$ with counting measure, then $L^p(X)$ become $\mathbb{F}^n$ with norm $\|\cdot\|_p$ given by

$$\|x\|_p = (\sum_{j=1}^{n} |x(j)|^p)^{\frac{1}{p}}, \text{ if } 1 \le p < \infty,$$

$$\|x\|_\infty = \sup_{1 \le j \le n} \{|x(j)|\}, \text{ if } p = \infty.$$

(10) If X and Y are normed spaces, then $X \times Y$ is linear space with addition and scalar multiplication defined coordinatewise and

$$\|(x, y)\| = \|x\| + \|y\|, x \in X, y \in Y,$$

defines a norm on it.
Examples. (1) l^p is Banach space for $1 \le p < \infty$.
(2) $L^p(E)$ is Banach space, where E is measurable subset of $\mathbb{F}$ and $1 \le p < \infty$.
(3) $l^\infty = \left\{ x : \mathbb{N} \to \mathbb{F} : \sup_{i=1,2,...} |x(i)| < \infty \right\}$ is Banach space.
PROOF. Clearly, l^∞ is vector space over $\mathbb{F}$.
Define norm on l^∞ as $\|x\| = \sup_i |x(i)|$.
Let $\{x_n\}$ be a cauchy sequence from l^∞.
Therefore, for given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that
$\|x_n - x_m\| < \epsilon, \quad \forall n, m \ge N$.

$\Longrightarrow \sup_i |x_n(i) - x_m(i)| < \epsilon, \quad \forall n, m \geq N.$
$\Longrightarrow |x_n(i) - x_m(i)| < \epsilon, \quad \forall n, m \geq N.$
$\Longrightarrow \{x_n(i)\}$ is a cauchy sequence in $\mathbb{F}$ and we know that $\mathbb{F}$ is complete.
$\Longrightarrow \{x_n(i)\} \to x(i)$ for some $x(i) \in \mathbb{F}$.
Now for fixed $n > N$ and taking $m \to \infty$ we get,
$\lim_{m\to\infty} |x_n(i) - x_m(i)| = |x_n(i) - x(i)| < \epsilon$.
$\Longrightarrow \sup_i |x_n(i) - x(i)| < \epsilon.$
$\Longrightarrow \|x_n - x\| < \epsilon, \quad \forall n > N.$
$\Longrightarrow \{x_n\} \to x \in l^\infty.$ $\because x : \mathbb{N} \to \mathbb{F}.$
$\therefore l^\infty$ complete normed space.
$\Longrightarrow l^\infty$ is Banach space.
(4) Show that $C_{00} = \{x \in l^p\}$ such that $x(i) = 0$ for all but finitely many i, is not Banach space.
PROOF. Clearly C_{00} is a subspace of l^∞.
Claim: C_{00} is not closed.
It is sufficient to show there exists a sequence which is not convergent in C_{00}.
Suppose $\{x_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, 0, 0, 0, ...\right\}$.
Since finitely many terms of this sequence are non-zero therefore $\{x_n\} \in C_{00}$.
Then, $\{x_n\}$ is Cauchy but it is not convergent in C_{00}.
Because as $n \to \infty$ the sequence $\{x_n\} \to \left\{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...\right\}$ but $\left\{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...\right\} \notin C_{00}$.
Hence, C_{00} is not closed.
$\Longrightarrow C_{00}$ is not Banach space. ■

Proposition. *If X is a normed space, then*
(a) the function $X \times X \to X$ defined by $(x, y) = x + y$ is continuous;
(b) the function $\mathbb{F} \times X \to X$ defined by $(\alpha, x) = \alpha x$ is continuous.
PROOF. (a) Let $f : X \times X \to X$ defined by $f(x, y) = x + y$.
To show: f is continuous.
Suppose $\{x_n\} \to x$ and $\{y_n\} \to y$ be any sequences in X.
That is, $\Longrightarrow \|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$ as $n \to \infty$.
Then, $(\{x_n\}, \{y_n\}) \to (x, y)$.
Consider,

$$\begin{aligned} \|(x_n + y_n) - (x + y)\| &= \|(x_n - x) + (y_n - y)\| \\ &\leq \|x_n - x\| + \|y_n - y\| \end{aligned}$$

As $n \to \infty \Longrightarrow \|(x_n + y_n) - (x + y)\| \to 0$.
$\Longrightarrow \{x_n + y_n\} \to x + y$ as $n \to \infty$.
$\Longrightarrow f(\{x_n\}, \{y_n\}) \to f(x, y)$.
$\Longrightarrow f$ is continuous.
(b) Let $g : \mathbb{F} \times X \to X$ defined by $g(\alpha, x) = \alpha x$.
To show: g is continuous.
Suppose $\{x_n\} \to x$ be any sequence in X and $\{\alpha_n\} \to \alpha$ be any sequence in $\mathbb{F}$.
$\Longrightarrow \|x_n - x\| \to 0$ and $|\alpha_n - \alpha| \to 0$ as $n \to \infty$.

Consider,

$$\begin{aligned}\|\alpha_n x_n - \alpha x\| &= \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\| \\ &= \|\alpha_n(x_n - x) + x(\alpha_n - \alpha)\| \\ &\leq |\alpha_n|\|x_n - x\| + \|x\||\alpha_n - \alpha|\end{aligned}$$

As $n \to \infty \implies \|\alpha_n x_n - \alpha x\| \to 0$.
$\implies \{\alpha_n x_n\} \to \alpha x$.
$\implies g(\{\alpha_n\}, \{x_n\}) \to g(\alpha, x)$ as $n \to \infty$.
$\implies g$ is continuous. ■

Lemma. *If p and q are seminorms on a vector space X, then following statements are equivalent.*
(a) $p(x) \leq q(x)$ for all x.
(b) $\{x \in X : q(x) < 1\} \subseteq \{x \in X : p(x) < 1\}$.
(b′) $p(x) < 1$ whenever $q(x) < 1$.
(c) $\{x : q(x) \leq 1\} \subseteq \{x : p(x) \leq 1\}$.
(c′) $p(x) \leq 1$ whenever $q(x) \leq 1$.
(d) $\{x : q(x) < 1\} \subseteq \{x : p(x) \leq 1\}$.
(d′) $p(x) \leq 1$ whenever $q(x) < 1$.

PROOF. (b) and (b'), (c) and (c'), (d) and (d') are equivalent.
Also (a) implies all the remaining conditions and that both (b) and (c) implies (d).
It remains to show $(d) \implies (a)$.
Given, $\{x : q(x) < 1\} \subseteq \{x : p(x) \leq 1\}$.
Let $q(x) = \alpha$. If $\epsilon > 0$, then $q(\frac{x}{\alpha+\epsilon}) = \frac{1}{\alpha+\epsilon} q(x) = \frac{\alpha}{\alpha+\epsilon} < 1$.
$p(\frac{x}{\alpha+\epsilon}) \leq 1$
$\implies \frac{1}{\alpha+\epsilon} p(x) \leq 1$.
$\implies p(x) \leq \alpha + \epsilon$.
Letting $\epsilon \to 0 \implies p(x) \leq \alpha$.
$\implies p(x) \leq q(x), \quad \forall x \in X$. ■

Definition. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on X. They are said to be equivalent norms if they define same topology.
That is, Let $\mathcal{T}_1$ be the topology induced by $\|\cdot\|_1$ and $\mathcal{T}_2$ is a topology induced by $\|\cdot\|_2$. Then, $\|\cdot\|_1$ and $\|\cdot\|_2$ are said to be equivalent if and only if $\mathcal{T}_1 = \mathcal{T}_2$.

Proposition. *If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on X, then these norms are equivalent if and only if there are positive constants c and C such that*
$c\|\cdot\|_1 \leq \|\cdot\|_2 \leq C\|\cdot\|_1$ for all x in X.

PROOF. Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.
Let $\mathcal{T}_1$ be the topology induced by $\|\cdot\|_1$ and $\mathcal{T}_2$ be the topology induced by $\|\cdot\|_2$.
$\therefore \mathcal{T}_1 = \mathcal{T}_2$. $\qquad \because \|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.
Let $B_1(0, 1)$ be an open ball in $\mathcal{T}_1$ centred at 0 and has radius 1.
$\implies \exists r_1 > 0$ such that $B_2(0, r_1) \subseteq B_1(0, 1)$ $\qquad \because \mathcal{T}_1 = \mathcal{T}_2 \implies \mathcal{T}_2 \subseteq \mathcal{T}_1$
$\implies \{x \in X : \|x - 0\|_2 < r_1\} \subseteq \{x \in X : \|x - 0\|_1 < 1\}$.
$\implies \left\{x \in X : r_1^{-1}\|x\|_2 < 1\right\} \subseteq \{x \in X : \|x\|_1 < 1\}$.
$\implies \{x \in X : q(x) < 1\} \subseteq \{x \in X : p(x) < 1\}$, where $q(x) = r_1^{-1}\|x\|_2$ and $p(x) = \|x\|_1$.
$\therefore p(x) \leq q(x) \quad \forall x \in X$. $\qquad \because$ by previous lemma.
$\implies \|x\|_1 \leq r_1^{-1}\|x\|_2$.

$\implies r_1\|x\|_1 \le \|x\|_2$.
Choose $c = r_1 \implies c\|x\|_1 \le \|x\|_2$. (1)
Similarly, $B_2(0,1)$ is open ball in $\mathcal{T}_2$ centred at 0 and has radius 1.
$\implies \exists r_2 > 0$ such that $B_1(0,r_2) \subseteq B_2(0,1)$. $\because \mathcal{T}_1 = \mathcal{T}_2 \implies \mathcal{T}_1 \subseteq \mathcal{T}_2$.
$\implies \{x \in X : \|x-0\|_1 < r_2\} \subseteq \{x \in X : \|x-0\|_2 < 1\}$.
$\implies \left\{x \in X : r_2^{-1}\|x\|_1 < 1\right\} \subseteq \{x \in X : \|x\|_2 < 1\}$.
$q(x) = r_2^{-1}\|x\|_1$ and $p(x) = \|x\|_2$.
$\implies p(x) \le q(x) \quad \forall x \in X$.
$\implies \|x\|_2 \le r_2^{-1}\|x\|_1$.
$\implies \|x\|_2 \le C\|x\|_1$, where $C = r_2^{-1}$. (2)
From (1) and (2) we have, $c\|x\|_1 \le \|x\|_2 \le C\|x\|_1$.
Conversely, suppose there exists constants c and C such that $c\|x\|_1 \le \|x\|_2 \le C\|x\|_1$.(1)
Let $\mathcal{T}_1$ and $\mathcal{T}_2$ are the topologies given by $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively.
Let $B_2(x_0,\epsilon) \in \mathcal{T}_2$.
Now, $B_2(x_0,\epsilon) = \{x \in X : \|x-x_0\|_2 < \epsilon\}$.
From (1) we have $\|x\|_2 \le C\|x\|_1$.
$\implies \{x \in X : \|x-x_0\|_1 < \epsilon/C\} \subseteq \{x \in X : \|x-x_0\|_2 < \epsilon\}$.
$\implies B_1(x_0,\epsilon/C) \subseteq B_2(x_0,\epsilon)$.
$\therefore B_2(x_0,\epsilon)$ is open in $\mathcal{T}_1$.
$\therefore \mathcal{T}_2 \subseteq \mathcal{T}_1$. (2)
Let $B_1(y_0,\epsilon)$ be an open set in $\mathcal{T}_1$.
Now, $B_1(y_0,\epsilon) = \{x \in X : \|y_0-x\|_1 < \epsilon\}$.
From (1) we have $c\|x\|_1 \le \|x\|_2$.
$\therefore \{x \in X : \|y_0-x\|_2 < c\epsilon\} \subseteq \{x \in X : \|y_0-x\|_1 < \epsilon\}$.
$\implies B_2(y_0,\epsilon) \subseteq B_1(y_0,\epsilon)$.
$\therefore B_1(y_0,\epsilon)$ is open in $\mathcal{T}_2$.
$\mathcal{T}_1 \subseteq \mathcal{T}_2$. (3)
From (2) and (3) $\mathcal{T}_1 = \mathcal{T}_2$.
$\therefore \|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. ■

