ENUMERATION OF CERTAIN ALGEBRAIC SYSTEMS AND RELATED RESULTS

A Thesis submitted to the
University of Pune
for the degree of
Doctor of Philosophy
in Mathematics

By

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Under the guidance of

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Certificate

Certifed that the work incorporated in the Thesis entitled "**Enumeration of certain algebraic systems and related results**" that is being submitted by **Mr. Ashok. N. Bhavale** has been carried out under my supervision and guidance. The material in the Thesis is his original work and material that has been obtained from the other sources has been duly acknowledged in the Thesis.

Date: December, 2013

Place: Pune

Research Guide: ______________________

Dr. B. N. Waphare

Declaration

I hereby declare that the work incorporated in the Thesis entitled "**Enumeration of certain algebraic systems and related results**" that is being submitted for the degree of Doctor of Philosophy in Mathematics to the University of Pune, is original and has not been previously submitted for any other degree of any University in India or Abroad.

Date: December, 2013

Place: Pune

Research Student: ______________________

Ashok. N. Bhavale

To my parents

Shri. Nivrutti Vithoba Bhavale

and

Smt. Bababai Nivrutti Bhavale

"They also serve who stand and wait . . ."

— John Milton.

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Preface

The theory of ordered sets is today a burgeoning branch of mathematics. It both draws upon and applies to several other branches of mathematics, including algebra, set theory, and combinatorics. The theory itself boasts an impressive body of fundamental and deep results as well as a variety of challenging problems, some of traditional heritage and some of fairly recent origin (see [16] to [19], [21] to [24], [27] to [31], [54], [55] and [58]).

Ordered sets have their roots in two trends of nineteenth century mathematics. On the one hand, ordered sets have entered into the study of those algebraic systems which originally arose from axiomatic schemes aimed at formalizing the "laws of thought"; Boole, Peirce, Schröder, and Huntington were among the earliest leaders of this trend. On the other hand, ordered sets were essential ingredients to the theory of sets, from its inception. It is not surprising that these two trends have influenced the subject in different ways.

The ordered sets of most interest to general algebra are lattices. It is lattice theory, however, that has stimulated the study of ordered sets as

abstract systems. The theory of lattices is bracketed under Universal Algebra, one of the major branches of Algebra.
Orders are everywhere in mathematics and related fields like computer science. Partial order and lattice theory have applications in distributed computing, programming language semantics and data mining.

Much of the combinatorial interest in ordered sets is inextricably linked to the combinatorial features of the diagrams associated with them. O. Ore[20] raised an open problem, namely, "Characterize those graphs which are orientable". It is also well known that a graph G is the comparability graph of an ordered set if and only if each odd cycle of G has a triangular chord (see [51] and [52]). In contrast little is known about this question (see [20]) : when is a graph the covering graph of an ordered set? Also, it is NP-complete to test whether a graph is a cover graph (see [57] and [60]). See also [12], [25], [39] to [43], [46], [47], [53] and [56] for the work done in this field.

Before 1940, G. Birkhoff[2] posed the following open problems.
(1) Compute for small n all non-isomorphic lattices/posets on a set of n elements.
(2) Find asymptotic estimates and bounds for the rate of growth of the number of non-isomorphic lattices/posets with n elements.
(3) Enumerate all finite lattices/posets which are uniquely determined (up to isomorphism) by their diagrams, considered purely as graphs.
It is known that these problems are NP-complete. Recently, Brinkmann

and McKay[14] obtained the number of non-isomorphic posets and lattices with at most 18 elements. The work of enumerating all non-isomorphic posets is still in progress. Thakare, Pawar and Waphare[13] enumerated the non-isomorphic lattices containing n elements and up to $n+1$ edges. See also [3], [5] to [9], [15], [32] to [36] for the work done in this field.

The work included in the Thesis is a contribution towards partial solutions to the above mentioned open problems. We will restrict ourselves to finite discrete structures such as posets, lattices and graphs.

The Thesis contains five chapters along with an appendix.

In the first chapter, we state the basic concepts, definitions and notations related to discrete structures such as posets, lattices and graphs. We deal with the origin and recent developments regarding the above mentioned open problems posed by G. Birkhoff. We also discuss the origin and recent developments in the theory of dismantlable lattices.

In the second chapter, we introduce and study posets dismantlable by doubly irreducibles. We obtain the structure theorem for posets dismantlable by doubly irreducibles. The motivation behind this study is due to Kelly and Rival[4], I. Rival[27] and Larose and Zadori[26].

We introduce the concept of the nullity of a poset/lattice and obtain some properties of nullity of lattices.

We introduce the concept of adjunct of ears and characterize the graphs which are orientable as posets dismantlable by doubly irreducibles.

We also prove, Whitney [44] type characterization of graphs, namely, "a finite loopless graph is connected if and only if it has an ear decomposition starting with a maximal path or a cycle". See also [45] and [48] to [50] in this regard.

In the third chapter, we introduce and study the concept of a basic block associated to a poset and the concept of a fundamental basic block associated to a dismantlable lattice. Using these concepts we enumerate certain classes of non-isomorphic lattices on n elements in the subsequent chapters.

In the fourth chapter, we obtain the recursive formulae for obtaining the number of fundamental basic blocks. We also enumerate the class of all non-isomorphic lattices on n elements in which the reducible elements are all comparable.

In the fifth chapter, we count the number of all non-isomorphic lattices of nullity up to three.

At the end, we provide an appendix in which we depict all the non-isomorphic basic blocks of nullity three.

All definitions, lemmas, theorems etc. are serially numbered sectionwise in each chapter. The figures are serially numbered. The Thesis ends with the sufficient number of relevant references (bibliography).

Acknowledgements

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I sincerely thank the entire non-teaching staff of the Department of Mathematics, University of Pune, Pune.

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I thank my parents and my brothers without whose support this work would have not been completed.

I take this opportunity to render affectionate gratitude to my wife Nalini and son Yash whose patience was tested by my erratic working hours.

Pune

December, 2013. Mr. A. N. Bhavale

Chapter 1

Preliminaries

In this chapter, we provide some basic definitions, concepts and notations which are used in the Thesis.

1.1 Basic concepts

We begin with the definition of a poset.

Definition 1.1.1. Let P be a nonempty set. If a binary relation " $\leq$ " is reflexive, anti-symmetric and transitive on P then $\leq$ is called *a partial order relation* on P.
The structure $(P, \leq)$ is called a *partially ordered set* or a *poset*.

Definition 1.1.2. Let $(P, \leq)$ be a finite poset. An element b in P *covers* an element a (or a is *covered by* b) in P if $a < b$ and there is no element c in P such that $a < c < b$.
If b covers a then it is denoted by $a \prec b$.

If $a \prec b$ then we say that $< a, b >$ is a *covering* or an *edge*; see Thakare, Pawar and Waphare [13].

The set of coverings in P is denoted by $E(P)$.

The *diagram or Hasse diagram* of a poset P represents the elements with small circles; the circles representing two elements x, y are connected by a straight line if and only if one covers the other; if x covers y, then the circle representing x is higher than the circle representing y. In a diagram the intersection of two lines does not indicate an element. Hasse diagrams are named after **Helmut Hasse** $(1898 - 1979)$.

A diagram is *planar* if no two lines intersect. A diagram which is not planar is called *non-planar*.

An *indegree* of an element x in a poset P is $|\{y \in P : y \prec x\}|$. Similarly, an *outdegree* of an element x in a poset P is $|\{z \in P : x \prec z\}|$. The sum of indegree and outdegree of an element $x \in P$ is the *degree* of x in P. A chain $x_1 < x_2 < \cdots < x_n$ in P is said to be *saturated* if $x_i \prec x_{i+1}$ for each i. The number of coverings in a chain is called *length of the chain*. A chain C in P is called *maximal* if there is no other chain in P which contains C. For $a < b$, the interval $[a, b]$ is the set $[a, b] = \{x \in P : a \leq x \leq b\}$ and $[a, b) = \{x \in P : a \leq x < b\}$; similarly (a, b) and $(a, b]$ can also be defined. The *width* of a poset P is a natural number n if there is an antichain in P containing n elements and all antichains in P have $\leq n$ elements.

An element x in a lattice L is *join-reducible (meet-reducible)* in L if there exist $y, z \in L$ both distinct from x, such that $y \vee z = x$ $(y \wedge z = x)$;

x is *join-irreducible (meet-irreducible)* if it is not join-reducible (meet-reducible); x is *doubly irreducible* if it is join-irreducible and meet-irreducible. Therefore, an element x is doubly irreducible in a lattice L if and only if x has at most one lower cover and x has at most one upper cover. The set of all meet-irreducible (join-irreducible) elements in L is denoted by $M(L)$ $(J(L))$. The set of all doubly irreducible elements in L is denoted by $Irr(L)$ and its complement in L is denoted by $Red(L)$. Thus, if $x \in Red(L)$ then x is either join reducible or meet reducible. A *subposet* Q of a poset P is a subset Q of P together with the restriction of the order relation on P to Q.

Definition 1.1.3. Let P and Q be posets. A map $\varphi : P \to Q$ is said to be (i) *order-preserving* if $x \leq y$ in P implies $\varphi(x) \leq \varphi(y)$ in Q; (ii) an *order-embedding* if $x \leq y$ in P if and only if $\varphi(x) \leq \varphi(y)$ in Q; (iii) an *order-isomorphism* if it is an order-embedding mapping P onto Q. When there exists an order isomorphism from P to Q, we say that P and Q are order-isomorphic and write $P \cong Q$. If two posets are not order-isomorphic then we say that they are *non-isomorphic*.

***Remark* 1.1.1.** (a) If $\varphi : P \to Q$ is an order-embedding then $\varphi(P) \cong P$. (b) An order-embedding is automatically a one-to-one map. Therefore an order-isomorphism is bijective.

Definition 1.1.4. An order-preserving map $g : P \to Q$ is a *retraction* of poset P onto subposet Q provided that $g(x) = x$ for all $x \in Q$. If there is a retraction of P onto Q, then Q is a *retract* of P.

Now we will see some definitions and terminologies of graph theory; see [41] for more details.

Definition 1.1.5. A *graph* G is a triple consisting of a *vertex set* $V(G)$, an *edge set* $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its *endpoints*.

A *loop* is an edge whose endpoints are equal. *Multiple edges* are edges having the same pair of endpoints. A vertex of a graph which is not an end of any edge is called *isolated*. A *simple graph* is a graph having no loops or multiple edges. Two vertices u and v are *adjacent* if they are endpoints of an edge e. We write $e = uv$ if u and v are endpoints of an edge e. If vertex v is an endpoint of edge e, then v and e are *incident*. The *degree* of a vertex v in a (loopless) graph G, written $d_G(v)$ or $d(v)$, is the number of edges incident to v. A *leaf* (or a *pendant vertex*) is a vertex of degree 1. A *path* is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. A graph with no cycle is called *acyclic*.

Definition 1.1.6. A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. We then write $H \subseteq G$ and say that "*G contains H*".

Definition 1.1.7. A graph G is *connected* if each pair of vertices in G belongs to a path; otherwise, G is *disconnected*.

The *components* of a graph G are its maximal connected subgraphs.

An *induced subgraph* is a subgraph obtained by deleting a set of vertices. The *nullity* (or *cyclomatic number* or *circuit rank* or the *first betti number*) of a graph G is given by $m - n + c$, where m is the number of edges in G, n is the number of vertices in G and c is the number of connected components of G. Note that, the nullity of a subgraph of a graph is less than or equal to the nullity of the graph.

Definition 1.1.8. Let G be a loopless connected graph. An *ear of a graph* G is an induced subgraph of G such that it is a maximal path in which all internal vertices are of degree 2 in G or it is a cycle in which all but one vertex have degree 2 in G. If G is a cycle (or path) itself then that cycle (or path) is the only ear of G. An ear of a graph G is called an *open ear* if the two endpoints do not coincide in G.
An ear which does not contain any internal vertex is called a *trivial ear*. Therefore a trivial ear is just an edge in G. An ear which is not an edge is called *non-trivial ear* in G. An ear $E : a - x_1 - x_2 - \cdots - x_r - b$ is said to be an *ear associated to the pair* (a, b) of length $r + 1$. Also for each i, we say x_i is associated to the pair (a, b).