Result 1. *Define an relation $\sim$ on set of all norms on a normed linear space X by $\|\cdot\|_1 \sim \|\cdot\|_2$ if and only if $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, then $\sim$ is equivalence relation.*
PROOF. Let $T = \{\|\cdot\| : (X, \|\cdot\|)$ is normed linear space$\}$.
(i) For $\|\cdot\| \in T$. Since $\|x\| = \|x\| = \|x\|$,, where $c = 1 = C$.
$\implies \|\cdot\| \sim \|\cdot\|$.
(ii) Let $\|\cdot\|_1 \sim \|\cdot\|_2 \implies \|\cdot\|_1$ is equivalent to $\|\cdot\|_2$.
$\therefore \exists c, C$ such that $c\|x\|_1 \le \|x\|_2 \le C\|x\|_1 \quad \forall x \in X$. (1)
From (1) we have $c\|x\|_1 \le \|x\|_2$.
$\implies \|x\|_1 \le \frac{1}{c}\|x\|_2$.
$\implies \|x\|_1 \le C'\|x\|_2$, where $\frac{1}{c} = C'$.
Also from (1) we have, $\|x\|_2 \le C\|x\|_1$.
$\implies \frac{1}{C}\|x\|_2 \le \|x\|_1$.
$\implies c'\|x\|_2 \le \|x\|_1$, where $\frac{1}{C} = c'$.
$\therefore c'\|x\|_2 \le \|x\|_1 \le C'\|x\|_2$.
$\implies \|\cdot\|_2$ is equivalent to $\|\cdot\|_1$.
(iii) Similarly transitivity holds.

$\therefore \sim$ is equivalence relation on T. ■

Remark 2. *There are two properties on a norm linear space one is topological property and another is metric property. The metric property depends on the precise norm but topological property depends only on the equivalence class of norms.*

Example. Let X be any Hausdorff space and let $C_b(X)$ = all continuous functions $f : X \to \mathbb{F}$ such that $\|f\| = \sup\{|f(x)| : x \in X\} < \infty$. For $f, g \in C_b(X)$, define $f + g : X \to \mathbb{F}$ by $(f + g)(x) = f(x) + g(x)$; for $\alpha \in \mathbb{F}$ define $(\alpha f)(x) = \alpha f(x)$. Then $C_b(X)$ is Banach space.

PROOF. Clearly $C_b(X)$ is vector space over $\mathbb{F}$(Exercise).

To show: $\|\cdot\| : C_b(X) \to \mathbb{F}$ defined by $\|f\| = \sup\{|f(x)| : x \in X\}$ is norm.

(i) $\|f\| \geq 0$.

If $\|f\| = 0 \implies \sup\limits_x \{|f(x)| : x \in X\} = 0$.

$\implies \sup\limits_{x\in X} |f(x)| = 0$.

$\implies |f(x)| = 0, \quad \forall x \in X$.

$\implies f(x) = 0, \quad \forall x \in X$.

$\implies f = 0$.

(ii) For $f, g \in C_b(X)$.

Consider,

$$\begin{aligned}
\|f+g\| &= \sup_x \{|(f+g)(x)| : x \in X\} \\
&= \sup_x \{|f(x)+g(x)| : x \in X\} \\
&\leq \sup_x \{|f(x)|+|g(x)| : x \in X\} \\
&= \sup_x \{|f(x)| : x \in X\} + \sup_x \{|g(x)| : x \in X\} \\
&= \|f\| + \|g\|
\end{aligned}$$

$\therefore \|f + g\| \leq \|f\| + \|g\|$.

(iii) For any $f \in C_b(X), \alpha \in \mathbb{F}$

$$\begin{aligned}
\|\alpha f\| &= \sup_x \{|(\alpha f)(x)| : x \in X\} \\
&= \sup_x \{|\alpha f(x)| : x \in X\} \\
&= |\alpha| \sup_x \{|f(x)| : x \in X\} \\
&= |\alpha| \|f\|
\end{aligned}$$

$$\therefore \|\alpha f\| = |\alpha| \|f\|$$

$\therefore C_b(X)$ is normed linear space.

Let $\{f_n\}$ be Cauchy sequence in $C_b(X)$.
$\Longrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}$ such that

$$\begin{aligned} \|f_n - f_m\| &< \epsilon, \quad \forall n, m \geq N \\ \Longrightarrow \sup_{x\in X} |(f_n - f_m)(x)| &< \epsilon \\ \Longrightarrow \sup_{x\in X} |f_n(x) - f_m(x)| &< \epsilon \\ \Longrightarrow |f_n(x) - f_m(x)| &< \epsilon, \quad \forall n, m \geq N \end{aligned}$$

For fixed $x \in X$,
$\Longrightarrow |f_n(x) - f_m(x)| < \epsilon, \quad \forall n, m \geq N$.
$\Longrightarrow \{f_n(x)\}$ is a Cauchy sequence in $\mathbb{F}$ and $\mathbb{F}$ is complete.
$\therefore f_n(x) \to f(x)$ as $n \to \infty$.
$\Longrightarrow f(x) = \lim_{x\to\infty} f_n(x)$.

$$\begin{aligned} |f_n(x) - f(x)| &= |f_n(x) - f_m(x) + f_m(x) - f(x)| \\ &\leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \\ &\leq \sup_x |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \\ &\leq \|f_n(x) - f_m(x)\| + |f_m(x) - f(x)| \\ &< \epsilon + |f_m(x) - f(x)|, \quad \forall n, m \geq N \end{aligned}$$

As $m \to \infty$
$|f(x) - f_n(x)| < \epsilon, \quad \forall n, m \geq N$.
This is true for all $x \in X$.
$\therefore \sup_{x\in X} |f_n(x) - f(x)| < \epsilon, \quad \forall n, m \geq N$.
$\Longrightarrow \|f_n - f\| < \epsilon, \quad \forall n, m \geq N$.
$\therefore f_n \to f$ uniformly. $\because N$ is independent of x.
$\therefore f$ must be continuous.
Now,

$$\begin{aligned} \|f\| &= \|f - f_n + f_n\| \\ &\leq \|f - f_n\| + \|f_n\| \\ &< \epsilon + \|f_n\| < \infty \end{aligned}$$

Therefore, $f_n \to f$ and $f \in C_b(X)$.
$\therefore C_b(X)$ is Banach space. ■

Proposition. *If X is locally compact space and $C_0(X)$ = all continuous functions $f : X \to \mathbb{F}$ such that for all $\epsilon > 0, \{x \in X : |f(x)| \geq \epsilon\}$ is compact, then $C_0(X)$ is*

closed subspace of $C_b(X)$ and hence Banach space.
PROOF. To show : $C_0(X)$ is subspace of $C_b(X)$.
It is given that for $f \in C_0(X)$, f is continuous. If we can show f is bounded that will prove $f \in C_b(X)$.
It is given that $\{x \in X : |f(x)| \geq \epsilon\}$ is compact and we know that continuous image of compact space is compact.
$\Longrightarrow \{f(x) : |f(x)| > \epsilon\}$ is compact.
$\Longrightarrow \{f(x) : |f(x)| > \epsilon\}$ is bounded.
Suppose $|f(x)| < m, \quad \forall x \in X$.
$\Longrightarrow \epsilon < |f(x)| < m$.
Choose $M = \max\{\epsilon, m\}$.
$\Longrightarrow |f(x)| < M$.
$\Longrightarrow f$ is bounded.
$\Longrightarrow f \in C_b(X)$.
$\Longrightarrow C_0(x)$ is subspace of $C_b(X)$(Exercise).
Let f be limit point of $C_0(X)$.
$\Longrightarrow \exists\{f_n\} \subset C_0(X)$ such that $f_n \to f$.
That is, for all $\epsilon > 0, \exists N \in \mathbb{N}$ such that $\|f_n - f\| < \epsilon/2, \quad \forall n, m \geq N$.
$\Longrightarrow \sup_x |f_n(x) - f(x)| < \epsilon/2$.
$\Longrightarrow |f_n(x) - f(x)| < \epsilon/2$.
$\|f\| = \|f - f_n + f_n\|$.
$\Longrightarrow \|f\| \leq \|f - f_n\| + \|f_n\|$.
Let $\epsilon > 0$ and $|f(x)| \geq \epsilon$.

$$\begin{aligned} \therefore \epsilon &\leq |f(x)| \\ &= |f(x) - f_n(x) + f_n(x)| \\ &\leq |f(x) - f_n(x)| + |f_n(x)| \\ &= \epsilon/2 + |f_n(x)| \end{aligned}$$

$\Longrightarrow \epsilon - \epsilon/2 \leq |f_n(x)|$.
$\Longrightarrow \epsilon/2 \leq |f_n(x)|$.
Therefore, If $\epsilon \geq f(x) \Longrightarrow |f_n(x)| \geq \epsilon/2$.
$\Longrightarrow \{x \in X : |f(x)| \geq \epsilon\} \subseteq \{x \in X : |f_n(x)| \geq \epsilon/2\}$.
$\Longrightarrow \{x \in X : |f(x)| \geq \epsilon\}$ is compact. $\because$ each $f_n \in C_0(X)$.
$\therefore f \in C_0(X)$.
$\Longrightarrow C_0(X)$ is closed subspace of Banach space and hence Banach space. ■

Proposition. *If p is a seminorm on X, $|p(x) - p(y)| \leq p(x - y)$ for all $x, y \in X$. If $\|\cdot\|$ is a norm, then $|\|x\| - \|y\|| \leq \|x - y\|$ for all $x, y \in X$.*
PROOF. For any $x, y \in X$, then
$p(x) = p(x - y + y) \leq p(x - y) + p(y)$.
$\Longrightarrow p(x) - p(y) \leq p(x - y)$.
Similarly, $p(y) - p(x) \leq p(x - y)$.
$\Longrightarrow -(p(x) - p(y)) \leq p(x - y)$.
$\Longrightarrow -p(x - y) \leq p(x) - p(y)$.
$\Longrightarrow -p(x - y) \leq p(x) - p(y) \leq p(x - y)$.

$\therefore |p(x) - p(y)| \leq p(x - y)$.
If $\|\cdot\|$ is norm then $|\|x\| - \|y\|| \leq \|x - y\|, \quad \forall x, y \in X$. ■

Definition. If X and Y are normed spaces, X and Y are said to be isometrically isomorphic if there is surjective linear isometry from X onto Y.

LINEAR OPERATORS ON NORMED SPACES

Let $\mathcal{B}(X, Y)$ = all continuous linear transformations from X to Y.

Proposition. *If X and Y are normed spaces and $A : X \to Y$ is a linear transformation, the following statements are equivalent.*
(a) $A \in \mathcal{B}(X, Y)$.
(b) *A is continuous at 0.*
(c) *A is continuous at some point.*
(d) *There is a positive constant c such that $Ax \leq c\|x\|$ for all $x \in X$.*
If $A \in \mathcal{B}(X, Y)$ and
$\|A\| = \sup\{\|Ax\| : \|x\| \leq 1\}$,
then

$$\begin{aligned} \|A\| &= \sup\{\|Ax\| : \|x\| = 1\} \\ &= \sup\{\|Ax\|/\|x\| : x \neq 0\} \\ &= \inf\{c > 0 : \|Ax\| \leq c\|x\| \text{ for } x \in X\} \end{aligned}$$

$\|x\|$ is called the norm of A and $\mathcal{B}(X, Y)$ become normed space if addition and scalar multiplication are defined pointwise.

Result. *$\mathcal{B}(X, Y)$ is Banach space, if Y is Banach space.*

PROOF. To show: $\mathcal{B}(X, Y)$ is Banach space.
Let $\{T_n\}$ be a cauchy sequence in $\mathcal{B}(X, Y)$.
Then, for every $\epsilon > 0, \exists N \in \mathcal{N}$ such that
$\|T_n - T_m\| < \epsilon, \quad \forall n, m \geq N$.
Now,

$$\begin{aligned} \|T_n(x) - T_m(x)\| &= \|(T_m - T_n)(x)\| \\ &\leq \|(T_m - T_n)\|\|x\| \\ &< \epsilon\|x\|, \quad \forall n, m \geq N \end{aligned}$$

For a fixed x and $x \neq 0$,
Choose $\epsilon_1 = \epsilon\|x\|$.
Then $\|T_n(x) - T_m(x)\| < \epsilon_1, \quad \forall n, m \geq N$.
$\implies \{T_n(x)\}$ is cauchy sequence in Y and Y is Banach and hence complete.
$\therefore \{T_n(x)\} \to T(x)$.
That is, $\lim_{n\to\infty} T_n(x) = T(x)$.
We have, $\|T_n(x) - T_m(x)\| < \epsilon_1, \quad \forall n, m \geq N$.
As $m \to \infty$
$\|T_n(x) - T(x)\| < \epsilon\|x\|, \quad \forall n \geq N$.

$$\implies \sup_x \{\|T_n(x) - T(x)\|\} < \epsilon. \qquad \because \|x\| < 1$$

$$\implies \sup_x \{\|(T_n - T)(x)\|\} < \epsilon.$$

$\implies \|T_n - T\| < \epsilon, \quad \forall n \geq N.$

$\implies \{T_n\} \to T$ as $n \to \infty$.
To show: $T \in \mathcal{B}(X,Y)$.
That is, To show T is continuous linear transformation.
Clearly, T is continuous.
Because T_n is sequence of continuous functions and converges to T hence T is continuous.
For any $x, y \in X$ and $\alpha \in \mathbb{F}$.

$$\begin{aligned} T(x+\alpha y) &= \lim_{n\to\infty} T_n(x+\alpha y) \\ &= \lim_{n\to\infty} (T_n(x) + \alpha T_n(y)) \\ &= \lim_{n\to\infty} T_n(x) + \alpha \lim_{n\to\infty} T_n(y) \\ &= T(x) + \alpha T(y) \end{aligned}$$

$\implies T$ is continuous linear transformation.
$\implies T \in \mathcal{B}(X,Y)$.
$\therefore \mathcal{B}(X,Y)$ is Banach space. ■
Note. A continuous linear operator is also called bounded linear operator.
Example 1. If (X, Ω, μ) is $\sigma-$finite measure space and $\phi \in L^\infty(X, \Omega, \mu)$, define $M_\phi : L^p(X, \Omega, \mu) \to L^p(X, \Omega, \mu), 1 \leq p \leq \infty$, by $M_\phi f = \phi f$ for all $f \in L^p(X, \Omega, \mu)$. Then $M_\phi \in \mathcal{B}(L^p(X, \Omega, \mu))$ and $\|M_\phi\| = \|\phi\|_\infty$.
Result 1. *Let X and Y be normed linear spaces. A linear mapping $T : X \to Y$ is said to be homeomorphism if and only if there exists constant $\alpha, \beta > 0$ such that $\alpha\|x\| \leq \|T(x)\| \leq \beta\|x\|. \quad \forall x \in X$.*
PROOF. Suppose $T : X \to Y$ is homeomorphism.
To prove: $\exists \alpha, \beta$ such that $\alpha\|x\| \leq \|T(x)\| \leq \beta\|x\|$.
Since T is homeomorphism means T and T^{-1} both are continuous.
T is continuous.
$\implies \|T(x)\| \leq \beta\|x\|$ for some $\beta > 0$.
T^{-1} is continuous.
$\implies \|T^{-1}(y)\| \leq \gamma\|y\|$ for some $\gamma > 0$ and $T(x) = y$.
$\implies \|x\| \leq \gamma\|T(x)\|$.
$\implies \frac{1}{\gamma}\|x\| \leq \|T(x)\|$.
$\implies \alpha\|x\| \leq \|T(x)\|$. where $\alpha = \frac{1}{\gamma}$.
$\therefore \alpha\|x\| \leq \|T(x)\| \leq \beta\|x\|$.
Conversely, $\exists \alpha, \beta$ such that $\alpha\|x\| \leq \|T(x)\| \leq \beta\|x\|$.
To show: T is homeomorphism.
Suppose $x \in \ker T$.
$\implies T(x) = 0$.
$\implies \|T(x)\| = 0$.
$\implies \alpha\|x\| = 0$. $\because 0 \leq \alpha\|x\| \leq \|T(x)\| = 0$ and $\alpha > 0$.
$\implies \|x\| = 0$. $\because \alpha > 0$.

$\implies x = 0$.
$\implies \ker T = \{0\}$.
$\implies T$ is injective.
Now consider, $T^{-1} : R(T) \to X$.
Clearly, T^{-1} is bijective mapping.
Also, T^{-1} is linear because for $y_1, y_2 \in R(T)$.
$\therefore y_1 = T(x_1), y_2 = T(x_2)$ for some $x_1, x_2 \in X$.
Now,

$$\begin{aligned} T^{-1}(y_1 + \alpha y_2) &= T^{-1}(T(x_1) + \alpha T(x_2)) \\ &= T^{-1}(T(x_1 + \alpha x_2)) \\ &= x_1 + \alpha x_2 \\ &= T^{-1}(y_1) + \alpha T^{-1}(y_2) \end{aligned}$$

Form given inequality we have $\|T(x)\| \leq \beta\|x\|$ for some $\beta > 0$ and $\forall x \in X$.
$\implies T$ is continuous.
Also, $\alpha\|x\| \leq \|T(x)\|$.
$\implies \alpha\|T^{-1}(y)\| \leq \|y\|$.
$\implies \|T^{-1}(y)\| \leq \frac{1}{\alpha}\|y\|$.
$\implies T^{-1}$ is continuous.
$\therefore T$ and T^{-1} both are continuous.
$\therefore T$ is homeomorphism. ■

Result 2. *Y is a subspace of a normed space X, then Y and it's closure are normed spaces with respect to induced norm.*

PROOF. Clearly, Y is normed space means $(Y, \|\cdot\|)$ is normed linear space, because X is normed linear space and Y is subspace of X.
To show: $\bar{Y}$ subspace of X.
Let $x, y \in \bar{Y}$, then there exist sequences $\{x_n\}$ and $\{y_n\}$ in Y such that $\{x_n\} \to x$ and $\{y_n\} \to y$ as $n \to \infty$.
Consider,

$$\begin{aligned} x + y &= \lim_{n\to\infty} \{x_n\} + \lim_{n\to\infty} \{y_n\} \\ &= \lim_{n\to\infty} \{x_n + y_n\} \\ &= \lim_{n\to\infty} \{z_n\}. \qquad \text{where } x_n + y_n = z_n. \end{aligned}$$

$\implies x + y \in \bar{Y}$.
Now for any $\alpha \in \mathbb{F}$ and $x \in \bar{Y} \implies \exists \{x_n\}$ of points in Y such that $\{x_n\} \to x$.
Consider,

$$\begin{aligned} \alpha x &= \alpha \lim_{n\to\infty} \{x_n\} \\ &= \lim_{n\to\infty} \{\alpha x_n\} \end{aligned}$$

$\implies \alpha x \in \bar{Y}$.
$\therefore \bar{Y}$ is subspace of X.
$\therefore (\bar{Y}, \|\cdot\|)$ is subspace of X. ■

Result 3. *Let Y be a closed subspace of a normed space X. For $x+Y$ in the quotient space X/Y, let $|||x+Y||| = \inf\{\|x+y\|/y \in Y\}$. Then $(X/Y, ||| \cdot |||)$ is normed space and $||| \cdot |||$ is called quotient norm on X/Y.*

PROOF. (i) $|||x+Y||| \geq 0$. $\|x+y\| \geq 0 \quad \forall y \in Y$.

If $|||x+Y||| = 0$.

$\Longrightarrow |||x+y||| = 0, \quad \forall y \in Y$.

$\Longrightarrow \inf\{\|x+y\|/y \in Y\} = 0$.

$\therefore \exists$ sequence $\{y_n\}$ from Y such that $\|x+y_n\| \to 0$ as $n \to \infty$.

Because, if $\inf\{a/a \in S\} = 0$ then there exists a sequence $\{a_n\}$ from S such that $\{a_n\} \to a$ as $n \to \infty$.

$\Longrightarrow \{x+y_n\} \to 0$ as $n \to \infty$.

$\Longrightarrow \{y_n\} \to -x \in \bar{Y} = Y$. $\because Y$ is closed.

$\Longrightarrow x \in Y$.

$\Longrightarrow x+Y = 0+Y$.

$\therefore |||x+Y||| = 0 \Longrightarrow x+Y = 0+Y$.

(ii) Let $x_1+Y, x_2+Y \in X/Y$ for some $x_1, x_2 \in X$.

To show : $|||(x_1+Y)+(x_2+Y)||| \leq |||x_1+Y||| + |||x_2+Y|||$.

Consider, $|||x_1+Y||| = \inf\{\|x_1+y\|/y \in Y\}$.

$\therefore \forall \epsilon/2 > 0$ there exists $y_1 \in Y$ such that $\|x_1+y_1\| \leq |||x_1+Y||| + \epsilon/2$.

Similarly, there exists $y_2 \in Y$ such that $\|x_2+y_2\| \leq |||x_2+Y||| + \epsilon/2$.

Now

$$\begin{aligned}\|(x_1+y_1)+(x_2+y_2)\| &\leq \|x_1+y_1\| + \|x_2+y_2\| \\ &\leq |||x_1+Y||| + \epsilon/2 + |||x_2+Y||| + \epsilon/2\end{aligned}$$

$\Longrightarrow \|(x_1+x_2)+(y_1+y_2)\| \leq |||x_1+y_1||| + |||x_2+y_2||| + \epsilon$.

Taking infimum on left hand side we get,

$\Longrightarrow \inf\{\|(x_1+x_2)+y/y \in Y\|\} \leq |||x_1+y_1||| + |||x_2+y_2||| + \epsilon$. $\because y = y_1+y_2 \in Y$.

$\Longrightarrow |||(x_1+x_2)+Y||| \leq |||x_1+y_1||| + |||x_2+y_2||| + \epsilon$.

$\therefore \forall \epsilon > 0, |||(x_1+Y)+(x_2+Y)||| \leq |||x_1+y_1||| + |||x_2+y_2||| + \epsilon$.

Now taking $\epsilon \to 0$.

$|||(x_1+Y)+(x_2+Y)||| \leq |||x_1+y_1||| + |||x_2+y_2|||$.

(iii) Let $\alpha \in \mathbb{F}$ and $x+Y \in X/Y$.