Hereafter by a path (or an ear) in a poset/lattice, we mean the path (or an ear) in the cover graph of that poset/lattice. As a simple observation, we have the following.

Proposition 1.1.1. *If an ear is a trivial ear in a poset P associated to a pair (a, b) then it is the only ear associated to (a, b) in P.*

A *cut-vertex* of a graph is a vertex whose deletion increases the number of components. We write $G - v$ or $G - S$ for the subgraph obtained by deleting a vertex v or set of vertices S respectively. A graph is said to be *k-connected* (or *k-vertex connected*) if there does not exist a set of $k-1$ vertices whose removal disconnects the graph.

Definition 1.1.9. A *tree* is a connected acyclic graph.

Note that, a tree on n vertices has $n-1$ edges. Also, a connected graph containing n vertices and $n-1$ edges is a tree. It is clear that the nullity of a tree is zero.

Definition 1.1.10. A *block of a graph* G is a maximal connected subgraph of G that has no cut-vertex. If G itself is connected and has no cut-vertex then G is a block.

***Remark* 1.1.2.** 1. An edge is a block if and only if it is a cut edge.
2. If a block has more than two vertices then it is 2-connected.
3. The blocks of a loopless graph are its isolated vertices, its cut edges and its maximal 2-connected subgraphs.
4. Two blocks in a graph share at most one vertex.

Definition 1.1.11. The *block cutpoint graph* of a graph G is a bipartite graph H in which one partite set consists of the cut-vertices of G and the other has a vertex b_i, for each block B_i of G and $\{v, b_i\}$ is an edge of H if and only if $v \in B_i$. *When G is connected, its block cutpoint graph is a tree whose leaves are blocks of G.*

Note that, a graph G that is not a single block has at least two blocks (called *leaf blocks* or *pendant blocks*) that each contain exactly one cut-vertex of G. Blocks of a graph can be found using a technique for searching graphs, viz., Depth First Search or Breadth First Search algorithms.

Definition 1.1.12. An *isomorphism* from a simple graph G to a simple graph H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. We say "G is *isomorphic to* H", written as $G \cong H$, if there is an isomorphism from G to H.

Definition 1.1.13. The *covering relation* of a partially ordered set P is the binary relation which holds between comparable elements that are immediate neighbours. The graph on P with edges as covering relations is called *cover graph*, denoted by $C(P)$.

Definition 1.1.14. For even $n \geq 4$, a subset $C = \{c_1, c_2, \ldots, c_n\}$ of P is a *crown* provided that $c_1 < c_n$, and $c_1 < c_2, c_2 > c_3, \ldots, c_{n-2} > c_{n-1}, c_{n-1} < c_n$ are the only (strict) comparability relations that hold in C and, in the case $n = 4$, there is no $a \in P$ such that $c_1 < a < c_2$ and $c_3 < a < c_4$.

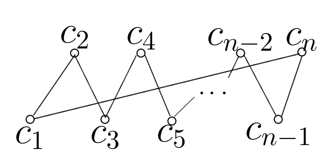


Fig.1

We recall the concept of linear sum of posets; see Stanley[11]. If P and Q are two disjoint posets, the *linear sum* (also known as *ordinal sum or direct sum*) $P \oplus Q$ is defined by taking the following order relation on $P \cup Q$: $x \leq y$ if and only if $x, y \in P$ and $x \leq y$ in P, or $x, y \in Q$ and

$x \leq y$ in Q or $x \in P$, $y \in Q$. If P and Q are finite posets, then a Hasse-diagram of $P \oplus Q$ is obtained by placing a diagram of P directly below a diagram of Q and then adding a line segment from each maximal element of P to each minimal element of Q. Further, if P and Q are lattices then $|E(P \oplus Q)| = |E(P)| + |E(Q)| + 1$.

1.2 Background and motivation

In this Thesis, we shall be concerned with the long standing open problem of enumerating some classes of lattices in the sense that given n, a positive integer, how many non-isomorphic lattices are possible with n vertices. It stems from the "Birkhoff's Open Problems" which are repeated in variant forms by several authors such as Stanley[11], Quackenbush[9] and others.

1.2.1 Birkhoff's open problems

1. Compute for small n all non-isomorphic posets/lattices on a set of n elements.

2. Find asymptotic estimates and bounds for the rate of growth of the number of non-isomorphic posets/lattices with n elements.

3. Enumerate all finite posets/lattices which are uniquely determined (up to isomorphism) by their diagrams, considered purely as graphs.

It is known that these problems are NP-complete. There were attempts to solve these problems by many authors. Today the number of all non-isomorphic posets on up to 16 elements is known. Chaunier and Legeros[3] (Order, 1992) enumerated all non-isomorphic posets with 13 elements. Lygeros and Zimmermann[7] enumerated all non-isomorphic posets with 14 elements and Brinkmann and Mckay[14] (Order, 2002) enumerated all non-isomorphic posets with 15 and 16 elements. The work of enumeration of all non-isomorphic (unlabelled) posets is still in progress for $n \geq 17$.

Nonetheless, we shall allude to the work of Kyuno[6] who gave an algorithm for finite lattices wherein he could obtain lattices of order ≤ 9. Independently, Kolhe[5] in his M.Phil. dissertation uses a rather ingenious algorithm so as to obtain the total number of all non-isomorphic lattices with 8 and 9 elements. According to Kolhe[5], the number of all non-isomorphic lattices with 9 elements is 1082, which however does not match with the number 1078 given in [15] for the same.

The following Theorem 1.2.1 gives the bounds for $L(n)$, the number of non-isomorphic lattices on $n + 2$ (labelled) elements.

Theorem 1.2.1. *If $L(n)$ is the number of non-isomorphic lattices on $n + 2$ (labelled) elements then*

$$\alpha^{n^{1.5}+O(n^{1.5})} < L(n) < \beta^{n^{1.5}+O(n^{1.5})},$$

where $\alpha = 2^{\sqrt{2}/4} \approx 1.2777$ and $\beta \approx 6.11343$.

In Theorem 1.2.1, the lower bound is due to W. Klotz and L. Lucht[35] and the upper bound is due to D. Kleitman and K. Winston[36].

1.2.2 Recent developments

The number of non-isomorphic (unlabelled) lattices on $n = 1$ to 18 elements are respectively $1, 1, 1, 2, 5, 15, 53, 222, 1078, 5994, 37622, 262776, 2018305, 16873364, 152233518, 1471613387, 15150569446, 165269824761$ (see Heitzig and Reinhold[15]). The number of distinct (labelled) posets (see Table 2) and distinct (labelled) lattices (see [15]) on $n \leq 18$ elements is also known.

***Remark* 1.2.1.** The number $P(n)$ of all non-isomorphic *unlabelled* posets (equivalently, T_0 *topologies*) with n elements for $n \leq 16$ is given as follows (see Table 1). The $P(n)$ values for $n = 0, 1, 2, 3, 4, 5, 6$ are respectively $1, 1, 2, 5, 16, 63, 318$ given by I. Rose and R. T. Sasaki, before 1940. (See page 4 of [2] and [14]).

n	**P(n)**	**Year**	**Researcher/s**
7	2,045	1972	J. Write
8	16,999	1977	S. K. Das
9	183,231	1984	R. H. Mohring
10	2,567,284	1990	J. C. Culberson and G. J. E. Rawlins
11	46,749,427	1990	J. C. Culberson and G. J. E. Rawlins
12	1,104,891,746	1991	C. Chaunier and N. Lygeros
13	33,823,827,452	1992	C. Chaunier and N. Lygeros
14	1,338,193,159,771	2000	N. Lygeros and P. Zimmermann
15	68,275,077,901,156	2002	G. Brinkmann and B. D. McKay
16	4,483,130,665,195,087	2002	G. Brinkmann and B. D. McKay

Table 1

***Remark* 1.2.2.** The number of all non-isomorphic *labelled* posets (equivalently, T_0 *topologies*) with n elements for $n \leq 18$ is given in Table 2 (see [14]). This number is also the number of *different partial order relations* on a set containing n elements.

n	**Labelled posets with n elements**
1	1
2	3
3	19
4	219
5	4231
6	130023
7	6129859
8	431723379
9	44511042511
10	6611065248783
11	1396281677105899
12	414864951055853499
13	171850728381587059351
14	98484324257128207032183
15	77567171020440688353049939
16	83480529785490157813844256579
17	122152541250295532286294128l269151
18	241939392597201176602897820148085023

Table 2

1.3 Dismantlable lattices

Definition 1.3.1. A finite lattice L of order n is called *dismantlable* if there exists a chain $L_1 \subset L_2 \subset \cdots \subset L_n (= L)$ of sublattices of L such that $|L_i| = i$, for all i.

Dismantlable lattices are introduced by Rival [10].
The following results can be found in Rival [10], Kelly and Rival [4].

Proposition 1.3.1. *[10]. Let L be a lattice, $A \subseteq Irr(L)$ then $L - A$ is a sublattice of L.*

Proposition 1.3.2. *[10]. If L is a dismantlable lattice then for any sublattice $S \subseteq L$, $Irr(S) \neq \phi$.*

Proposition 1.3.3. *[10]. A sublattice of a dismantlable lattice is dismantlable.*

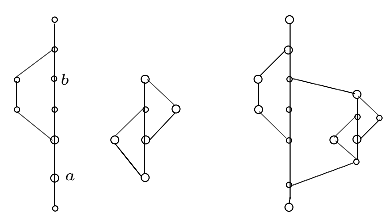
Proposition 1.3.4. *[4]. A finite dismantlable lattice which is not a chain, contains at least two incomparable doubly-irreducible elements.*

Theorem 1.3.5. *[4]. A finite lattice is dismantlable lattice if and only if it contains no crown.*

The concept of *adjunct operation of lattices* was firstly introduced by Thakare, Pawar and Waphare[13] to achieve a constructive characterization of dismantlable lattices. If L_1 and L_2 are two disjoint lattices and (a, b) are a pair of elements in L_1 such that $a < b$ and $a \nprec b$, define the partial order $\leq$ on $L = L_1 \cup L_2$ with respect to the pair (a, b) as follows: $x \leq y$ in L if $x, y \in L_1$ and $x \leq y$ in L_1, or $x, y \in L_2$ and

$x \leq y$ in L_2, or $x \in L_1$, $y \in L_2$ and $x \leq a$ in L_1, or $x \in L_2$, $y \in L_1$ and $b \leq y$ in L_1.

It is easy to see that L is a lattice containing L_1 and L_2 as sublattices. The procedure for obtaining L in this way is called *adjunct operation (or adjunct sum) of* L_1 *with* L_2. The pair (a, b) is called as *an adjunct pair* and L as *adjunct* of L_1 with L_2 with respect to the adjunct pair (a, b) and write $L = L_1]_a^b L_2$. A diagram of L is obtained by placing a diagram of L_1 and a diagram of L_2 side by side in such a way that the largest element 1 of L_2 is at the lower position than b and the least element 0 of L_2 is at the higher position than a and then by adding the coverings $< 1, b >$ and $< a, 0 >$. This clearly gives $|E(L)| = |E(L_1)| + |E(L_2)| + 2$. This also implies that the adjunct operation preserves all the covering relations of the individual lattices L_1 and L_2.



L_1 $\quad$ L_2 $\quad$ $L_1 \,]_a^b \, L_2$

Fig.2

A lattice L is called *adjunct of lattices* $L_1, L_2, \ldots, L_k$, if it is of the form $L = (\ldots((L_1]_{a_1}^{b_1} L_2)]_{a_2}^{b_2} L_3)]_{a_3}^{b_3} \ldots)]_{a_{k-1}}^{b_{k-1}} L_k$. Hereafter, we write this representation as $L = L_1]_{a_1}^{b_1} L_2]_{a_2}^{b_2} L_3]_{a_3}^{b_3} \ldots]_{a_{k-1}}^{b_{k-1}} L_k$ or $L = L_1]_{\alpha_1} L_2]_{\alpha_2} L_3]_{\alpha_3} \ldots]_{\alpha_{k-1}} L_k$, where $\alpha_i = (a_i, b_i)$, $\forall\, i,\ 1 \leq i \leq k-1$.

Note that, if L is adjunct of k chains then L contains $k-1$ adjunct pairs (including repetition, if any).

Following is the characterization obtained by Thakare, Pawar and

Waphare[13].

Theorem 1.3.6. *[13]. A finite lattice is dismantlable if and only if it is an adjunct of chains.*

The above characterization is similar to a structure theorem for planar lattices; see. Baker, Fishburn and Roberts[1]. Note that, a representation of a dismantlable lattice as an adjunct of chains is not unique. However, the number of chains in any adjunct representation of a dismantlable lattice remains the same. More explicitly,

Lemma 1.3.7. *[13]. If L is a dismantlable lattice then the number of chains in every adjunct representation of L is the same.*

Corollary 1.3.8. *[13]. A dismantlable lattice with n elements has $n+r-2$ coverings if and only if it is an adjunct of r chains.*

Corollary 1.3.9. *[13]. If L is a dismantlable lattice with n elements $(n \geq 3)$ then $n-1 \leq |E(L)| \leq 2n-4$.*

Lemma 1.3.10. *[13]. Let L be a dismantlable lattice with an adjunct representation $L = C_1]_{a_1}^{b_1} C_2]_{a_2}^{b_2} C_3]_{a_3}^{b_3} \dots]_{a_{k-1}}^{b_{k-1}} C_k$. Then*
(i) $M(L) = L - \{a_1, a_2, \dots, a_{k-1}\}$ and
(ii) $J(L) = L - \{b_1, b_2, \dots, b_{k-1}\}$.

Interestingly, in each adjunct representation of a lattice L, an adjunct pair (a, b) occurs in the same number of times.

Theorem 1.3.11. *[13]. An adjunct pair (a, b) occurs r times in an adjunct representation of a dismantlable lattice L if and only if there exist exactly $r+1$ maximal chains $C_0, C_1, C_2, \dots, C_r$ in $[a, b]$ such that*

$x \wedge y = a$ and $x \vee y = b$ for any $x \in C_i - \{a, b\}, y \in C_j - \{a, b\}$ and $i \neq j$.

Corollary 1.3.12. *[13]. Let L be a dismantlable lattice and*

$$L = C_1]_{a_1}^{b_1} C_2]_{a_2}^{b_2} \ldots]_{a_{k-1}}^{b_{k-1}} C_k = C_1']_{p_1}^{q_1} C_2']_{p_2}^{q_2} \ldots]_{p_{k-1}}^{q_{k-1}} C_k'$$

be any two adjunct representations of L. Then there is a permutation π of $1, 2, \ldots, k-1$ such that $(a_i, b_i) = (p_{\pi(i)}, q_{\pi(i)})$, for all i.

Theorem 1.3.13. *[13]. If L is a dismantlable lattice with n elements and $n + k$ coverings then $n - 2k - 2 \leq |Irr(L)| \leq n - 2$.*

Lemma 1.3.14. *[13]. Every lattice with n elements and $n+r$ coverings with $-1 \leq r \leq 3$ is dismantlable.*

The concept of a *block*(of a lattice) is introduced by Thakare, Pawar and Waphare[13]. Let L be a finite lattice which is not a chain. Then L contains a unique maximal sublattice which is a block, denoted by B. The lattice L has the form $C_1 \oplus B$ or $B \oplus C_2$ or $L = C_1 \oplus B \oplus C_2$, where C_1, C_2 are chains, hence $|E(L)| - |L| = |E(B)| - |B|$. Thus, a lattice is a block if 0 and 1 are reducible elements in it.
In the next Chapter, we extend some of the above mentioned results to posets that are dismantlable by doubly irreducibles.

Chapter 2

Dismantlable posets

In this Chapter, we introduce and study posets dismantlable by doubly irreducibles. We also study some graph theoretical aspects such as cover graphs, orientability and an ear decomposition. I. Rival [10] introduced the concept of a dismantlable lattice. I. Rival [27] introduced the concept of a poset dismantlable by irreducibles. In the first section, we introduce the concept of a poset dismantlable by doubly irreducibles. We also introduce the operations, '1-sum' and '2-sum' of posets. Using these operations, we obtain the structure theorem for posets dismantlable by doubly irreducibles. Further, we try to give the inter-connections among these three concepts. In the second section, we study the concept of nullity of lattices and obtain various properties of the nullity of lattices. This concept is extensively used in the subsequent chapters. In the third section, we introduce the concept of adjunct of ears and characterize the graphs which are orientable as posets dismantlable by doubly irreducibles, thereby we try to give a

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partial solution to the open problem, "Characterize those graphs which are orientable", raised by O. Ore[20]. In the last section, we prove Whitney type characterizations of graphs (see [44]), namely, "a loopless graph is connected if and only if it has an ear decomposition". We begin with a simple well known characterization of a doubly irreducible element in a lattice.

Proposition 2.0.15. *An element other than* 0 *and* 1 *is doubly irreducible in a lattice* L *if and only if it has exactly one upper cover and exactly one lower cover in* L.

Proof. Suppose an element a in a lattice L is doubly irreducible, where $a \neq 0$ and $a \neq 1$. Therefore a is neither meet reducible nor join reducible. If a has at least two upper covers say b and c then $b \wedge c = a$, a contradiction. Similarly if a has at least two lower covers say e and f then $e \vee f = a$, a contradiction. Also we get a contradiction, if a has no upper or no lower cover in L, since $0 \leq a \leq 1$. Hence a has exactly one upper cover and exactly one lower cover in L.
Conversely, suppose a has exactly one upper cover and exactly one lower cover in L. Then a is neither meet reducible nor join reducible. Hence a is doubly irreducible in L. □

Brucker and Gely[37] characterized dismantlable lattices as follows.

Theorem 2.0.16. *[37]. A lattice* L *is dismantlable lattice if and only if there exists a chain of lattices* $L_1 \subset L_2 \subset \cdots \subset L_n = L$ *such that* L_1 *is a singleton and* $L_{i-1} = L_i \setminus \{x\}$ *where* x *is doubly irreducible element of* L_i.

The following definition is due to Duffus and Rival[25].

Definition 2.0.2. An element a of a poset P is *irreducible* in P if a is an isolated element or a has precisely one upper cover or precisely one lower cover in P.

Let $I(P)$ denote the set of all elements irreducible in a poset P.

Definition 2.0.3. An n-element *poset P is dismantlable by irreducibles* if the elements of P can be labelled $a_1, a_2, \ldots, a_n$ so that $a_i \in I(P - \{a_1, a_2, \ldots, a_{i-1}\})$ for each $i = 1, 2, \ldots, n-1$.

Equivalently, a finite poset P is dismantlable by irreducibles if P is one element or $P = \{x_1, x_2, \ldots, x_n\}$ such that for all $i = 1, 2, \ldots, n-1$, x_i is an irreducible element in the subposet of P induced by $\{x_i, x_{i+1}, \ldots, x_n\}$. That means, an n-element poset P is dismantlable by irreducibles if there exists a chain $P_1 \subset P_2 \cdots \subset P_n (= P)$ of subposets of P such that P_1 is one element and $P_{i-1} = P_i \setminus \{x\}$, where x is an irreducible element in P_i, for all i.

Proposition 2.0.17. *[25]. Let P be a finite connected poset. If P contains no crown then P is dismantlable by irreducibles.*

The converse of Proposition 2.0.17 is not true. For example, a cube (or a Boolean lattice $\mathbf{2}^3$) is dismantlable by irreducibles. Using Proposition 2.0.17, we get the following.

Corollary 2.0.18. *Every dismantlable lattice is dismantlable by irreducibles.*

In fact, we have the following result.

Theorem 2.0.19. *Every finite ordered set with a smallest element is dismantlable by irreducibles.*

Proof. Let P be a finite ordered set with a smallest element 0. We prove the result by induction on $n = |P| \geq 1$. Clearly, if $n = 1$ or 2 then we are done. Suppose $n > 2$ and the result is true for all the posets of order $r < n$. Let a be an atom of P. Let $P' = P - \{a\}$. Then P' is a poset with the same smallest element 0 and $|P'| = n - 1$. Hence by induction hypothesis, P' is dismantlable by irreducibles. Now a is irreducible in P, since it has exactly one lower cover, which is 0. As $P' = P \setminus \{a\}$, P is dismantlable by irreducibles. □

2.1 Posets dismantlable by doubly irreducibles

2.1.1 Introduction

The theories of dismantlable lattices (see [10]) and posets dismantlable by irreducibles (see [25], [26] and [27]) motivate us to define the following.

Definition 2.1.1. An element a of a poset P is *doubly irreducible* in P if a has at the most one upper cover and at the most one lower cover in P.

For example, any element in a chain is a doubly irreducible element and no element in the cube $\mathbf{2}^3$ is doubly irreducible.
Let $DI(P)$ denotes the set of all doubly irreducible elements in a poset P. Now we introduce the posets dismantlable by doubly irreducibles.

Definition 2.1.2. An n-element poset P is said to be a *poset dismantlable by doubly irreducibles* if there exists a chain $P_1 \subset P_2 \cdots \subset P_n (= P)$ of subposets of P such that P_1 has one element and $P_{i-1} = P_i \setminus \{x\}$, where x is a doubly irreducible element in P_i, for all i.

Equivalently, a finite poset P is dismantlable by doubly irreducibles if P has one element or $P = \{x_1, x_2, \ldots, x_n\}$ such that for all $i = 1, 2, \ldots, n-1$, x_i is a doubly irreducible element in the subposet of P induced by $\{x_i, x_{i+1}, \ldots, x_n\}$.
That means, an n-element poset P is dismantlable by doubly irreducibles if the elements of P can be labelled $a_1, a_2, \ldots, a_n$ so that $a_i \in DI(P - \{a_1, a_2, \ldots, a_{i-1}\})$ for each $i = 1, 2, \ldots, n-1$.
For example, a chain and an antichain are dismantlable posets by doubly irreducibles. A crown is not a poset dismantlable by doubly irreducibles, since it does not contain a doubly irreducible element.
We say a poset P is *connected* if $C(P)$ is connected. Therefore, a *component* of a poset P is a maximal connected subposet of P. Using the definition of a poset dismantlable by doubly irreducibles, it is clear that, a poset P is dismantlable by doubly irreducibles if and only if all the components of P are dismantlable by doubly irreducibles.

***Remark* 2.1.1.** A lattice dismantlable by irreducibles need not be dismantlable by doubly irreducibles. For example, a cube $\mathbf{2}^3$. Note that, a cube $\mathbf{2}^3$ contains a crown. Therefore, by Proposition **??**, a finite lattice need not be a lattice dismantlable by doubly irreducibles. However, by Theorem 2.0.16 and by Proposition 2.0.15, every dismantlable lattice is dismantlable by doubly irreducibles.

Using the definitions of a doubly irreducible element and an irreducible element, we have the following result.

Lemma 2.1.1. *Every doubly irreducible element in a poset is an irreducible element.*

As a consequence of the above Lemma 2.1.1, we get the following result.

Corollary 2.1.2. *If a poset is dismantlable by doubly irreducibles then it is dismantlable by irreducibles.*

Recall that, Thakare, Pawar and Waphare [13] introduced the concept of an adjunct operation for lattices. We extend this concept to posets by introducing "adjunct of posets". For this, we introduce the concepts of "1-sum" and "2-sum" for posets.