Consider,

$$\begin{aligned}\|\alpha(x+Y)\| &= \inf\{\|\alpha(x+y)\|/y \in Y\} \\ &= \inf\{|\alpha|\|x+y\|/y \in Y\} \\ &= |\alpha| \inf\{\|x+y\|/y \in Y\} \\ &= |\alpha| |||x+Y|||\end{aligned}$$

$\therefore ||| \cdot |||$ is norm on X/Y. ∎

Result 4. *A sequence $\{x_n+Y\}$ converges to $x+Y$ if and only if $\exists \{y_n\}$ such that $\{x_n+y_n\} \to x$.*

PROOF. Suppose $\exists \{y_n\}$ such that $\{x_n+y_n\} \to x$.

$$\begin{aligned}|||(x_n+Y)-(x+Y)||| &= |||(x_n-x)+Y||| \\ &= |||(x_n-x)+Y||| \\ &= \inf\{\|(x_n-x)+y\|/y \in Y\} \\ &\leq \|x_n-x+y_n\|\end{aligned}$$

Now, as $n \to \infty \implies$ RHS of above inequality goes to 0.
$\therefore \{x_n + Y\} \to x + Y$ as $n \to \infty$.
Conversely, suppose $\{x_n + Y\} \to x + Y$ as $n \to \infty$.
$|||(x_n + Y) - (x + Y)||| = \inf\{\|x_n - x + y\|/y \in Y\}$.
$\therefore \exists y_n \in Y$ such that $\|x_n - x + y_n\| = |||(x_n + Y) - (x + Y)|||$.
$\implies \|x_n - x + y_n\| < |||(x_n + Y) - (x + Y)||| + \frac{1}{n}$.
Now, for $n \to \infty$.
$\implies \{x_n + y_n\} \to x$.

Definition. A series $\sum_{n=1}^{\infty} x$ is said to be convergent if the partial sum S of sequence $\{x_n\}$ is converges in X, where $S = x_1 + x_2 + ... + x_m$.

Definition. Let X is normed linear space. If $x_n \in X$ and $\sum \|x_n\| < \infty$, then $\sum x_n$ is called absolutely convergent.

Theorem. *A normed space X is a Banach space if and only if every absolutely convergent series in it convergent.*

PROOF. Let X is Banach space.
Suppose that X is Banach space and $\sum x_n$ is absolutely convergent series in X.
$\implies \sum_{n=1}^{\infty} \|x_n\| < \infty$.
Let $t_n = \|x_1\| + \|x_2\| + ... + \|x_n\|$, and $y_n = x_1 + x_2 + ... + x_n$.
Then, for $n \geq m \geq 1$,

$$\begin{aligned} \|y_n - y_m\| &= \|x_{m+1} + x_{m+2} + ... + x_n\| \\ &\leq \|x_{m+1}\| + \|x_{m+2}\| + ... + \|x_n\| \\ &= t_n - t_m \end{aligned}$$

Since $\{t_n\}$ converges and hence it is Cauchy sequence.
$\implies t_n - t_m \leq \|t_n - t_m\| < \epsilon$.
$\implies \|y_n - y_m\| < \epsilon$.
$\implies \{y_n\}$ is a Cauchy sequence in X and hence convergent in X. $\because X$ is Banach space.
$\therefore \sum x_n$ converges.
Conversely, suppose every absolutely convergent series in X converges.
To show: X is Banach space.
Let $\{x_n\}$ be Cauchy sequence in X.
$\therefore \|x_n - x_{n_1}\| < 1, \quad \forall n \geq n_1$.
Choose $n_2, n_3, ...$, successively, such that $n_r > n_{r-1}$ and
$\|x_n - x_{n_r}\| < \frac{1}{r^2}$ for all $n \geq n_r, r = 2, 3, ...$
Then, $\|x_{n_{r+1}} - x_{n_r}\| < \frac{1}{r^2}$ for $r = 1, 2, ...$
Let $z_r = x_{n_{r+1}} - x_{n_r}, r = 1, 2, ...$
Then,
$\sum_{r=1}^{\infty} \|z_r\| = \sum_{r=1}^{\infty} \|x_{n_{r+1}} - x_{n_r}\| \leq \sum_{r=1}^{\infty} \frac{1}{r^2} < \infty$.
Thus, $\sum z_r$ is absolutely convergent.
Suppose $\sum z_r$ converges to $x \in X$.

$\implies z_1 + z_2 + ... + z_{r-1} = x_{n_2} - x_{n_1} + x_{n_3} - x_{n_2} + ... + x_{n_r} - x_{n_{r-1}}$.
$\implies \sum_{j=1}^{r-1} z_j = x_{n_r} - x_{n_1}$.
$\implies x_{n_r} = \sum_{j=1}^{r-1} z_j + x_{n_1}$.
$\implies x_{n_r} \to x + x_{n_1}$ as $r \to \infty$.
$\therefore \{x_{n_r}\}$ is convergent.
$\implies \{x_n\}$ is convergent. $\quad\because \{x_{n_r}\}$ is convergent subsequence of Cauchy sequence $\{x_n\}$.
$\therefore X$ is Banach space.
$\therefore \sum_{n=1}^{\infty} x_n$ is summable. ■

Corollary. *T is homeomorphism from X onto Y. Then X is complete if and only if Y is complete.*

PROOF. Suppose T is homeomorphism from X onto Y.
$\therefore \quad \exists \alpha, \beta > 0$ such that $\alpha\|x\| \leq \|T(x)\| \leq \beta\|x\|$. (1)
Suppose X is complete.
To show: Y is complete.
Let $\{y_n\}$ be Cauchy sequence in Y.
$\therefore \quad \exists \{x_n\}$ in X such that $T(x_n) = y_n$.
From inequality (1) we have,
$\alpha\|x_n - x_m\| \leq \|T(x_m) - T(x_n)\|$.
$\implies \alpha\|x_n - x_m\| \leq \|y_m - y_n\|$.
$\implies \|x_n - x_m\| \leq \frac{1}{\alpha}\|y_m - y_n\| < \epsilon$.
$\implies \|x_n - x_m\| < \epsilon$.
$\implies \{x_n\}$ is a Cauchy sequence in X and X is complete.
$\implies \{x_n\} \to x$ for some $x \in X$.
Again from inequality (1) we have, $\|T(x_n - x)\| \leq \beta\|x_n - x\|$.
$\implies \|T(x_n) - T(x)\| \leq \beta\|x_n - x\| < \epsilon_1, \quad \forall n \geq N_1 \in \mathbb{N}$.
$\implies \|T(x_n) - T(x)\| < \epsilon_1$.
$\implies \|y_n - y\| < \epsilon_1, \quad \forall n \geq N_1 \in \mathbb{N}$, where $y = T(x)$.
$\therefore \{y_n\} \to y$ and $y \in Y$.
$\implies Y$ is complete.
Similarly converse holds so left for exercise. ■

Theorem. *Let Y is closed subspace of X. Then X is a Banach space if and only if Y and X/Y are Banach spaces.*

PROOF. Let X is Banach space.
We know that closed subspace of Banach Space is Banach space.
$\implies Y$ is Banach space.
To show: X/Y is Banach space.
Let $\{x_n + Y\}$ be a absolutely convergent sequence in X/Y. That is, $|\|x_n + Y\|| < \infty$.
To show: $\{x_n + Y\}$ is convergent in X/Y.

That is we have to show $\sum_{n=1}^{\infty} x_n + Y < \infty$.
By definition of $|||\cdot|||, \exists \{y_n\}$ in Y such that
$\|x_n + y_n\| < |||x_n - y_n||| + \frac{1}{n^2}$.
$\implies \sum_{n=1}^{\infty} \|x_n + y_n\| < \infty$. $\quad \because \sum_{n=1}^{\infty} |||x_n + Y||| < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.
Let $\sum_{n=1}^{\infty} x_n + y_n = C$.
Consider,

$$\begin{aligned} |||\sum_{n=1}^{m} x_n + Y - C + Y||| &= |||\sum_{n=1}^{m} x_n + y_n - C + Y||| \\ &= |||\sum_{n=1}^{m}(x_n + y_n - C) + Y||| \\ &\leq \|\sum_{n=1}^{m} x_n + y_n - C\| \end{aligned}$$

As $m \to \infty$ RHS of above inequality goes to 0.
$\implies |||\sum_{n=1}^{m} x_n + Y - C + Y||| \to 0$ as $m \to \infty$.
$\implies \sum_{n=1}^{\infty} x_n + Y = C + Y$.
$\therefore X/Y$ is Banach space.
Conversely, Suppose Y and X/Y are Banach spaces.
To show: X is Banach space.
Let $\{x_n\}$ be Cauchy sequence in X.
For given $\epsilon > 0$, $\quad \exists N \in \mathbb{N}$ such that
$\|x_m - x_n\| < \epsilon, \quad \forall n, m \geq N$.
Consider, $|||x_n + Y - (x_m + Y)||| = |||x_n - x_m + Y||| \leq \|x_n - x_m\| < \epsilon, \quad \forall n, m \geq N$.
$\therefore \{x_n + Y\}$ is Cauchy sequence in X/Y.
$\therefore \{x_n + Y\} \to \{x + Y\}$ for some $x \in X$.
$\implies \exists y_n \in Y$ such that $\{x_n + y_n\} \to x$.
Consider,

$$\begin{aligned} \|y_n - y_m\| &= \|y_n + x_n - x + x_m - x_n - x_m - y_m + x\| \\ &\leq \|y_n + x_n - x\| + \|x_m - x_n\| + \|x_m + y_m - x\| \\ &< \epsilon, \quad \forall n, m \geq N. \end{aligned}$$

$\implies \{y_n\}$ is Cauchy sequence in Y.
Given Y is Banach space.
$\therefore \{y_n\} \to y$ as $n \to \infty$ for some $y \in Y$.
$\therefore x_n = x_n + y_n - y_n$.
$\implies \{x_n\} \to x - y$ as $n \to \infty$. $\quad \because \{x_n + y_n\} \to x$ and $\{y_n\} \to y$.
$\{x_n\} \to x - y$ and $x - y \in X$.

X is Banach space. ■

FINITE DIMENSIONAL NORMED SPACES

Proposition. *If X is finite dimensional vector space over $\mathbb{F}$, then any two norms on X are equivalent.*

PROOF. Let $\{e_1, e_2, ..., e_d\}$ be a Hamal basis for X.

$\therefore$ for any $x \in X, x = \sum_{j=1}^{d} x_j e_j$.

Define $\|x\|_\infty = \max\{|x_j| : 1 \leq j \leq d\}$, then $\|\cdot\|_\infty$ is a norm on X.

Let $\|\cdot\|$ be another norm on X.

To show: $\|\cdot\|$ and $\|\cdot\|_\infty$.

Consider,

$$\begin{aligned} \|x\| &= \|\sum_{j=1}^{d} x_j e_j\| \\ &\leq \sum_{j=1}^{d} \|x_j e_j\| \\ &\leq \sum_{j=1}^{d} |x_j| \|e_j\| \qquad \because x_j's \text{ are scalar.} \\ &\leq \max_{1\leq j\leq d} |x_j| \sum_{j=1}^{d} \|e_j\| \end{aligned}$$

$\therefore \|x\| \leq C\|x\|_\infty$, where $C = \sum_{j=1}^{d} \|e_j\|$. (1)

Let $\mathcal{T}_1$ be topology defined by $\|\cdot\|_\infty$ and $\mathcal{T}_2$ be the topology defined by $\|\cdot\|$.

Claim: $\mathcal{T}_1 \supseteq \mathcal{T}_2$.

Let $B_2(x_0, r)$ be an open set in $\mathcal{T}_2$.

$\therefore B_2(x_0, r) = \{x \in X : \|x - x_0\| < r\}$

$\implies \left\{x \in X : \|x - x_0\|_\infty < \frac{r}{C}\right\} \subseteq \{x \in X : \|x - x_0\| < r\}$.

$\implies B_1(x_0, \frac{r}{C}) \subseteq B_2(x_0, r)$.

$\implies B_2(x_0, r)$ is open in $\mathcal{T}_1$.

$\implies \mathcal{T}_1 \supseteq \mathcal{T}_2$.

Consider, $B = \{x \in X : \|x\|_\infty \leq 1\}$.

Clearly, B is compact $\mathcal{T}_1$ compact.

Claim: B is $\mathcal{T}_2$ compact.

Let $\{U_\alpha\}$ be open cover for B with respect to $\mathcal{T}_2-$topology.

$\implies \{U_\alpha\}$ be a open cover for B with respect to $\mathcal{T}_1-$topology.

$\implies \{U_i\}_{i=1}^{n}$ covers B. $\because B$ is $\mathcal{T}_1$ compact.

$\implies B$ is $\mathcal{T}_2$ compact.

Let $A = \{x \in X : \|x\|_\infty < 1\}$.

Clearly, A is $\mathcal{T}_1$ open.

$\implies A$ is open in $(B, \mathcal{T}_2)$.

$\implies \exists$ some $U \in \mathcal{T}_2$ such that $B \cap U = A$.

Let $x \in B \cap U \implies x \in B$ and $x \in U$.
$\implies \|x\|_\infty \leq 1$.
$\therefore x \in U$ and U is open in $\mathcal{T}_2$.
$\therefore \exists r > 0$ such that $\{x \in X : \|x\| < r\} \subseteq U$.
$\therefore \|x\|_\infty \leq 1$ and $\|x\| < r$.
$\implies \|x\|_\infty < 1$. (2)
Claim: $\|x\| < r \implies \|x\|_\infty < 1$.
Let $\|x\| < r$ and $\alpha = \|x\|_\infty$.
$\therefore \|\frac{x}{\alpha}\|_\infty = 1$.
If possible, $\alpha \geq 1 \implies \frac{1}{\alpha} \leq 1$.
$\therefore \|\frac{x}{\alpha}\| \leq r$.
$\therefore \|\frac{x}{\alpha}\| \leq r$ and $\|\frac{x}{\alpha}\|_\infty = 1$.
$\implies \|\frac{x}{\alpha}\|_\infty < 1$. $\because$ By (2).
$\longrightarrow\!\longleftarrow$.
$\therefore$ We must have $\alpha < 1$.
$\implies \|x\|_\infty < 1$.
$\implies \|x\|_\infty \leq \frac{1}{r}\|x\|$.
$\implies r\|x\|_\infty \leq \|x\|$.
$\implies c\|x\|_\infty \leq \|x\|$, where $r = c$. (3)
$\therefore$ From (1) and (3) we get,
$c\|x\|_\infty \leq \|x\| \leq C\|x\|_\infty$.
$\therefore \|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent.
Now, assume that $\|\cdot\|_1$ and $\|\cdot\|_2$ are any two norms on X.
Then, $\|\cdot\|_1$ equivalent to $\|\cdot\|_\infty$ and $\|\cdot\|_2$ equivalent to $\|\cdot\|_\infty$.
$\implies \|\cdot\|_1$ equivalent to $\|\cdot\|_2$. ■

Theorem. *Let X be finite dimensional norm linear space and M be a linear manifold in X. Then M is closed.*

PROOF. Since $(X, \|\cdot\|)$ is norm linear space.
$\implies (M, \|\cdot\|)$ be normed linear space.
Let $\{e_1, e_2, ..., e_n\}$ be Hamal basis for M.
$\therefore$ For any $x \in M$, $x = \sum_{j=1}^{n} x_j e_j$, where $x_j \in F$ for all $1 \leq j \leq n$.
Define another norm on M as $\|x\|_\infty = \max\{|x_j| : 1 \leq j \leq n\}$.
Let $\{x_n\}$ be Cauchy sequence in M.
$\implies$ For any $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that,
$\|x_n - x_m\|_\infty < \epsilon, \quad \forall n, m \geq N$.
$\implies \max\left\{|x_{n_j} - x_{m_j}| : 1 \leq j \leq n\right\} < \epsilon, \quad \forall n, m \geq N$.
$\implies |x_{n_j} - x_{m_j}| < \epsilon, \quad \forall n, m \geq N$.
$\implies \left\{x_{n_j}\right\}$ is Cauchy sequence in $\mathbb{F}$.
$\therefore \left\{x_{n_j}\right\} \to x_j$ as $n_j \to \infty$, and it is true for all $1 \leq j \leq n$.
Let $x = x_1e_1 + x_2e_2 + ... + x_ne_n$.
Here $\{x_{n_1}\} \to x_1, \{x_{n_2}\} \to x_2, ..., \{x_{n_n}\} \to x_n$.
$\therefore \{x_n\} \to x$.
Because, $\|x_n - x\|_\infty = \max\left\{|x_{n_j} - x_j| : 1 \leq j \leq n\right\}$.

RHS goes to 0 and $n_j \to \infty$.
$\implies \{x_n\} \to x$ and $x \in M$.
$\therefore M$ is complete with respect to $\|\cdot\|_\infty$.
$\implies M$ is complete with respect to $\|\cdot\|$ and hence closed. ■

Theorem. *Let X and Y are normed spaces and X is finite dimensional. Then prove that every linear transformation $T : X \to Y$ is continuous.*

PROOF. Let $\{e_1, e_2, ..., e_d\}$ be a Hamal basis for X.

$\therefore$ For any $x \in X, x = \sum_{j=1}^{d} x_j e_j$.

Define a norm on X as $\|x\| = \max\{|x_j| : 1 \le j \le d\}$.
Now,

$$\begin{aligned}
\|T(x)\| &= \|T(\sum_{j=1}^{d} x_j e_j)\| \\
&= \|x_1 T(e_1) + x_2 T(e_2) + ... + x_d T(e_d)\| \\
&\le |x_1|\|T(e_1)\| + |x_2|\|T(e_2)\| + ... + |x_d|\|T(e_d)\| \\
&\le \max_{1\le j\le d} |x_j|(\|T(e_1)\| + \|T(e_2)\| + ... + \|T(e_d)\| \\
&\le \|x\|C, \text{ where, } C = \|T(e_1)\| + \|T(e_2)\| + ... + \|T(e_d)\| \\
\implies \|T(x)\| = C\|x\|
\end{aligned}$$

$\therefore T$ is continuous. ■

QUOTIENT AND PRODUCT OF NORM LINEAR SPACES

Let X be a normed space, M be a linear manifold in X, and let $Q : X \to X/M$ be a natural map defined by $Qx = x + M$. Then

$$|||x + M||| = \inf\{\|x + y\| : y \in M\}$$

is norm on X/M, provided M is closed(Why?).

Theorem. *If $M \le X$ and $|||x + Q|||$ is norm on X/M, Then*
(a) $|||Qx||| \le \|x\|, \quad \forall x \in X$ *and hence Q is continuous.*
(b) *If X is Banach space then, X/M is Banach.*
(c) *A subset W of X/M is open relative to norm if and only if $Q^{-1}(W)$ is open in X.*
(d) *If U is open in X, then $Q(U)$ is open in X/M.*

PROOF. (a) For all $x \in X$,
Consider,

$$\begin{aligned}
|||Qx||| &= |||x + M||| \\
&= \inf\{\|x + y\| : y \in M\} \\
&\le \|x\|
\end{aligned}$$

(b) Already done.
(c) From part (a) we have, $Q : X \to X/M$ defined by $Qx = x + M$ is continuous.
$\therefore$ Inverse image of open set in X/M under mapping Q is open in X.
$\implies$ If W is open in X/M then $Q^{-1}(W)$ is open in X.
Conversely, Suppose $W \subseteq X/M$ such that $Q^{-1}(W)$ is open in X.
To show: W is open in X/M.

Let $x_0 \in Q^{-1}(W)$.
$\implies \exists r > 0$, such that $\{x \in X : \|x - x_0\| < r\} \subseteq Q^{-1}(W)$.
Let $y = x - x_0$, then
$\{x_0 + y : \|y\| < r\} \subseteq Q^{-1}(W)$.
Let $B_r = \{x/\|x\| < r\}$.
$\therefore x_0 + B_r = \{x_0 + y/\|y\| < r\} \subseteq Q^{-1}(W)$.
$\implies x_0 + B_r \subseteq Q^{-1}(W)$.
Let $T = \{x + M/|||x + M||| < r\}$.
Claim: $Q(B_r) = T$.
Let $y \in Q(B_r) \implies \exists x \in B_r$ such that $Q(x) = y$.
$\implies y = x + M$.
Now, $x \in B_r \implies \|x\| < r$.
$\therefore |||x + M||| \leq \|x\| < r$.
$\therefore Q(B_r) \subseteq T$.
Suppose, $x + M \in T$.
$\implies |||x + M||| < r$.
$\exists y \in M$ such that $\|x + y\| < r$.
Now $Q(x + y) = x + y + M = x + M$.
$\implies x + M = Q(x + y) \in Q(B_r)$.
$\therefore T \subseteq Q(B_r)$.
$\therefore T = Q(B_r)$.
Already we have $x_0 + B_r \subseteq Q^1(W)$.
$\implies \{x_0 + y/\|y\| < r\} \subseteq Q^{-1}(W)$.
$\implies \{x/\|x - x_0\| < r\} \subseteq Q^{-1}(W)$.
$\implies Q(\{x/\|x - x_0\| < r\}) \subseteq W$.
$\implies \{x + M/|||x - x_0 + M||| < r\} \subseteq W$.
$\implies \{x + M/|||(x + M) - (x_0 + M)||| < r\} \subseteq W$.
$\implies B(x_0 + M, r) \subseteq W$.
$\therefore W$ is open in X/M.
(d) Let U be an open set in X, $QU = U/M$.

$$\begin{aligned}
\therefore Q^{-1}(QU) &= Q^{-1}(U/M) \\
&= \{x \in X : Qx \in U/M\} \\
&= \{x \in X : x + M \in U/M\} \\
&= \{u \in U : u + M \in U/M\} \\
&= U + M \\
&= \cup\{U + y : y \in M\}
\end{aligned}$$

$\implies Q^{-1}(QU)$ is open. $\because$ Each $U + y$ is open in X.
$\therefore QU$ is open. $\because$ By part (c). ■

Proposition. *If X is a normed space, $M \leq X$ and N is a finite dimensional subspace of X, then $M + N$ is closed subspace of X.*
PROOF. Consider, $Q : X \to X/M$.
Then, $QN = N/M$.
$\dim QN = \dim N/M = \dim N - \dim M \leq \dim N < \infty$.

$\therefore QN$ is finite dimensional.
QN is closed in $X/M \because$ Every finite dimensional subspace of norm linear space is closed
Also, we have Q is continuous mapping from X to X/M.
$\Longrightarrow Q^{-1}(Q(N))$ is closed in X.

$$\begin{aligned} \therefore Q^{-1}(QN) &= Q^{-1}(N/M) \\ &= \{x \in X/x + M \in N/M\} \\ &= N + M \end{aligned}$$

$\Longrightarrow N + M$ is closed subspace in X. ■
Let $\{X_i : i \in I\}$ be collection of normed linear space. Then $\prod_{i \in I} X_i$ is a vector space.
Let $\|\cdot\|$ is norm on X is norm on each X_i. For each $1 \leq p < \infty$, define

$$\oplus_p X_i = \left\{ x \in \prod_{i \in I} X_i : \|x\| = [\sum_i \|x\|^p]^{1/p} < \infty \right\},$$
$$\oplus_\infty X_i = \left\{ x \in \prod_{i \in I} X_i : \|x\| = \sup_i \|x\| < \infty \right\},$$
$$\oplus_0 X_n = \left\{ x \in \prod_{i \in I} X_i : \|x(n)\| \to 0 \right\}.$$

Then, $\oplus_p X_i$ and $\oplus_\infty X_i$ are normed linear spaces and $\oplus_0 X_i$ is subspace of $\oplus_\infty X_i$.
Proposition. *Let $\{X_i : i \in I\}$ be a collection of normed spaces and let*
$X = \oplus_p X_i, \quad 1 \leq p \leq \infty$.
(a) *X is normed space and the projection $P_i : X \to X_i$ is a continuous linear map with $\|P_i(x)\| \leq \|x\|$ for each $x \in X$.*
(b) *X is Banach space if and only if each X_i is Banach space.*
(c) *Each projection P_i is open map of X onto X_i.*
PROOF. We shall prove the above result for $1 \leq p \leq \infty$.
(a) Let $x, y \in X$ and $\alpha \in \mathbb{F}$.