2.1.2 1-sum and 2-sum of posets

Definition 2.1.3. Let P_1 and P_2 be two disjoint posets. Let $a \in P_1$. Define a partial order on $P = P_1 \cup P_2$ with respect to a as follows.
For $x, y \in P$, we say that $x \leq y$ in P if $x, y \in P_1$ and $x \leq y$ in P_1 or $x, y \in P_2$ and $x \leq y$ in P_2 or $x \in P_1, y \in P_2$ and $x \leq a$ in P_1.
It is easy to see that P is a poset containing P_1 and P_2 as subposets.
The procedure for obtaining P in this way is called an *up 1-sum of* P_1 *with* P_2 *with respect to* a and write $P = P_1]_a P_2$.
A diagram of P is obtained by placing a diagram of P_1 and a diagram of P_2 side by side in such a way that the minimal elements of P_2 are at higher positions than a and then by adding the coverings $< a, x >$ for all $x \in S$, the set of all minimal elements of P_2. This clearly gives

$|E(P)| = |E(P_1)| + |E(P_2)| + |S|$.

Dually, one can define a *down 1-sum of posets*. If P is a down 1-sum of P_1 with P_2 with respect to a in P_1 then write $P = P_1]^a P_2$.

We call the element a as an *adjunct element* of the 1-sum.

We say that P is a *1-sum of posets P_1 and P_2 with respect to an element* $a \in P_1$ if P is either an up 1-sum or a down 1-sum of P_1 and P_2 with respect to a.

A 1-sum $P_1]_a P_2$ or $P_1]^a P_2$ is called a *trivial 1-sum* if P_2 is a chain and a is respectively maximal or minimal element of P_1; otherwise, we say that the 1-sum is *non-trivial*.

Definition 2.1.4. A *2-sum of the posets P_1 and P_2 with respect to a pair* (a, b) with $a < b$ but $a \not\prec b$ in P_1, is the poset $P = P_1 \cup P_2$ with a partial order defined on P, which is the union of the partial orders in $P_1]_a P_2$ and $P_1]^b P_2$. The pair (a, b) is called an *adjunct pair* of the 2-sum. We denote the 2-sum of the posets P_1 and P_2 with respect to a pair (a, b) by $P_1]_a^b P_2$.

The figure(Fig.2) shows the 2-sum of the two posets L_1 and L_2.

If a poset P is obtained by either 1-sum or 2-sum of the posets $P_1, P_2, \ldots, P_k$ then we say that P is an *adjunct of the posets* $P_1, P_2, \ldots, P_k$ and we write $P = (\cdots((P_1]_{\alpha_1} P_2)]_{\alpha_2}) P_3 \cdots]_{\alpha_{k-1}}) P_k$ or $P = P_1]_{\alpha_1} P_2]_{\alpha_2} P_3 \cdots]_{\alpha_{k-1}} P_k$, where for each i, α_i is either an adjunct element or an adjunct pair. If for some i, α_i is an adjunct element a correspond to an up 1-sum (or a down 1-sum) then the notation $]_{\alpha_i}$ is considered as $]_a$(or $]^a$) and if it is an adjunct pair (a, b) then the notation $]_{\alpha_i}$ is considered as $]_a^b$.

Note that, the operation 1-sum or 2-sum of posets preserves the existing coverings of the posets.

Lemma 2.1.3. *Let L_1 and L_2 be lattices and let $L = L_1]_a L_2$, where a is an adjunct element. Then L is a lattice if and only if a is 1 of L_1. Further, $L = L_1 \oplus L_2$.*

Proof. Suppose $L = L_1]_a L_2$ is a lattice. Let b be 1 of L_1. Let c be 1 of L_2. If $a \neq b$ then $b \vee c$ does not exist in L, a contradiction. Therefore $a = b$ and hence a must be 1 of L_1. Conversely, if a is 1 of L_1 then $L = L_1 \oplus L_2$. Thus L is a lattice. □

Dually, it follows from Lemma 2.1.3 that, if L_1 and L_2 are lattices and $L = L_1]^a L_2$. Then L is a lattice if and only if a is 0 of L_1. Further, $L = L_2 \oplus L_1$. Thus in such a situation, the 1-sum coincides with a linear sum.

Theorem 2.1.4. *Let L be a lattice. Then the following statements are equivalent.*

1. L is dismantlable.

2. L is dismantlable by doubly irreducibles.

3. L is obtained by 2-sum of chains.

Proof. (1) implies (2) follows from Theorem 2.0.16 and Proposition 2.0.15. To prove (2) implies (1), suppose L is dismantlable by doubly irreducibles. Therefore, there is a doubly irreducible element say x in L. As L is a lattice, $L' = L - \{x\}$ is again a lattice dismantlable by doubly irreducibles. Applying the same arguments to L' as applied to L and continuing in this way, we can have a chain of sublattices

$L_0 \subset L_1 \subset \cdots \subset L_n(= L)$ of L with $L_{i-1} = L_i \setminus \{x\}$ for each i and L_0 is empty, where x is doubly irreducible element of L_i. Therefore, by Theorem 2.0.16, L is dismantlable. Hence, L is dismantlable if and only if L is dismantlable by doubly irreducibles.

Now, (1) if and only if (3) follows from the Theorem 1.3.6.

Hence, we have (2) if and only if (3). □

Lemma 2.1.5. *[38]. Let P and Q be dismantlable (by irreducibles) ordered sets. Then $P \times Q$ is dismantlable (by irreducibles).*

The above Lemma 2.1.5 is not true for lattices dismantlable by doubly irreducibles. Since, a boolean lattice $\mathbf{2}^3$ is not a lattice dismantlable by doubly irreducibles, whereas M_2 (see Fig.5) and the 2-chain are lattices dismantlable by doubly irreducibles.

2.1.3 Structure theorem

We now prove a structure theorem for posets dismantlable by doubly irreducibles.

Theorem 2.1.6. *A connected poset P is dismantlable by doubly irreducibles if and only if P is obtained by (non-trivial) 1-sum or 2-sum of chains.*

Proof. Suppose P is a poset dismantlable by doubly irreducibles. Therefore there exists a chain $P_1 \subset P_2 \cdots \subset P_n(= P)$ of subposets of P such that P_1 is one element and $P_{i-1} = P_i \setminus \{x\}$, where x is doubly irreducible element in P_i, for all i.

Using induction on $n = |P| \geq 1$.

If $n = 1$ then P is the 1-chain and we are done. Now suppose $n > 1$ and the result is true for all posets of order $< n$.

If P is a chain then we are done. Therefore, suppose P is not a chain. As P is dismantlable by doubly irreducibles, there is at least one doubly irreducible element in P. Let x_0 be a doubly irreducible element in P and π be a maximal path in $C(P)$ containing x_0, consisting of doubly irreducible elements in P. Suppose $\pi : x_1 - x_2 - \cdots - x_m$. If x_1 or x_m is pendant in $C(P)$ then denote it by x (Note that, both x_1 and x_m can not be pendant as P is connected but not a chain); otherwise, denote x_1 by x.

Let $P_{n-1} = P_n \setminus \{x\}$. Now P_{n-1} is a poset dismantlable by doubly irreducibles. Also, $|P_{n-1}| = n-1 < n$. Therefore, by induction hypothesis, P_{n-1} is obtained by (non-trivial) 1-sum or 2-sum of chains. Suppose P_{n-1} has an adjunct representation

$$P_{n-1} = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2 \cdots]_{\alpha_k}C_k, \qquad (*)$$

where for each i, C_i is a chain and α_i is either an adjunct element or an adjunct pair.

If x is a pendant vertex in $C(P)$ then $P = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2 \cdots]_{\alpha_j}C'_j \cdots]_{\alpha_k}C_k$, where $C'_j = C_j \cup \{x\}$ with $x \prec x_2$ whenever $x = x_1$ and $x_2 \in C_j$, and $x_{m-1} \prec x$ whenever $x = x_m$ and $x_{m-1} \in C_j$.

Now, suppose x is not pendant in $C(P)$, $y \prec x$ and $x_m \prec z$ in P.

Case : 1. Suppose $y \not\prec z$ in P_{n-1}. If $\pi : x$ (that is, $m = 1$) then $P = P_{n-1}]_y^z\{x\}$; Otherwise, $P = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2 \cdots]_{\alpha_j}C'_j \cdots]_{\alpha_k}C_k$, where $C'_j = C_j \cup \{x\}$ with $x \prec x_2$ whenever $x_2 \in C_j$.

Case : 2. Suppose $y \prec z$ in P_{n-1} (that is, $m = 1$ and there is no another

path from y to z in P). Let $y \in C_i$ and $z \in C_j$.
If $i \leq j$ then $P = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2 \cdots]_{\alpha_i}C_i \cdots]_{\alpha_j}C'_j \cdots]_{\alpha_k}C_k$,
where $C'_j = C_j \cup \{x\}$ with $x \prec z$.
If $i > j$ then $P = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2 \cdots]_{\alpha_j}C_j \cdots]_{\alpha_i}C'_i \cdots]_{\alpha_k}C_k$,
where $C'_i = C_i \cup \{x\}$ with $y \prec x$.
Thus P is obtained by (non-trivial) 1-sum or 2-sum of chains.
Conversely, suppose P is obtained by (non-trivial) 1-sum or 2-sum of chains. Let $P = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2 \cdots]_{\alpha_k}C_k$ where for each i, α_i is either an adjunct element or an adjunct pair and C_i is a chain.
Again, using induction on $n = |P| \geq 1$. If $n = 1$ then P is 1-chain and we are done. Now suppose $n > 1$ and the result is true for all posets of order $< n$.
Let $x \in C_k$. Clearly x is doubly irreducible in the chain C_k and hence in P. Let $P' = P \setminus \{x\}$. Now P' is connected and $|P'| = n-1 < n$ and
$P' = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2 \cdots]_{\alpha_{k-1}}C_{k-1}$, if $C_k = \{x\}$ and
$P' = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2 \cdots]_{\alpha_{k-1}}C_{k-1}]_{\alpha_k}C'_k$, if $|C_k| > 1$, where $C'_k = C_k - \{x\}$.
Thus P' is obtained by (non-trivial) 1-sum or 2-sum of chains. Therefore, by induction hypothesis P' is dismantlable by doubly irreducibles.
Now $P' = P \setminus \{x\}$, where x is doubly irreducible in P. Therefore P is dismantlable by doubly irreducibles. □

Theorem 2.1.7. *If P is a poset dismantlable by doubly irreducibles and C is any maximal chain in P then we get chains $C_1, C_2, \ldots, C_k$ in P such that $P = C]C_1]C_2 \cdots]C_k$.*

Proof. We prove the result using induction on $|P| \geq 1$. If $|P| = 1$ then we are done. Now suppose $|P| > 1$ and the result is true for all posets

Q with $|Q| < |P|$. Let C be a maximal chain in P. Let x be a doubly irreducible element in P. Let $P' = P \setminus \{x\}$.

Case 1 : Suppose $x \in C$ and $C - \{x\}$ is a maximal chain in P'. Now P' is a poset dismantlable by doubly irreducibles. Therefore, by induction hypothesis, we get chains $C_1, C_2, \ldots, C_k$ in P' such that $P' = (C - \{x\})]C_1]C_2 \cdots]C_k$. Hence $P = C]C_1]C_2 \cdots]C_k$ as required.

Case 2 : Suppose $x \in C$ and $C - \{x\}$ is not a maximal chain in P'. Then there exist $a, b \in C$ such that $a \prec x \prec b$ and a maximal chain C_0 in $[a, b]$ such that $x \notin C_0$. But then $C' = (C \cap (a]) \cup C_0 \cup (C \cap [b))$ is a maximal chain in P'. Therefore, by induction hypothesis, we get chains $C'_1, C'_2, \ldots, C'_k$ in P' such that $P' = C']C'_1]C'_2 \cdots]C'_k$. But then it is easy to see that, $P = C]_a^b C_0]C'_1]C'_2 \cdots]C'_k$ as required.

Case 3 : Suppose $x \notin C$. Then C remains a maximal chain in P'. Therefore, by induction hypothesis, we get chains $C''_1, C''_2, \ldots, C''_k$ in P' such that $P' = C]C''_1]C''_2 \cdots]C''_k$. Consider an ear E in P containing x. If $E \neq \{x\}$ and $y \in E$ with $y \neq x$ then change C''_i to $C'''_i = C''_i \cup \{x\}$, where $y \in C''_i$, we get $P = C]C''_1]C''_2 \cdots]C'''_i \cdots]C''_k$.

If $E = \{x\}$ then $P = C]C''_1]C''_2 \cdots]C''_k]\{x\}$ whenever x is pendant or there is an element $z \neq x$ such that $a \prec z \prec b$, where a and b are the elements of P such that $a \prec x \prec b$. If there is no z in P such that $z \neq x$ and $a \prec z \prec b$ then we must have C''_i containing both a and b. Replace C''_i by $C'''_i = C''_i \cup \{x\}$, we get the required result. □

By Theorem 2.1.6, if a poset P is dismantlable by doubly irreducibles then P is obtained by (non-trivial) 1-sum or 2-sum of chains. That is, $P = C_0]_{\alpha_1} C_1]_{\alpha_2} C_2 \cdots]_{\alpha_k} C_k$, where for each i, α_i is either an adjunct element or an adjunct pair and C_i is a chain. Henceforth, we call such a representation of P as an "adjunct representation" of P into chains. It is obvious that there may be different adjunct representations to a poset dismantlable by doubly irreducibles. However, the number of chains in each adjunct representation is the same. Moreover, the adjunct elements and the adjunct pairs are also the same, having the same multiplicity. More precisely, we have the following.

Theorem 2.1.8. *Let P be a poset dismantlable by doubly irreducibles. Then an element $x \in P$ occurs k times as a base of up 1-sum(i.e., as an adjunct subscript) in an adjunct representation of P if and only if $k = 0$ whenever x is a maximal element of P, and*
$k = |\{B : B$ is a block in $C(P)$ containing x such that x is not the largest element of $B\}| - 1.$ $(*)$

Proof. Suppose x occurs k times as a base of up 1-sum in an adjunct representation with corresponding chains $C_1, C_2, \ldots, C_k$ and C is the chain in the representation containing x. It is clear that, if x is a maximal element of the poset P then x can not become a base of any up 1-sum, since otherwise we get a contradiction to the maximality of x. Hence $k = 0$. Now, suppose x is not a maximal element of P, that means, there is a block in $C(P)$ containing x as an element other than the largest element of the block. Note that, if the chain C corresponds to an up 1-sum and x is the largest element of C then $k = 0$, as the 1-sums in the representation are non-trivial, but then x becomes a maximal element of P, a contradiction. Hence, we can select an element y such that $x \prec y$, $y \notin C_i$, for all i, $1 \leq i \leq k$, and y appears before joining any of C_i's in the representation. Let x_i be the smallest element of C_i, for all i, $1 \leq i \leq k$. Let B be the block containing x and y. Let B_i be the block containing x and x_i, for all i, $1 \leq i \leq k$.

Now, we claim that, all these blocks B, B_i's are distinct and these are the only blocks containing x as a non-largest element.

As y appears in the representation before x_i, for all i, and the chains in the representation are pairwise disjoint, it is easy to see that $B \neq B_i$, for all i and $B_i \neq B_j$, for all $i \neq j$. Now, suppose B' is a block containing x, $B' \neq B$, $B' \neq B_i$, for all i, and x is not the largest element of B'. Note that, any block other than B and having the common vertex x must correspond to an up 1-sum at x; it means that, the 1-sum corresponding to B' is, or can be exchanged with, one of the 1-sums used in the representation. In any case, the block $B' = B_i$ for some i. Thus, $k = |\{B : B$ is a block in $C(P)$ containing x such that x is not the largest element of $B\}| - 1$.

Conversely, we prove using induction on $|P| = n$ that, for a poset P dismantlable by doubly irreducibles, if k is the number satisfying $(*)$ for an element x then x occurs k times as a base of up 1-sum in any adjunct representation of P.

If $n = 1$ or 2 then we are done. Suppose $n > 2$. Let $x \in P$ be an element and k be the number satisfying $(*)$. Let $R = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2 \cdots]_{\alpha_m}C_m$ be an adjunct representation of P.

If $|C_m| > 1$ then replacing C_m by $C_m \setminus \{y\}$ in R, where $y \in C_m$ and $y \neq x$, we get a poset P' with representation R' such that $|P'| < n$, $x \in P'$ and the number k satisfies $(*)$ for x in P'. By induction, x

occurs k times as a base of up 1-sum in R' and hence in R as required. Now, suppose $|C_m| = 1$, say $C_m = \{y\}$. Note that, if $x = y$ then $k = 0$ and clearly the result is true. Let $x \neq y$ and consider the poset $P' = P \setminus \{y\}$. Note that, if x is a maximal element of P then x is also a maximal element of P' and it follows by induction that, there is no up 1-sum in R with base x.

Now assume that x is not a maximal element of P.

Case 1 : Suppose C_m is used for 1-sum in R. If C_m corresponds to a down 1-sum then same k satisfies $(*)$ for x in P' and we get that x occurs k times as a base of up 1-sum in R' and hence in R. Similarly, we get the required result by induction whenever the 1-sum is up 1-sum at the base other than x.

Suppose C_m corresponds to up 1-sum with base x. Then $k-1$ satisfies $(*)$ for x in P'. Hence by induction x occurs $k-1$ times as a base of up 1-sum in R' and hence x occurs k times as a base of up 1-sum in R, as required.

Case 2 : Suppose C_m is used for 2-sum in R, say (a, b) is the corresponding adjunct pair. Let B be a block containing the interval $[a, b]$ in P. Clearly, if $x \notin B$ then the same k satisfies $(*)$ for x in P' and the result follows by induction. Finally, if $x \in B$ and x is not a largest element of B then in P', there is a unique block B_1 in P' which is contained in B such that $x \in B_1$ and x is not the largest element of B_1. Hence, in this situation also the same k satisfies $(*)$ for x in P' and the result follows by induction. □

Dually, an element $x \in P$ occurs k times as a base of down 1-sum(i.e., as an adjunct superscript) in an adjunct representation of P if and only if $k = 0$ whenever x is a minimal element of P, and $k = |\{B : B$ is a block in $C(P)$ containing x such that x is not the smallest element of $B\}| - 1$. Thus, using Theorem 2.1.8, we have the following.

Corollary 2.1.9. *Let P be a poset dismantlable by doubly irreducibles. Let $a \in P$. Then the number of times a occurs as an adjunct subscript in any adjunct representation of P is the same. The same holds for adjunct superscripts.*

It is known that, if L is a dismantlable lattice then the number of chains in every adjunct representation of L is the same (see [13]). In Theorem 2.1.10, we prove that the number of the chains in any adjunct representation of P remains same.

Theorem 2.1.10. *If P is a poset dismantlable by doubly irreducibles then the number of chains in any adjunct representation of P remains same.*

Proof. Let P be a poset dismantlable by doubly irreducibles. Without loss, we assume that P is connected. By Theorem 2.1.6, $P = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2 \cdots]_{\alpha_k}C_k$, where for each i, α_i is an adjunct element or an adjunct pair and C_i is a chain. If r_1 is the number of adjunct elements and r_2 is the number of adjunct pairs in P then $k = r_1 + r_2$. Suppose P can also written as $C'_0]_{\beta_1}C'_1]_{\beta_2}C'_2 \cdots]_{\beta_l}C'_l$, where for each j, β_j is an adjunct element or an adjunct pair and C'_j is a chain. If s_1 is the number of adjunct elements and s_2 is the number of adjunct pairs in this representation of P then $l = s_1 + s_2$.

Now $|E(P)| = \sum_{i=0}^{k} |E(C_i)| + r_1 + 2r_2 = \sum_{j=0}^{l} |E(C'_i)| + s_1 + 2s_2$. Therefore we have $|P| - (k+1) + r_1 + 2r_2 = |P| - (l+1) + s_1 + 2s_2$ which implies that $k - r_1 = l - s_1$. But by the above Corollary 2.1.9, $r_1 = s_1$. Therefore $k = l$. Thus, the number of chains in both the representations of P is same. □

In order to prove the Theorem 2.1.12, we first prove the following.

Theorem 2.1.11. *Let P be a poset having a maximum of k internally disjoint maximal chains from a to b. Then for any chain C, the poset $P]_a^b C$ has a maximum of $k+1$ internally disjoint maximal chains from a to b.*
Further, for any 1-sum with C or 2-sum with C at a pair other than (a, b) to P produces a poset in which the maximum number of internally disjoint maximal chains from a to b is k.

Proof. The first part follows from the definition of 2-sum. Now we prove the second part. It is clear that any 1-sum of P with C produces a poset in which the maximum number of internally disjoint maximal chains from a to b is k, since this 1-sum does not produce a chain from a to b. Let $P' = P]_c^d C$, where $(c, d) \neq (a, b)$ in P. If either c or d but not both belong to the interval $[a, b]$ then we are done. Therefore, suppose both $c, d \in [a, b]$. Let k be the maximum number of internally disjoint maximal chains, say $C_1, C_2, \ldots C_k$ from a to b in P. Now $c, d \in [a, b]$ implies that either $c, d \in C_i$ for some i, or $c \in C_i$ (or $d \in C_i$) for some i, or $c, d \notin C_i$ for any i.
Case 1 : Suppose $c, d \in C_i$ for some i.

If $|(c,d)\cap C_i| \geq |C|$ then we are done; otherwise, replace C_i by the chain $C_i' = ([a,c]\cap C_i)\oplus C\oplus([d,b]\cap C_i)$.

Case 2 : Suppose $c \in C_i$ for some i.

Let $d \in C'$, where $C' : x_1 \prec x_2 \prec \cdots x_{r-1} \prec x_r$ is a maximal chain such that $e, f \in C_i, e \prec x_1, x_r \prec f$ and $C_i \cap C' = \emptyset$, since those k chains are internally disjoint maximal chains. It is clear that $c \leq e$. If $|(c,f)\cap C_i| \geq |C\oplus([d,f)\cap C')|$ then we are done; otherwise, replace C_i by the chain $C_i' = ([a,c]\cap C_i)\oplus C\oplus([d,f)\cap C')\oplus([f,b]\cap C_i)$.

Case 3 : Suppose $d \in C_i$ for some i.

This case is similar to Case 2 above.

Case 4 : Suppose $c, d \notin C_i$ for any i.

Let $c, d \in C''$, where $C'' : y_1 \prec y_2 \prec \cdots y_{s-1} \prec y_s$ is a maximal chain such that $g, h \in C_i, g \prec y_1, y_s \prec h$ and $C_i \cap C'' = \emptyset$ for some i, since those k chains are internally disjoint maximal chains. Again, if $|(g,h)\cap C_i| \geq |((g,c]\cap C'')\oplus C\oplus([d,h)\cap C'')|$ then we are done; otherwise, replace C_i by the chain $C_i' = ([a,g]\cap C_i)\oplus((g,c]\cap C'')\oplus C\oplus([d,h)\cap C'')\oplus([h,b]\cap C_i)$. In any case, we get k internally disjoint maximal chains from a to b. Hence the proof is complete. □

Theorem 2.1.12. *Let P be a poset dismantlable by doubly irreducibles. A pair (a,b) of elements $a, b \in P$ with $a < b$ and $a \not\prec b$ in P occurs r times in an adjunct representation of P if and only if the maximum number of internally disjoint maximal chains from a to b in P is $r+1$.*

Proof. Let $R = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2\cdots]_{\alpha_m}C_m$ be an adjunct representation for P, and the pair $\alpha = \alpha_{i_1} = \alpha_{i_2}\cdots = \alpha_{i_r}$ occurs r times in the adjunct representation R. Consider the subposet $P' = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2\cdots]_{\alpha_{i_1-1}}C_{i_1-1}$.