(i)

$$\begin{aligned}
\|x\| &= 0 \\
\iff (\sum_i \|x(i)\|^p)^{1/p} &= 0 \\
\iff \sum_i \|x(i)\|^p &= 0 \\
\iff \|x(i)\|^p &= 0 \\
\iff \|x(i)\| &= 0 \\
\iff x(i) &= 0, \quad \forall i \\
\iff x &= 0
\end{aligned}$$

(ii)

$$\begin{aligned}
\|x+y\| &= (\sum_i \|(x+y)(i)\|^p)^{1/p} \\
&= (\sum_i \|x(i)+y(i)\|^p)^{1/p} \\
&\leq (\sum_i \|x(i)\|^p)^{1/p} + (\sum_i \|y(i)\|^p)^{1/p} \qquad \because \text{By Minkowski's Inequality} \\
&\leq \|x\| + \|y\|
\end{aligned}$$

(iii)

$$\begin{aligned}
\|\alpha x\| &= (\sum_i \|\alpha x(i)\|^p)^{1/p} \\
&= (|\alpha|^p \sum_i \|x(i)\|^p)^{1/p} \\
&= |\alpha|(\sum_i \|x(i)\|^p)^{1/p} \\
&= |\alpha|\|x\|
\end{aligned}$$

$\therefore X = \oplus_p X_i$ is normed linear space.
Let $P_i : X \to X_i$ be projection mapping defined by $P_i(x) = x(i)$.
(i) For $x, y \in X$.

$$\begin{aligned}
P_i(x+y) &= (x+y)(i) \\
&= x(i) + y(i) \\
&= P_i(x) + P_i(y)
\end{aligned}$$

(ii) For $x \in X$ and $\alpha \in \mathbb{F}$.

$$\begin{aligned} P_i(\alpha x) &= (\alpha x)(i) \\ &= \alpha x(i) \\ &= \alpha P_i(x) \end{aligned}$$

(iii) For any $x \in X$.

$$\begin{aligned} \|P_i(x)\| &= \|x(i)\| \\ &\leq (\sum_i \|x(i)\|^p)^{1/p} \\ &\leq \|x\| \end{aligned}$$

$\Longrightarrow P_i$ is continuous and linear mapping.
(b) To show: X is Banach space if and only if each X_i is Banach space.
Suppose X is Banach space.
$\Longrightarrow \prod_i X_i$ is Banach space.
Let $\{x_n(i)\}$ be a Cauchy sequence in X.
Choose $x_n = (0, 0, ..., 0, x_n(i), 0...)$.
$\Longrightarrow \{x_n\}$ is a Cauchy sequence in X.
$\Longrightarrow \{x_n\} \to x \in X$. $\because X$ is Banach space.
$\Longrightarrow \{x_n(i)\} \to x(i) \in X_i$, where $x = (0, 0, ..., x(i), 0, 0, ...)$.
$\Longrightarrow X_i$ is Banach space for each i.
Conversely, suppose X_i is Banach space for each i.
Let $\{x_n\}$ be Cauchy sequence in X.
For any $\epsilon > 0, \exists N \in \mathbb{N}$ such that,
$\|x_n - x_m\| < \epsilon, \quad \forall n, m \geq N$.
$(\sum_i \|(x_n - x_m)(i)\|^p)^{1/p} < \epsilon, \quad \forall n, m \geq N$.
$\|x_n(i) - x_m(i)\| \leq (\sum_i \|(x_n - x_m)(i)\|^p)^{1/p} < \epsilon, \quad \forall n, m \geq N$.
$\Longrightarrow \{x_n(i)\}$ is Cauchy sequence in X_i.
$\Longrightarrow \{x_n(i)\} \to x(i) \in X_i$.
$\therefore \{x_n\} \to x \in X$, where $x = (x(1), x(2), ..., x(i), ...)$.
$\therefore X$ is Banach space.
(c) To show: Exercise. ■

LINEAR FUNCTIONALS

Definition. Let X be vector space over a field $\mathbb{F}$. The a linear mapping $f : V \to \mathbb{F}$ is called linear functional.

Definition. A hyperplane in X is a linear manifold M in X such that $\dim(X/M) = 1$.

Proposition. (a) *A linear manifold in X is a hyperplane if and only if it is the kernel of a non-zero linear functional.*
(b) *Two linear functionals have the same kernel if and only if one is a non-zero multiple of the other.*

PROOF. (a) Let M be a linear manifold which is hyperplane in X.
Consider, the map $Q : X \to X/M$ defined by $Q(x) = x + M$.
Also consider a non-zero linear isomorphism $T : X/M \to \mathbb{F}$, then $f \equiv T \circ Q$ is linear functional from X to $\mathbb{F}$.
Now,

$$\begin{aligned}
\ker f &= \{x \in X : f(x) = 0\} \\
&= \{x \in X : (T \circ Q)(x) = 0\} \\
&= \{x \in X : T(Q(x)) = 0\} \\
&= \{x \in X : T(x + M) = 0\} \\
&= \{x \in X : x + M = M\} \\
&= \{x \in X : x \in M\} \\
&= M
\end{aligned}$$

$\therefore M$ is kernel of non-zero linear functional $T \circ Q$.
Conversely, Assume $f : X \to \mathbb{F}$ be a non-zero linear functional.
To show: $\ker f$ is hyperplane.
By rank nullity theorem we have,
$\dim X = \dim(\ker f) + \dim(\mathrm{Im} f)$.
$\implies \dim X - \dim(\ker f) = \dim(\mathrm{Im} f)$.
$\implies \dim(X/\ker f) = \dim(\mathrm{Im} f)$.
$\implies \dim(X/\ker f) = 1$.
$\implies \ker f$ is a hyperplane.
(b) Let $f : X \to \mathbb{F}$ and $g : X \to \mathbb{F}$ be two non-zero linear functionals and assume that $\ker f = \ker g$.
Since f is non-zero functional hence there exist some element $x_0 \in X$ such that $f(x_0) = 1$.
$\therefore g(x_0) \neq 0$.
Let $\beta = g(x_0)$ and $\alpha = f(x)$.
Consider, $f(x - \alpha x_0) = f(x) - \alpha f(x_0)$.
$\implies f(x - \alpha x_0) = \alpha - \alpha = 0$.
$\implies x - \alpha x_0 \in \ker f = \ker g$.
$\implies x - \alpha x_0 \in \ker g$.
$\implies g(x - \alpha x_0) = 0$.
$\implies g(x) - \alpha g(x_0) = 0$.
$\implies g(x) = \beta f(x), \quad \forall x \in X$, where $g(x_0) = \beta$.
Conversely, Suppose if $g = \beta f$.
$x \in \ker g$.
$\iff x \in \ker \beta f$.
$\iff \beta f(x) = 0$.
$\iff f(x) = 0$. $\beta \neq 0$.
$\iff x \in \ker f$.
$\therefore \ker g = \ker f$. ■

Proposition. *If X is a normed space and M is a hyperplane in X, then either M is closed or M is dense.*
PROOF. Suppose M is a hyperplane in X.
$\implies \dim(X/M) = 1$.

We know that, $M \subseteq \text{cl}M$.
$\implies \dim(X/\text{cl}M) \leq \dim(X/M) = 1$.
$\implies$ Either $\dim(X/\text{cl}M) = 0$ or $\dim(X/\text{cl}M) = 1$.
If $\dim(X/\text{cl}M) = 0 \implies X = \text{cl}M$.
$\implies M$ is dense in X.
If $\dim(X/\text{cl}M) = 1 = \dim(X/M)$.
$\implies \text{cl}M = M$. $M \subseteq \text{cl}M$.
$\implies M$ is closed. ■

Theorem. *If X is normed space and $f : X \to \mathbb{F}$ is a linear functional, then f is continuous if and only if $\ker f$ is closed.*

PROOF. Let $f : X \to \mathbb{F}$ be a linear functional.
Suppose f is continuous.
We know that, $\ker f = \{x \in X : f(x) = 0\}$.
That is, $\ker f = f^{-1}(\{0\})$.
$\therefore \ker f$ is closed. $\because \{0\}$ is closed in $\mathbb{F}$ and f is continuous.
Conversely, suppose $\ker f$ is closed.
Define $Q : X \to X/\ker f$ by $Q(x) = x + \ker f$.

$$\begin{aligned} |||Q(x)||| &= |||x + \ker f||| \\ &= \inf\{\|x + y\| : x \in \ker f\} \\ &\leq \|x\| \end{aligned}$$

$\therefore Q$ is continuous.
Consider an isomorphism $T : X/\ker f \to \mathbb{F}$, then $T \circ Q : X \to \mathbb{F}$ is continuous.

$$\begin{aligned} \ker T \circ Q &= \{x \in X : (T \circ Q)(x) = 0\} \\ &= \{x \in X : T(Q(x)) = 0\} \\ &= \{x \in X : T(x + \ker f) = 0\} \\ &= \{x \in X : x + \ker f = \ker f\} \\ &= \{x \in X : x \in \ker f\} \\ &= \ker f \end{aligned}$$

Let $T \circ Q = g$.
$\implies \ker g = \ker f$.
$\implies f = \beta g$ for some scalar $\beta \in \mathbb{F}$.
$\implies f$ is continuous. $\because g$ is continuous. ■

THE HAHN-BANACH THEOREM

Definition. If X is a vector space, a sublinear functional is a function $q : X \to \mathbb{R}$ such that
(a) $q(x + y) \leq q(x) + q(y)$ for all $x, y \in X$.
(b) $q(\alpha x) = \alpha q(x)$ for $x \in X$ and $\alpha \geq 0$.