Now, the elements a, b satisfy $a, b \in P'$, $a < b$ and $a \not\prec b$ in P'. As the pair (a, b) is not used in the representation of P', and noting that any 1-sum or 2-sum does not disturb the existing covering relation, there are no two internally disjoint maximal chains from a to b in P'. Select a maximal chain C_0' in P' from a to b. The chain C_0' together with the chains $C_s' = C_{i_s} \cup \{a, b\}$ for $s = 1, 2, \ldots, r$ form $r+1$ internally disjoint maximal chains from a to b in P. Now, the fact that there is no set of $r+2$ internally disjoint maximal chains from a to b in P follows by Theorem 2.1.11. Therefore $C_0', C_1', \ldots, C_r'$ are the required chains.
To prove the converse, we use induction on $n = |P| \geq 1$. If $n \leq 4$ then the result holds obviously. Now, suppose $n > 4$ and assume that the result is true for all posets dismantlable by doubly irreducibles having less than n elements. Let (a, b) be a pair of elements $a, b \in P$ with $a < b$ and $a \not\prec b$ in P. Suppose the maximum number of internally disjoint maximal chains from a to b in P is $r+1$.
Using Theorem 2.1.6, we have $P = C_0]_{\alpha_1} C_1]_{\alpha_2} C_2 \cdots]_{\alpha_k} C_k$, where for each i, α_i is either an adjunct element or an adjunct pair and C_i is a chain. Then $Q = P - C_k = C_0]_{\alpha_1} C_1]_{\alpha_2} C_2 \cdots]_{\alpha_{k-1}} C_{k-1}$ is a subposet of P. Moreover, by Theorem 2.1.6, Q is a poset dismantlable by doubly irreducibles with $|Q| < n$.
If α_k is an adjunct pair (a, b) then the maximum number of internally disjoint maximal chains from a to b in Q is r. Therefore, by induction hypothesis, the pair (a, b) occurs $r-1$ times in some adjunct representation R of Q and hence r times in the adjunct representation $R]_a^b C_k$ of P. If α_k is not an adjunct pair (a, b) then the proof follows from

Lemma 2.1.11. □

The following Corollary 2.1.13 immediately follows from Theorem 2.1.12 (Note that, the Corollary 2.1.13 also follows from Corollary 2.1.9 and Theorem 2.1.10).

Corollary 2.1.13. *Let P be a poset dismantlable by doubly irreducibles. Let $a, b \in P$ be such that $a < b$ but $a \nprec b$. Then the number of times (a, b) occurs as an adjunct pair in any adjunct representation of P remains the same.*

The following Corollary 2.1.14 follows from Theorem 2.1.10, Corollary 2.1.9 and Corollary 2.1.13.

Corollary 2.1.14. *Let P be a poset dismantlable by doubly irreducibles. Let $C_0]_{\alpha_1}C_1]_{\alpha_2}C_2\cdots]_{\alpha_k}C_k$ and $C'_0]_{\beta_1}C'_1]_{\beta_2}C'_2\cdots]_{\beta_k}C'_k$ be any two adjunct representations of P. Then there is a permutation π of $\{1, 2, \ldots, k\}$ such that $\alpha_i = \beta_{\pi(i)}$ for all i, $1 \leq i \leq k$.*

In order to prove Theorem 2.1.16, we need the following.

Lemma 2.1.15. *Let P be a poset dismantlable by doubly irreducibles and let B be a pendant block in $C(P)$ with cut-vertex x. Then $P - \{B - \{x\}\}$ is also a poset dismantlable by doubly irreducibles.*

Proof. Suppose P is a poset dismantlable by doubly irreducibles and B is a pendant block in $C(P)$ with cut-vertex x. By Theorem 2.1.6, $P = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2\cdots]_{\alpha_k}C_k$, where for each i, α_i is either an adjunct element or an adjunct pair and C_i is a chain. We use induction on $n = |P|$. If $n \leq 3$ then we are done. Now, suppose $n \geq 4$ and the result

is true for all posets containing $< n$ elements. Let $P' = P - C_k$. Then $|P'| < n$ and P' is also a poset dismantlable by doubly irreducibles.
Case 1 : Suppose $B \cap C_k = \emptyset$. Now B is a pendant block in $C(P)$ with cut-vertex x. Therefore, $x \notin C_k$ and if α_k is an adjunct element then $\alpha_k \notin B$. Also, if $\alpha_k = (a, b)$ is an adjunct pair then $a, b \notin B$, since otherwise, $C_k \subset B$, which is not possible. Hence B remains a pendant block in P'. Therefore, by induction hypothesis, $Q = P' - \{B - \{x\}\}$ is a poset dismantlable by doubly irreducibles. Therefore using Theorem 2.1.6, $P = Q]_{\alpha_k} C_k$ is also a poset dismantlable by doubly irreducibles.
Case 2 : Suppose $B \cap C_k \neq \emptyset$. Then $C_k \subset B$. Consider $B' = B - C_k$. If B' is a block itself then it is a pendant block in $C(P')$ with cut-vertex x. Therefore, by induction hypothesis, $Q' = P' - \{B' - \{x\}\}$ is a poset dismantlable by doubly irreducibles. Therefore using Theorem 2.1.6, $P = Q']_{\alpha_k} C_k$ is also a poset dismantlable by doubly irreducibles.
Suppose B' is not a block itself. Suppose there are $t \geq 2$ blocks in $C(B')$. If B_1 is a pendant block in $C(B')$ with cut-vertex x_1 then B_1 is also a pendant block in $C(P')$ with cut-vertex x_1. Therefore, by induction hypothesis, $P'_1 = P' - \{B_1 - \{x_1\}\}$ is a poset dismantlable by doubly irreducibles.
Now, suppose B_2 is a pendant block in $C(B'_1)$, where $B'_1 = B' - (B_1 - \{x_1\})$ with cut-vertex x_2 then B_2 is also a pendant block in $C(P'_1)$ $(\subset C(P'))$ with cut-vertex x_2. Therefore, by induction hypothesis, $P'_2 = P'_1 - \{B_2 - \{x_2\}\}$ is a poset dismantlable by doubly irreducibles.
Continuing in this way, suppose B_t is a pendant block in $C(B'_{t-1})$, where $B'_{t-1} = B'_{t-2} - (B_{t-1} - \{x_{t-1}\})$ with cut-vertex x_t then B_t is also

a pendant block in $C(P'_{t-1})$ ($\subset C(P')$) with cut-vertex x_t. Therefore, by induction hypothesis, $P'_t = P'_{t-1} - \{B_t - \{x_t\}\}$ is a poset dismantlable by doubly irreducibles. Note that $P'_t = Q$. Hence the proof is complete. □

We now prove one more characterization of a poset dismantlable by doubly irreducibles.

Theorem 2.1.16. *Let P be a connected poset. Then P is dismantlable by doubly irreducibles if and only if every block in $C(P)$ is a poset dismantlable by doubly irreducibles.*

Proof. Suppose P is a connected poset dismantlable by doubly irreducibles. Let B be a block in $C(P)$. If $B = P$ then we are done; Otherwise, $C(P)$ contains at least two pendant blocks. We use induction on $n = |P|$. If $n \leq 3$ then we are done. Now, suppose $n \geq 4$ and the result is true for all posets containing $< n$ elements. Let B' and B'' be pendant blocks in $C(P)$.

If $B \neq B'$ then consider $P' = P - \{B' - \{x\}\}$, where x is a cut-vertex of B' (Note that, if $B = B'$ then one can consider $P' = P - \{B'' - \{y\}\}$, where y is a cut-vertex of B''). By Lemma 2.1.15, P' is a poset dismantlable by doubly irreducibles. Also P' is connected. Now $|P'| < n$ and B is a block in $C(P')$. Therefore by induction hypothesis, every block in $C(P')$ is a poset dismantlable by doubly irreducibles. Hence B is a poset dismantlable by doubly irreducibles.

Conversely, suppose every block in $C(P)$ is a poset dismantlable by doubly irreducibles. If there is only one block then we are done. Again, we use induction on $n = |P|$. If $n \leq 3$ then we are done. Now, suppose $n \geq 4$ and the result is true for all posets containing $< n$ elements. Let B be a pendant block in $C(P)$ with cut-vertex x.

Case 1 : Suppose B is an edge $\{x, y\}$. Let $P' = P - \{y\}$. Then P' contains all the blocks of P except B. Now P' is connected and $|P'| < n$. Therefore by induction hypothesis, P' is a poset dismantlable by doubly irreducibles. By Theorem 2.1.6, suppose $P' = C_0]_{\alpha_1} C_1]_{\alpha_2} C_2 \cdots]_{\alpha_k} C_k$, where for each i, α_i is either an adjunct element or an adjunct pair and C_i is a chain. If x is doubly irreducible in P then

$P = C_0]_{\alpha_1} C_1]_{\alpha_2} C_2 \cdots]_{\alpha_j} C'_j \cdots]_{\alpha_k} C_k$, where $C'_j = C_j \cup \{y\}$ with $x \in C_j$; Otherwise, $P = P']_x \{y\}$ or $P = P']^x \{y\}$. Therefore by Theorem 2.1.6, P is a poset dismantlable by doubly irreducibles.

Case 2 : Suppose B is not an edge. Now B is a poset dismantlable by doubly irreducibles. By Theorem 2.1.6, let $B = C'_0]_{\beta_1} C'_1]_{\beta_2} C'_2 \cdots]_{\beta_l} C'_l$, where for each i, β_i is an adjunct element or an adjunct pair and C'_i is a chain. Now β_l is not an adjunct element, since otherwise, we get a contradiction to the fact that B is a pendant block. As $B - C'_l$ is also a (connected) poset dismantlable by doubly irreducibles, every block in $C(B - C'_l)$ is also a poset dismantlable by doubly irreducibles. Let $P'' = P - C'_l$. Note that it can be assumed that $x \notin C'_l$. Now except B all the blocks in $C(P)$ are also the blocks in $C(P'')$. Also, all the blocks in $C(B - C'_l)$ are also the blocks in $C(P'')$.

Now, all the blocks in $C(P'')$ are posets dismantlable by doubly irreducibles and P'' is connected with $|P''| < n$. Therefore by induction hypothesis, P'' is a poset dismantlable by doubly irreducibles. As

$P = P'']_{\beta_l} C'_l$, using Theorem 2.1.6, P is a poset dismantlable by doubly irreducibles. □

Theorem 2.1.17. *If a poset P is dismantlable by doubly irreducibles then for all $a, b \in P$ having an upper bound, the supremum $a \vee b$ exists.*

Proof. Suppose P is a poset dismantlable by doubly irreducibles. By Theorem 2.1.6, let $P = C_0]_{\alpha_1} C_1]_{\alpha_2} C_2 \cdots]_{\alpha_k} C_k$ where for each i, α_i is an adjunct element or an adjunct pair and C_i is a chain.

We prove the result using induction on $n = |P| \geq 1$. If $n \leq 2$ then we are done. Suppose $n > 2$ and the result is true for all posets of order $< n$. Let $Q = P - C_k$. Then Q is a poset dismantlable by doubly irreducibles with $|Q| < n$. Therefore, for all $a, b \in Q$ having an upper bound in Q, the supremum $a \vee b$ exists. Let $a, b \in P$ be such that a and b have an upper bound in P. Let $C_k : x_1 \prec x_2 \prec \cdots \prec x_r$.

Case 1 : Suppose $a, b \in Q$. By induction hypothesis, if a and b have an upper bound in Q then supremum exists in Q. Therefore, if a and b have no upper bound in C_k then we are done; Otherwise, C_k either corresponds to an up 1-sum or 2-sum. If $c \in P$ is such that $c \prec x_1$ then c is also an upper bound of both a and b in Q. Hence supremum of a and b, say d exists in Q. If C_k corresponds to an up 1-sum then d remains as supremum of a and b in P. Now, if C_k corresponds to 2-sum and $\alpha_k = (x, y)$ then x, y are also upper bounds for both a and b in Q. In this case, d remains the supremum of a and b in P, since $d \leq x$.

Case 2 : If $a, b \in C_k$ then we are done.

Case 3 : Without loss, suppose $a \in Q$ and $b \in C_k$. Suppose a and b have an upper bound in P.

Subcase i : If C_k corresponds to down 1-sum then $x_r \prec \alpha_k$. Now $a, \alpha_k \in Q$ and have an upper bound in Q. Therefore by induction hypothesis, $a \vee \alpha_k$ exists in Q. Now $a \vee b = a \vee \alpha_k$, since $b \leq \alpha_k$. Hence $a \vee b$ exists in P.

Subcase ii : If C_k corresponds to up 1-sum then $\alpha_k \prec x_1$ and hence $\alpha_k < b$. Also $a \leq \alpha_k$. For if, suppose $a > \alpha_k$ or $a || \alpha_k$ then $a || b$ and a and b can not have an upper bound in P, a contradiction. Therefore $a < b$ and hence $a \vee b = b$ exists in P.

Subcase iii : If C_k corresponds to 2-sum and $\alpha_k = (x, y)$ then $b < y$ and hence $a \vee b = a \vee y$. Note that by induction hypothesis, $a \vee y$ exists in Q. Thus for all $a, b \in P$ having an upper bound, $a \vee b$ exists in P. □

Dually, if a connected poset P is dismantlable by doubly irreducibles then for all $a, b \in P$ having a lower bound, the infimum $a \wedge b$ exists.

The converse of Theorem 2.1.17 is not true, since in a crown, for any two elements having an upper bound, supremum exists but it is not a poset dismantlable by doubly irreducibles.

Theorem 2.1.18. *If a poset P is dismantlable by doubly irreducibles then any subposet Q of P which is a lattice is dismantlable by doubly irreducibles.*

Proof. Suppose P is a poset dismantlable by doubly irreducibles. Therefore by Theorem 2.1.6, $P = C_0]_{\alpha_1} C_1]_{\alpha_2} C_2 \cdots]_{\alpha_k} C_k$, where for each i, α_i is either an adjunct element or an adjunct pair and C_i is a chain. Let Q be a subposet of P which is a lattice. We use induction on $n = |P|$. If $n \leq 4$ then we are done. Now, suppose $n \geq 5$ and the result is true for all posets containing $< n$ elements.

If there exists $x \in C_k$ such that $x \notin Q$ then Q is a subposet of $P' = P - \{x\}$ which is a lattice. Also, P' is a poset dismantlable by doubly irreducibles with $|P'| < n$. Therefore by induction hypothesis, Q is a poset dismantlable by doubly irreducibles.

Now, suppose $C_k \subseteq Q$. Let $P'' = P - \{y\}$, where $y \in C_k$. Then $Q' = Q - \{y\}$ is a subposet of P'' which is also a lattice, since $y \in Irr(P)$. Now $|P''| < n$. Therefore by induction hypothesis, Q' is a poset dismantlable by doubly irreducibles. As $Q' = Q \setminus \{y\}$ and y is a doubly irreducible element in Q, Q is a poset dismantlable by doubly irreducibles. Hence the proof. □

The converse of Theorem 2.1.18 is not true, since in a crown, any subposet which is a lattice is dismantlable by doubly irreducibles but crown is not a poset dismantlable by doubly irreducibles.

The following Lemma 2.1.19 follows from the fact that 1-sum and 2-sum operations preserve the existing coverings of the posets.

Lemma 2.1.19. *Let P_0, P_1 and P_2 be posets. If $P = (P_0]_{\alpha_1} P_1)]_{\alpha_2} P_2$, where α_1, α_2 are adjunct pairs lying in P_0 then $P = (P_0]_{\alpha_2} P_2)]_{\alpha_1} P_1$.*

Corollary 2.1.20. *Let $P = C_0]_{\alpha_1} C_1]_{\alpha_2} C_2 \cdots]_{\alpha_i} C_i \cdots]_{\alpha_j} C_j \cdots]_{\alpha_k} C_k$, where C_0 is a maximal chain containing all the reducible elements of P. Then for any $i \neq j$, $P = C_0]_{\alpha_1} C_1]_{\alpha_2} C_2 \cdots]_{\alpha_j} C_j \cdots]_{\alpha_i} C_i \cdots]_{\alpha_k} C_k$.*

Proof. The proof clearly follows by Lemma 2.1.19. Since, in particular, if $P = (P_0]_{\alpha_1} C_1)]_{\alpha_2} C_2$, where α_1, α_2 are adjunct pairs lying in poset P_0 and C_1, C_2 are chains then $P = (P_0]_{\alpha_2} C_2)]_{\alpha_1} C_1$. □

Lemma 2.1.21. *If a lattice contains at most eight pairs of incomparable reducible elements then L is dismantlable.*

Proof. If L is not dismantlable then by Theorem 1.3.5, it contains a crown. But a crown contains at least nine pairs of incomparable reducible elements. Therefore this is not possible by hypothesis. Hence L is dismantlable. □

By Lemma 2.1.21, the following Corollary 2.1.22 follows immediately.

Corollary 2.1.22. *A lattice in which all the reducible elements are comparable is dismantlable.*

2.1.4 Nullity of a poset

Recall that, the *nullity* of a graph G is given by $m - n + c$, where m is the number of edges in G, n is the number of vertices in G and c is the number of connected components of G.
We define *nullity of a poset* as the nullity of its cover graph.
We now obtain some properties of nullity of posets.

Theorem 2.1.23. *Let P be a poset. Let $x \in Irr(P)$. Then nullity($P - \{x\}$) = nullity(P) if and only if (i) There are no $y, z \in Red(P)$ such that $y \prec x \prec z$ or (ii) There are $y, z \in Red(P)$ such that $y \prec x \prec z$ and there is no other (directed) path from y to z.*

Proof. Let $P' = P - \{x\}$. Suppose $x \in Irr(P)$ and nullity(P') = nullity(P). If x satisfies the condition (i) then we are done. If not, then there are $y, z \in Red(P)$ such that $y \prec x \prec z$. Suppose there is another path from y to z then nullity$(P') = |E(P')| - |P'| + 1 = (|E(P)| - 2) - (|P| - 1) + 1 = |E(P)| - |P| =$ nullity$(P) - 1$, a contradiction. Therefore x must satisfy condition (ii).

Conversely, suppose $x \in Irr(P)$ and the condition (i) or the condition (ii) holds. Suppose the condition (i) is true. If $y \prec x \prec z$ in P then either $y \in Irr(P)$ or $z \in Irr(P)$. In any case, nullity$(P') = |E(P')| - |P'| + 1 = (|E(P)| - 1) - (|P| - 1) + 1 = |E(P)| - |P| + 1 =$ nullity(P). Now suppose condition (ii) is true. But then also nullity(P') = nullity(P), since $y \prec z$ in P'. □

Theorem 2.1.24. *A connected poset P dismantlable by doubly irreducibles is of nullity k if and only if the number of adjunct pairs in P counted with multiplicity is k.*

In particular, if there is no adjunct pair in P then $C(P)$ is a tree.

Proof. Suppose the poset P is dismantlable by doubly irreducibles. Then by Theorem 2.1.6, P is obtained by (non-trivial) 1-sum or 2-sum of chains. That is, $P = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2 \cdots]_{\alpha_l}C_l$, where for each i, α_i is either an adjunct element or an adjunct pair and C_i is a chain. Let r be the number of adjunct elements (counted with multiplicity) and s be the number of adjunct pairs (counted with multiplicity) in the above adjunct representation of P. Then $l = r + s$. We know that, the number of edges (or coverings) in a chain is one less than the number of elements in it. Also, 1-sum by a chain increases the number of coverings by one and 2-sum by a chain increases the number of coverings by two. Therefore, if m is the number of coverings in P then $m = (|P| - (l + 1)) + r + 2s$. As P is connected and the nullity of P is

k, $k = m - |P| + 1$. Therefore $k = s$.

Conversely, suppose the number of adjunct pairs in P counted with multiplicity is k. Then $s = k$ and $l = r + k$. As P is connected, the nullity of P is $m - |P| + 1$. But $m = (|P| - (l + 1)) + r + 2s = (|P| - (r + k + 1)) + r + 2k = |P| + k - 1$. Hence the nullity of P is k. In particular, if there is no adjunct pair in P then $s = 0$ and hence the nullity of P is 0. Therefore $C(P)$ is a tree. □

Theorem 2.1.25. *A poset P is obtained by (non-trivial) 1-sum of chains if and only if $C(P)$, the covering graph of P is a tree.*

Proof. Suppose a poset P is obtained by (non-trivial) 1-sum of chains. Therefore by Theorem 2.1.6, P is a poset dismantlable by doubly irreducibles and there is no adjunct pair in P. Hence by Theorem 2.1.24, the nullity of P is 0. Thus $C(P)$ is a tree.

Conversely, suppose P is a poset for which $C(P)$ is a tree. Let x be a pendant vertex of $C(P)$. Then $C(P) - \{x\}$ is also a tree. Now x is a pendant vertex of $C(P)$ if and only if x must have either one lower cover say a, but no upper cover in P or one upper cover say b, but no lower cover in P. (Note that, if x has at least one lower cover and at least one upper cover then the degree of x in $C(P)$ would be at least two. Also, if x has no lower cover and no upper cover in P then x would be isolated and hence $C(P)$ would not be connected.) Therefore $C(P) - \{x\} = C(P - \{x\})$. Now we use induction on $n = |P| \geq 1$. If $n = 1$ then we are done. If $n > 1$ and suppose the result is true for all posets of order $< n$. Let $Q = P - \{x\}$. Then $|Q| = n - 1 < n$. Also $C(Q) = C(P) - \{x\}$ which is a tree. Therefore by induction

hypothesis, Q is obtained by (non-trivial) 1-sum of chains. Suppose
$Q = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2 \cdots]_{\alpha_l}C_l,$ $(*)$
where for each i, α_i is an adjunct element and C_i is a chain.
Then $P = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2 \cdots]_{\alpha_i}C_i' \cdots]_{\alpha_l}C_l$, where $C_i' = C_i \cup \{x\}$ with $a \prec x$ whenever $a \in C_i$ in $(*)$ and $x \prec b$ whenever $b \in C_i$ in $(*)$. Thus P is also obtained as (non-trivial) 1-sum of chains. Hence the proof is complete. □

Using Theorem 2.1.24 and Theorem 2.1.6, we get the following.

Corollary 2.1.26. *If P is a poset such that $C(P)$ is a tree then P is dismantlable by doubly irreducibles.*

2.2 Nullity of a lattice

Recall that, nullity of a poset is the nullity of its cover graph. Therefore, in particular, the nullity of a lattice L is given by $|E(L)| - |L| + 1$.
Note that, a lattice is always connected.
We now obtain some properties of lattices with respect to nullity.

Theorem 2.2.1. *Let L be a lattice. Let $x \in L$. Then $L' = L - \{x\}$ is a sublattice of L, maintaining the nullity if and only if the element x is doubly irreducible satisfying*
(i) There are no $y, z \in Red(L)$ such that $y \prec x \prec z$ or
(ii) There are $y, z \in Red(L)$ such that $y \prec x \prec z$ and there is no other (directed) path from y to z.

Proof. Suppose $L' = L - \{x\}$ is a sublattice of L and nullity(L') = nullity(L). If x is meet reducible in L then there are $a, b \in L$ with

$a \wedge b = x$. But then $L' = L - \{x\}$ will not be sublattice of L, since $a \wedge b$ will not be maintained in L', which is a contradiction. Hence x is not meet reducible in L. Similarly, it can be proved that x is not join reducible in L. Hence $x \in Irr(L)$.
The remaining proof follows from Theorem 2.1.23.
Conversely, suppose x is doubly irreducible element satisfying the condition (i) or the condition (ii). As $x \in Irr(L)$, by Proposition 1.3.1, $L' = L - \{x\}$ is a sublattice of L. The remaining proof follows from Theorem 2.1.23. □

Let P be a poset and $x \in P$. We denote an element y by x^- if $y \prec x$ and by x^+ if $x \prec y$. Recall that, the *indegree* of an element x in a poset P is $|\{y \in P : y \prec x\}|$ and the *outdegree* of an element x in a poset P is $|\{z \in P : x \prec z\}|$.