The Hahn-Banach Theorem. *Let X be a vector space over $\mathbb{R}$ and let q be a sublinear functional on X. If M is a linear manifold in X and $f : M \to \mathbb{R}$ is a linear functional such that $f(x) \leq q(x)$ for all $x \in M$, then there is a linear functional $F : X \to \mathbb{R}$ such that $F|M = f$ and $F(x) = f(x)$ for all $x \in M$.*

Lemma. *Let X is vector space over $\mathbb{C}$.*
(a) if $f : X \to \mathbb{R}$ is an $\mathbb{R}-$linear functional, then $\tilde{f}(x) : f(x) - if(ix)$ is a $\mathbb{C}-$linear

functional and $f = \text{Re } \tilde{f}$.
(*b*) *If* $g : X \to \mathbb{C}$ *is* $\mathbb{C}-$*linear,* $f = \text{Re } g$*, and* $\tilde{f}$ *as defined in (a), then* $\tilde{f} = g$.
(*c*) *If* p *is a seminorm on* X *and* f *and* $\tilde{f}$ *are as in (a), then* $|f| \leq p(x)$ *for all* $x \in X$ *if and only if* $|\tilde{f}(x)| \leq p(x)$ *for all* $x \in X$.
(*d*) *If* X *is a normed space and* f *and* $\tilde{f}$ *are as in (a), then* $\|f\| = \|\tilde{f}\|$.
PROOF. If we can show that $\tilde{f}(ix) = i\tilde{f}$ then our proof will be over.

$$\begin{aligned}\tilde{f}(ix) &= f(ix) - if(-x) \\ &= f(ix) + if(x) \\ &= i[f(x) - if(ix)] \\ &= i\tilde{f}\end{aligned}$$

$\Longrightarrow \tilde{f}$ is $\mathbb{C}-$linear.
(b) We know that $\text{Re } g = f = \text{Re } \tilde{f}$.
It is sufficient to show $\text{Im } g = \text{Im } \tilde{f}$.
Consider,

$$\begin{aligned}\text{Im } g(x) &= -\text{Re } ig(x) \\ &= -\text{Re } g(ix) \\ &= -f(ix) \\ &= \text{Im } \tilde{f}(x)\end{aligned}$$

$\therefore \tilde{f} = g$.
(c) Let $|f(x)| \leq p(x) \quad \forall x \in X$.
Choose θ such that, $\tilde{f}(x) = e^{i\theta}|\tilde{f}(x)|$.

$$\begin{aligned}|\tilde{f}(x)| &= e^{-i\theta}\tilde{f}(x) \\ &= \tilde{f}(e^{-i\theta}x) \\ &= \text{Re } \tilde{f}(e^{-i\theta}x) \\ &= f(e^{-i\theta}x) \\ &\leq p(e^{-i\theta}x) \\ &\leq p(x)\end{aligned}$$

$f(x) = \text{Re } \tilde{f}(x) \leq |\tilde{f}(x)| \leq p(x)$.
$-f(x) = f(-x) = \text{Re } \tilde{f}(-x) \leq |\tilde{f}(x)| \leq p(x)$.
$\therefore |f(x)| \leq p(x)$, for all $x \in X$.
(d) Follows from (c). ■

Corollary 1. *Let* X *be vector space, let* M *be a linear manifold in* X*, and let* $p : X \to [0, \infty)$ *be a seminorm. If* $f : M \to \mathbb{F}$ *is a linear functional such that* $|f(x)| \leq p(x)$ *for all* $x \in M$*, then there is a linear functional* $F : X \to \mathbb{F}$ *such that* $F|M = f$ *and* $|F(x)| \leq p(x)$ *for all* $x \in X$.
PROOF. Case 1: If $\mathbb{F} = \mathbb{R}$.
We have $f(x) \leq |f(x)| \leq p(x), \quad \forall x \in M$.
$\Longrightarrow f(x) \leq p(x), \quad \forall x \in M$.
$\therefore$ By Hahn Banach theorem there is linear functional $F : X \to \mathbb{R}$ such that $F|M = f$ and $F(x) \leq p(x)$.
Also, $-F(x) = F(-x) \leq |F(-x)| \leq p(-x) = p(x)$.

$\therefore |F(x)| \leq p(x)$.
Case 2: If $\mathbb{F} = \mathbb{C}$.
Let $f_1 = \text{Re } f$. Then by Case 1 there is an extension $F_1 : X \to \mathcal{R}$ such that $F_1|M = f_1$ and $|F_1(x)| \leq p(x)$.
Let $F = F_1(x) - iF_1(ix), \quad \forall x \in X$.
Also, by part (c) of previous lemma we have get $|F(x)| \leq p(x), \quad \forall x \in X$. ■

Corollary 2. *If X is a normed space, M is a linear manifold in X, and $f : M \to \mathbb{F}$ is bounded linear functional, then there is an F in X^* such that $F|M = f$ and $\|F\| = \|f\|$.*

PROOF. Let $f : M \to \mathbb{F}$ is bounded linear functional.
$\therefore |f(x)| \leq \|f\|\|x\|$ for all $x \in X$.
Take $p(x) = \|f\|\|x\|$.
Clearly, p is seminorm on X.
$\therefore |F(x)| \leq p(x), \quad \forall x \in X$.
Therefore, by corollary 1 there exist a linear functional $F : X \to \mathbb{F}$ such that $F|M = f$ and $|F(x)| \leq p(x), \quad \forall x \in X$.
$\implies |F(x)| \leq \|f\|\|x\|$, for all $x \in X$.
Therefore, by definition of norm of function $\|F\| = \|f\|$. ■

Corollary 3. *If X is normed space, $\{x_1, x_2, ..., x_d\}$ is linearly independent subset of X, and $\alpha_1, \alpha_2, ..., \alpha_d$ are arbitrary scalars, then there is an f in X^* such that $f(x_j) = \alpha_j$ for $1 \leq j \leq d$.*

PROOF. Let $M = \langle x_1, x_2, ..., x_d \rangle$.
That is, M is generated by $\{x_1, x_2, ..., x_d\}$.
Define, a linear functional $g : M \to \mathbb{F}$ such that $g(\sum_{j=1}^{d} \beta_j x_j) = \sum_{j=1}^{d} \beta_j \alpha_j$.
Let $x_1, x_2 \in M \implies x_1 = \sum_{j=1}^{d} \beta_j x_j, x_2 = \sum_{j=1}^{d} \gamma_j x_j$ and $\alpha \in \mathbb{F}$.
Consider,

$$\begin{aligned}
g(x_1 + \alpha x_2) &= g(\sum_{j=1}^{d} \beta_j x_j + \alpha \sum_{j=1}^{d} \gamma_j x_j) \\
&= g(\sum_{j=1}^{d} (\beta_j + \alpha \gamma_j) x_j) \\
&= \sum_{j=1}^{d} (\beta_j + \alpha \gamma_j) \alpha_j \\
&= \sum_{j=1}^{d} \beta_j \alpha_j + \sum_{j=1}^{d} \alpha \gamma_j \alpha_j \\
&= g(\sum_{j=1}^{d} \beta_j x_j) + \alpha g(\sum_{j=1}^{d} \gamma_j x_j) \\
&= g(x_1) + \alpha g(x_2)
\end{aligned}$$

$\therefore g$ is linear functional from finite dimensional vector space M to $\mathbb{F}$.
Therefore, g is continuous.

$\therefore$ by corollary 2, g can be extended to a linear functional $f \in X^*$ such that
$f|M = g$ and $f(x_j) = g(x_j)$.
Therefore, $f(x_j) = g(x_j) = \alpha_j$. ■

Corollary 4. *If X is normed space and $x \in X$, then*

$$\|x\| \;=\; \sup\{|f(x)| : f \in X^* \text{ and } \|f\| \le 1\}.$$

PROOF. Let $\alpha = \sup\{|f(x)| : f \in X^* \text{ and } \|f\| \le 1\}$.
For any $f \in X^*$ and $\|f\| \le 1$.
Therefore, f is bounded linear functional on X.
Hence we can write,

$$\begin{aligned} |f(x)| &\le \|f\|\|x\|, \quad \forall x \in X \\ &\le \|x\|, \qquad \because \|f\| \le 1 \end{aligned}$$

Taking supremum on both side we get,
$\alpha \le \|x\|, \quad \forall x \in X$.
Consider, $M = \{\beta x : \beta \in \mathbb{F}\}$
Clearly M is linear manifold in X.
Define a linear functional on M such that
$g(\beta x) = \beta\|x\|$.
$\|g\| = \sup\{|g(x)| : \|x\| \le 1\} = 1$.
$\therefore g$ is bounded and $\|g\| = 1$.
By corollary 2, there exists a linear functional $f : X \to \mathbb{F}$ such that $f|M = g$ and $\|f\| = \|g\|$.
$\therefore \|f\| = 1$ and $f|M = g \implies g(x) = f(x), \quad \forall x \in M$.
$\implies f(x) = \|x\|, \quad \forall x \in M$.
So we can choose, $f(x) = \|x\|, \quad \forall x \in X$.
$\therefore \alpha = \|x\|$. ■

Corollary 5. *If X is a normed space $M \le X, x_0 \in X - M$ and $d = \text{dist}(x_0, M)$, then there is an f in X^* such that $f(x_0) = 1, f(x) = 0$ for all $x \in M$, and $\|f\| = d^{-1}$.*
PROOF. Suppose $Q : X \to X/M$ defined by $Q(x) = x + M$.
For $x_0 \in X/M, d = \text{dist}(x_0, M) = \|x_0 + m\|$.
$\therefore$ By corollary 4, there exist $g \in (X/M)^*$ such that $g(x_0 + M) = d$ and $\|g\| = 1$.
Consider, $f = d^{-1} g \circ Q : X \to \mathbb{F}$.
$\therefore f$ is continuous linear functional.
Now,

$$\begin{aligned} |f(x)| = |d^{-1} g \circ Q(x)| & \\ &= d^{-1}|g(Q(x))| \\ &\le d^{-1}|||Q(x)||| \\ &\le d^{-1}\|x\| \end{aligned}$$

Therefore, by definition of $\|f\|$, $\|f\| \le d^{-1}$.
$\therefore \|g\| = 1$ there exist a sequence $\{x_n + M\}$ in X/M such that $|g(x_n + M)| \to 1$ for $|||x_n + M||| \le 1$.
Consider, $\{y_n\}$ be sequence in M such that $\|x_n + y_n\| \le 1$.

$$\begin{aligned} |f(x)| &= d^{-1}|g \circ Q(x_n)| \\ &= d^{-1}|g(x_n + M)| \\ &\to d^{-1} \end{aligned}$$

Therefore, $\|f\| = d^{-1}$.
Now,

$$\begin{aligned} f &= d^{-1}g \circ Q \\ \therefore f(x_0) &= d^{-1}g \circ Q(x_0) \\ &= d^{-1}g(x_0 + M) \\ &= d^{-1}d \\ &= 1 \end{aligned}$$

Also, for $x \in M$,

$$\begin{aligned} f &= d^{-1}g \circ Q \\ \therefore f(x) &= d^{-1}g \circ Q(x) \\ &= d^{-1}g(x + M) \\ &= d^{-1}g(0 + M) \\ &= d^{-1}g(M) \\ &= 0 \end{aligned}$$

■

Theorem. *If X is a normed space and M is a linear manifold in X, then*

$$\text{cl}M = \cap \{\ker f : f \in X^* \text{ and } M \subseteq \ker f\}.$$

PROOF. Let $N = \cap \{\ker f : f \in X^* \text{ and } M \subseteq \ker f\}$.
For $f \in X^*, \ker f$ is closed. $\because f$ is continuous linear functional.
So $M \subseteq \ker f$.
$\Longrightarrow \text{cl}(M) \subseteq \text{cl}(\ker f)$.
$\Longrightarrow \text{cl}(M) \subseteq \ker f$.
So if $f \in X^*$ and $M \subseteq \ker f$ then $\text{cl}(M) \subseteq \ker f \Longrightarrow \text{cl}(M) \subseteq N$.
Let $y \notin \text{cl}(M)$.
$\Longrightarrow \text{dist}(y, M) \geq 0$.
Let $d = \text{dist}(y, M)$.
By corollary 5, there exists $f \in X^*$ such that $f(y) = 1$ and $f(x) = 0$.
$\Longrightarrow M \subseteq \ker f$ and $y \notin \ker f, \quad \forall x \in M$.
$\therefore y \notin N \Longrightarrow N \subseteq \text{cl}(M)$.
Therefore, $N = \text{cl}(M)$.
$\Longrightarrow \text{cl}M = \cap \{\ker f : f \in X^* \text{ and } M \subseteq \ker f\}$. ■

Corollary *If X is normed space and M is a linear manifold in X, then M is dense in X if and only if the only bounded linear functional on X that annihilates M is zero function.*
PROOF. Let M is dense in X.
$\Longrightarrow \text{cl}(M) = X$.
$\Longrightarrow f \in X^*$ and $M \subseteq \ker f$ gives as $\ker f = X$.
$\Longrightarrow f = 0$ and $f(M) = 0$.

Only zero function annihilates M.
Conversely, suppose only bounded linear functional that annihilates M is zero function.
That is, $f \in X^*$ and $M \subseteq \ker f$.
$\implies f = 0 \implies \ker f = X$.
$\therefore$ By definition of cl(M).
$\therefore$ cl$(M) = X \implies M$ is dense in X. ∎
THE OPEN MAPPING THEOREM AND CLOSED GRAPH THEOREM
The Open Mapping Theorem: *IF X, Y are Banach spaces and $A : X \to Y$ is a continuous linear surjection, $A(G)$ is open in Y whenever G is open in X.*
PROOF. ∎
The Inverse Mapping Theorem: *If X and Y are Banach spaces and $A : X \to Y$ is a bounded linear transformation that is bijective, then A^{-1} is bounded.*
PROOF. Suppose A is bounded linear transformation.
$\implies A$ is continuous.
$\therefore$ By open mapping theorem A is open mapping.
To show: A^{-1} is bounded.
That is to show: A^{-1} is bounded.
Here $A^{-1} : Y \to X$ is linear map and for U open in X.
$(A^{-1})^{-1}(U) = A(U)$ is open in Y. $\because A$ is open map.
$\implies A^{-1}$ is continuous and hence bounded. ∎
The Closed Graph Theorem: *If X and Y are Banach spaces and $A : X \to Y$ is a linear transformation such that graph of A,*

$$\text{gra}A \;=\; \{x \oplus Ax \in X \oplus_1 Y : x \in X\}.$$

is closed, then A is continuous.
PROOF. Since X and Y are Banach spaces.
$\implies X \oplus_1 Y$ is Banach space.
Also, gra$A$ is closed subset of $X \oplus_1 Y$.
$\therefore$ gra$A$ is Banach space.
Let $G = \text{gra}A$. Define a mapping $P : G \to X$ by $P(x \oplus Ax) = x$.
Claim 1: P is linear mapping.
For any $x_1 \oplus Ax_1, x_2 \oplus Ax_2 \in G$.

$$\begin{aligned} P(x_1 \oplus Ax_1 + x_2 \oplus Ax_2) &= P((x_1 + x_2) \oplus A(x_1 + x_2)) \\ &= x_1 + x_2 \\ &= P(x_1 \oplus Ax_1) + P(x_2 \oplus Ax_2) \end{aligned}$$

Also, For $x \oplus Ax \in G$ and any scalar $\alpha \in \mathbb{F}$.

$$\begin{aligned} P(\alpha(x \oplus Ax)) &= P(\alpha x \oplus A\alpha x) \\ &= \alpha x \\ &= \alpha P(x \oplus Ax) \end{aligned}$$

Therefore, P is linear.
Claim 2: P is bijective.

$$\begin{aligned}\ker P &= \{x \oplus Ax \in G : P(x \oplus Ax) = 0\}\\ &= \{x \oplus Ax \in G : x = 0\}\\ &= \{0 \oplus A(0) \in G\}\\ &= \{0\}\end{aligned}$$

$\therefore P$ is injective.
Clearly, for any $x \in X, \exists x \oplus Ax \in G$ such that $P(x \oplus Ax) = x$.
$\therefore P$ is surjective.
$\implies P$ is bijective.
Claim 3: P is continuous.

$$\begin{aligned}\|P(x \oplus Ax)\| &= \|x\|\\ &\leq \|x\| + \|Ax\| \qquad \because \|x_1 \oplus x_2\| = \|x_1\| + \|x_2\|.\\ &\leq \|x \oplus Ax\|\end{aligned}$$

$\therefore \|P(x \oplus Ax)\| \leq \|x \oplus Ax\| \implies P$ is continuous.
$\therefore$ By claim 1, 2, 3, P is linear, bijective and continuous(bounded) function.
So $P : G \to X$ is a bounded linear and bijective map and G and X are Banach spaces.
By Inverse mapping theorem,
$P^{-1} : X \to G$ is bounded.
Define a mapping $T : G \to Y$ by $T(x \oplus Ax) = Ax$.
Similarly, we can show T is linear, bijective and continuous(bounded).
Now, $T \circ P^{-1} : X \to Y$ is linear and continuous mapping which is same as given mapping A.
$\therefore A$ is continuous. ■

Principle of Uniform Boundedness: *Let X be a Banach space and Y a normed space. If $\mathcal{A} \subseteq \mathbb{B}(X, Y)$ such that for each x in X, $\sup\{\|Ax\| : A \in \mathcal{A}\} < \infty$, then* $\sup\{\|A\| : A \in \mathcal{A}\} < \infty$.
PROOF. Let $M(x) = \sup\{\|Ax\| : A \in \mathcal{A}\} < \infty$.
$\implies \|Ax\| \leq M(x), \quad \forall x \in X$.
If possible, Let $\sup\{\|A\| : A \in \mathcal{A}\}$ is infinite.
Then, there exists a sequence $\{A_n\} \subseteq \mathcal{A}$ such that $\|A_n\| \to \infty$ as $n \to \infty$.
That is, $\exists \{x_n\} \subseteq X$ such that $\|x_n\| = 1$ and $\|Ax_n\| > 4^n$.
Let $y_n = 2^{-n}x_n$.
$\therefore \|y_n\| = \|2^{-n}x_n\| = 2^{-n}\|x_n\| = 2^{-n}$.
$\|A_n y_n\| = \|A_n 2^{-n} x_n\| = 2^{-n}\|A_n x_n\| > 2^{-n}4^n = 2^n$.
$\therefore \|A_n y_n\| > 2^n$.
Claim: There exists subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that
(a) $\|A_{n_{k+1}} y_{n_{k+1}}\| > 1 + k + \displaystyle\sum_{j=1}^{k} M(y_{n_j})$.
(b) $\|y_{n_{k+1}}\| < 2^{-k-1} \sup\left\{\|A_{n_j}\| : 1 \leq j \leq k\right\}$.
Consider, $\displaystyle\sum_k \|y_{n_{k+1}}\| = \sum_k 2^{-n_k} < \infty$.
$\implies \displaystyle\sum_k y_{n_{k+1}}$ converges. $\because X$ is Banach space.

Let $y = \sum_k y_{n_k}$.
Consider,

$$\begin{aligned}
\|A_{n_{k+1}} y\| &= \|A_{n_{k+1}} \sum_k y_{n_k}\| \\
&= \|\sum_{j=1}^{k} A_{n_{k+1}} y_{n_j} + A_{n_{k+1}} y_{n_{k+1}} + \sum_{j=k+2}^{\infty} A_{n_{k+1}} y_{n_j}\| \\
&= \|A_{n_{k+1}} y_{n_{k+1}} - (-\sum_{j=1}^{k} A_{n_{k+1}} y_{n_j} - \sum_{j=k+2}^{\infty} A_{n_{k+1}} y_{n_j})\| \\
&\geq \|A_{n_{k+1}} y_{n_{k+1}}\| - \|\sum_{j=1}^{k} A_{n_{k+1}} y_{n_j} + \sum_{j=k+2}^{\infty} A_{n_{k+1}} y_{n_j}\| \\
&\geq \|A_{n_{k+1}} y_{n_{k+1}}\| - \|\sum_{j=1}^{k} A_{n_{k+1}} y_{n_j}\| - \|\sum_{j=k+2}^{\infty} A_{n_{k+1}} y_{n_j}\| \\
&\geq \|A_{n_{k+1}} y_{n_{k+1}}\| - \sum_{j=1}^{k} \|A_{n_{k+1}} y_{n_j}\| - \sum_{j=k+2}^{\infty} \|A_{n_{k+1}} y_{n_j}\| \\
&\geq \|A_{n_{k+1}} y_{n_{k+1}}\| - \sum_{j=1}^{k} \|A_{n_{k+1}} y_{n_j}\| - \sum_{j=k+2}^{\infty} \|A_{n_{k+1}}\| \|y_{n_j}\| \qquad (1)
\end{aligned}$$

Claim 1: $-\sum_{j=1}^{k} \|A_{n_{k+1}} y_{n_j}\| \geq -\sum_{j=1}^{k} M(y_{n_j})$.
We know, $M(x) = \sup\{\|Ax\| : A \in \mathcal{A}\}$.
$\therefore \|A_{n_{k+1}} y_{n_j}\| \leq M(y_{n_j})$.
$-M(y_{n_j}) \leq -\|A_{n_{k+1}} y_{n_j}\|$.
$\implies -\sum_{j=1}^{k} M(y_{n_j}) \leq -\sum_{j=1}^{k} \|A_{n_{k+1}} y_{n_j}\|$.
Hence claim 1.
Claim 2: $-\sum_{j=k+2}^{\infty} \|A_{n_{k+1}}\| \|y_{n_j}\| \geq -\sum_{j=k+2}^{\infty} 2^{-j}$.
From the part (b) we have,
$\|y_{n_{k+1}}\| < 2^{-k-1} [\sup\{\|A_{n_j}\| : 1 \leq j \leq k\}]^{-1}$.
Replace k by $k+1$.
$\|y_{n_{k+2}}\| < 2^{-k-2} [\sup\{\|A_{n_j}\| : 1 \leq j \leq k+1\}]^{-1}$.
In particular, for $j = k+1$.
$\|y_{n_{k+2}}\| < 2^{-k-2} \|A_{n_{k+1}}\|^{-1}$.
$\implies -\|y_{n_{k+2}}\| > -2^{-k-2} \|A_{n_{k+1}}\|^{-1}$.
$\implies -\|A_{n_{k+1}}\| \|y_{n_{k+2}}\| > -2^{-k-2}$.
Similarly, Replace k by $k+2$.
$\|y_{n_{k+3}}\| < 2^{-k-3} [\sup\{\|A_{n_j}\| : 1 \leq j \leq k+1\}]^{-1}$.
$\implies -\|A_{n_{k+1}}\| \|y_{n_{k+3}}\| > -2^{-k-3}$.

Similarly, $\implies -\|A_{n_{k+1}}\|\|y_{n_{k+4}}\| > -2^{-k-4}$.
$\therefore -\|A_{n_{k+1}}\|\|y_{n_j}\| > -2^{-j}, \quad \forall j = k+2, k+3, ...$
$\therefore -\sum_{j=k+2}^{\infty} \|A_{n_{k+1}}\|\|y_{n_j}\| > -\sum_{j=k+2}^{\infty} 2^{-j}$.
Hence claim 2.
From inequality (1)

$$\begin{aligned}
\|A_{n_{k+1}}y\| &\geq \|A_{n_{k+1}}y_{n_{k+1}}\| - \sum_{j=1}^{k} \|A_{n_{k+1}}y_{n_j}\| - \sum_{j=k+2}^{\infty} \|A_{n_{k+1}}\|\|y_{n_j}\| \\
&> 1 + k + \sum_{j=1}^{k} M(y_{n_j}) - \sum_{j=1}^{k} M(y_{n_j}) - \sum_{j=k+2}^{\infty} 2^{-j} \\
&> 1 + k - \sum_{j=k+2}^{\infty} 2^{-j} \\
&> 1 + k - \tfrac{1}{2^{k+1}} \\
&> k + (1 - \tfrac{1}{2^{k+1}}) \\
&> k
\end{aligned}$$

$\therefore \|A_{n_{k+1}}y\| > k, \quad \forall k.$ (2)
$\rightarrow\leftarrow$ Because, $M(y) = \sup\{\|Ay\| : A \in \mathcal{A}\} < \infty$.
From inequality (2) as $k \to \infty$.
$\|A_{n_{k+1}}y\| \to \infty$.
But by our given condition,
$\|A_{n_{k+1}}y\| < \infty, \quad \forall k$.
$\therefore$ Our assumption was wrong.
Therefore we must have $\{\|A\| : A \in \mathcal{A}\} < \infty$. ■

♣♣♣