Theorem 2.2.2. *Let L be a block. Let $x \in Irr(L)$. Then indegree and outdegree of any reducible element in L and $L - \{x\}$ are the same if and only if either there are no $y, z \in Red(L)$ with $y \prec x \prec z$ or there are $y, z \in Red(L)$ with $y \prec x \prec z$ and there is no other (directed) path from y to z.*

Proof. Suppose indegree and outdegree of any reducible element in L and $L - \{x\}$ are the same. As L is a block, let $y \prec x \prec z$. Now either at least one of y and z belongs to $Irr(L)$ or $y, z \in Red(L)$. If at least one of y and z belongs to $Irr(L)$ then x satisfies the first condition. If $y, z \in Red(L)$ then either there is another path from y to z in L or there is no other path from y to z in L. If there is another path from y

to z in L then outdegree of y and indegree of z decrease by one if x is removed from L, since $x \in Irr(L)$, which is not possible by assumption. Thus there are $y, z \in Red(L)$ with $y \prec x \prec z$ and there is no other path from y to z.

Conversely, suppose either there are no $y, z \in Red(L)$ with $y \prec x \prec z$ or there are $y, z \in Red(L)$ with $y \prec x \prec z$ and there is no other path from y to z. Let $a \in Red(L)$. Let m and n be the indegree and the outdegree of a in L respectively.

Case : 1. Suppose $a \prec x$.

Let $x \prec b$. If $b \in Red(L)$ then x must satisfy the second condition. Therefore there is no other path from a to b in L. Hence $a \prec b$ in $L - \{x\}$. Also if $b \notin Red(L)$ then x satisfies the first condition. But then $b \in Irr(L)$. As $x \in Irr(L)$ and $a \prec x \prec b$, $a \prec b$ in $L - \{x\}$.

Thus removal of x from L does not change the values of m and n.

Case : 2. Suppose $x \prec a$.

Proof in this case is similar to Case 1 above.

Case : 3. Suppose neither $a \prec x$ nor $x \prec a$.

Let $x^- \prec x \prec x^+$. Clearly $a \neq x^-$ as well as $a \neq x^+$. Therefore $x^- \prec x^+$ in $L - \{x\}$. Thus, the removal of x from L does not change the values of m and n. Hence the proof is complete. □

In Chapter 5, we enumerate the number of non-isomorphic lattices of nullity up to three. In this regard, we prove the following result.

Theorem 2.2.3. *Any lattice of nullity at most four is dismantlable.*

Proof. By Lemma 1.3.14, every lattice with n elements and $n + r$ coverings (or edges) with $-1 \leq r \leq 3$ is dismantlable. If L is a lattice on n

elements, containing m edges and having nullity k then $k = m - n + 1$. If $m = n + r$ and $-1 \leq r \leq 3$ then $k = r + 1$ and $0 \leq k \leq 4$. Hence the proof is complete. □

Theorem 2.2.4. *A dismantlable lattice L containing n elements is of nullity k if and only if L is adjunct of $k+1$ chains.*

Proof. Suppose a dismantlable lattice L containing n elements is of nullity k. If L contains m edges then the nullity $k = m - n + 1$ and hence $m = n + k - 1$. Therefore by Corollary 1.3.8, L is adjunct of $k + 1$ chains. Conversely, suppose L is adjunct of $k + 1$ chains. Again by Corollary 1.3.8, the number of edges in L is $m = n + k - 1$. Thus $k = m - n + 1$ and hence the nullity of L is k. □

Definition 2.2.1. Let $\mathscr{L}(n, k)$ be the the class of all non-isomorphic dismantlable lattices on n elements such that each lattice in it is of nullity k. Let $\mathscr{L}'(n, k)$ be the subclass of $\mathscr{L}(n, k)$ such that the reducible elements in each lattice in it are all comparable.

The purpose behind studying the class $\mathscr{L}(n, k)$ is to find the cardinality of this class. Recall that, a chain is the only lattice of nullity 0. Therefore $\mathscr{L}(n, 0)$ consists of the chain on n elements. Thakare, Pawar and Waphare [13] enumerated the classes $\mathscr{L}(n, 1)$ and $\mathscr{L}(n, 2)$. In Chapter 5, we enumerate the class $\mathscr{L}(n, 3)$. In Chapter 4, in the last section, we enumerate the class $\mathscr{L}'(n, k)$.

It is clear that the reducible elements in a lattice of nullity one are comparable. We now prove that the reducible elements in a lattice of nullity two are also comparable.

Theorem 2.2.5. *Let $L \in \mathscr{L}(n,2)$. Then the reducible elements in L are all comparable.*

Proof. As $L \in \mathscr{L}(n,2)$, L is a dismantlable lattice of nullity 2. Therefore by Theorem 2.2.4, $L = (C_0]_{a_1}^{b_1} C_1)]_{a_2}^{b_2} C_2$, where $C_0,\ C_1$ and C_2 are chains and a_1, b_1, a_2, b_2 are the only reducible elements (which may not all be distinct) of L. Clearly $a_1,\ b_1 \in C_0$. As far as the positions of a_2 and b_2 are concerned we have the following four cases.
Case (1): If $a_2,\ b_2 \in C_0$ then we are done.
Case (2): If $a_2,\ b_2 \in C_1$ then choose $C_0' = [0, a_1] \oplus C_1 \oplus [b_1, 1]$ and $C_1' = C_0 \cap (a_1, b_1)$. Let $L' = (C_0']_{a_1}^{b_1} C_1')]_{a_2}^{b_2} C_2$. Then $L = L'$ with C_0' containing all reducible elements.
Case (3): If $a_2 \in C_0$ and $b_2 \in C_1$ then $a_2 \leq a_1$. For if, suppose $a_2 > a_1$. But then we get either $a_2 || b_2$ or $a_2 > b_2$, whenever $a_2 \in C_0 \cap (a_1, b_1)$ or $a_2 \in C_0 \cap [b_1, 1]$ respectively. This is not possible. Again, if we choose $C_0' = [0, a_1] \oplus C_1 \oplus [b_1, 1]$ and $C_1' = C_0 \cap (a_1, b_1)$ then $L = L' = (C_0']_{a_1}^{b_1} C_1')]_{a_2}^{b_2} C_2$ with C_0' containing all reducible elements.
Case (4): If $a_2 \in C_1$ and $b_2 \in C_0$ then $b_1 \leq b_2$ and the remaining proof is similar to that is given in Case (3). Hence the proof is complete. □

In the following result, we obtain the bounds on the number of reducible elements of a dismantlable lattice depending on the nullity.

Lemma 2.2.6. *For $k \geq 1$, if $L \in \mathscr{L}(n,k)$ then $2 \leq |Red(L)| \leq 2k$.*

Proof. Let L be a lattice in $\mathscr{L}(n,k)$ containing r reducible elements. Now the nullity of L is $k \geq 1$. Therefore by Theorem 2.2.4, L is adjunct sum of $k+1$ chains. Therefore adjunct representation of L consists of

k adjunct pairs, say $\alpha_i = (a_i, b_i)$, $1 \leq i \leq k$. By Lemma 1.3.10, $Red(L) = \{a_i, b_i | 1 \leq i \leq k\}$. Now these reducible elements may not all be distinct. Therefore $2 \leq |Red(L)| \leq 2k$. □

Definition 2.2.2. Let $\mathscr{L}(n,k,r) = \{L \in \mathscr{L}(n,k) : |Red(L)| = r\}$. Let $\mathscr{L}'(n,k,r) = \{L \in \mathscr{L}'(n,k) : |Red(L)| = r\}$.

By Lemma 2.2.6, it follows that, for given $k \geq 1$, $\{\mathscr{L}(n,k,r) : 2 \leq r \leq 2k\}$ forms a partition of the class $\mathscr{L}(n,k)$.
It is clear that $\mathscr{L}(n,k,r) = \emptyset$ if and only if $r = 1$ and $n < k+r$. Therefore, if $\mathscr{L}(n,k,r) \neq \emptyset$ then $n \geq k+r$.
We now obtain a lower bound for the nullity of a dismantlable lattice depending on the number of reducible elements.

Proposition 2.2.7. *For any lattice in $\mathscr{L}(n,k,r)$, $k \geq [\frac{r+1}{2}]$.*

Proof. Let $L \in \mathscr{L}(n,k,r)$. We have $r \geq 0$ but $r \neq 1$. If $r = 0$, that is, if L is a chain then its nullity is 0 and we are done. Now suppose $r \geq 2$. By Lemma 2.2.6, $r \leq 2k$. That is, $\frac{r}{2} \leq k$. Therefore $k \geq \frac{r}{2}$, if r is even and $k \geq \frac{r+1}{2}$, if r is odd. □

In the following Proposition 2.2.8, we see for which k and r, the class $\mathscr{L}(n,k,r)$ coincides with the class $\mathscr{L}'(n,k,r)$.

Proposition 2.2.8. *If $k \leq 2$ or $r \leq 3$ then $\mathscr{L}(n,k,r) = \mathscr{L}'(n,k,r)$ for all $n \geq k+r$. Moreover for $r \geq 4$, the classes need not be equal.*

Proof. For $k = 0$ or 1 the proof is obvious. For $k = 2$, the proof follows from Theorem 2.2.5. Now suppose $k \geq 3$. By Lemma 2.2.6, $2 \leq r \leq 2k$. Also, using Proposition 1.3.4, a lattice which is not a chain contains at least two comparable reducible elements (one of them is a meet reducible element, say p and the other is a join reducible element, say q with $p < q$). Therefore for $r = 2$, the proof is obvious. Suppose $r = 3$ and let x, y, z be the reducible elements in a lattice $L \in \mathscr{L}(n, k, r)$. Suppose x is a meet reducible and $y > x$ is a join reducible element. If z is comparable with x and y then we are done. If $z||x$ then $z \vee x = y$, since otherwise $z \vee x$ is a reducible element other than x, y, z, which is not possible. Now $z \wedge x$ is a reducible element other than x, y, z, which is not possible. Similarly if $z||y$ then $z \wedge y = x$ and $z \vee y$ is a reducible element other than x, y, z, which is not possible. If $r = 4$ then for $k \geq 3$, $D_k \in \mathscr{L}(k+5, k, r)$ but $D_k \notin \mathscr{L}'(k+5, k, r)$ (see Fig.3).

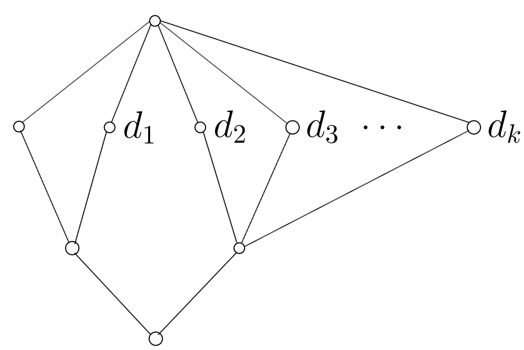


Fig.3 (D_k)

If $r \geq 5$ then $E_k = (C_{r-4} \oplus D_{k-(r-4)})]^1_{x_1}\{d_{k-r+3}\} \ldots]^1_{x_{r-4}}\{d_k\}$ is the basic block of nullity $k \geq 4$, where C_{r-4} is a chain $x_1 \prec x_2 \prec \ldots \prec x_{r-4}$. Clearly for $k \geq 4$, $E_k \in \mathscr{L}(k+r+5, k, r)$ but $E_k \notin \mathscr{L}'(k+r+5, k, r)$. $\square$

Let $L \in \mathscr{L}'(n, k, r)$. Let $(a_1, b_1), (a_2, b_2), \ldots, (a_l, b_l)$ be the distinct adjunct pairs in the adjunct representation of L, containing C as a chain containing all the r reducible elements of L. By Corollary 2.1.20, without loss, we can assume that $(a_1, b_1) < (a_2, b_2) < \ldots < (a_l, b_l)$ with respect to the dictionary order defined on $C \times C$. Let n_i be the multiplicity of an adjunct pair (a_i, b_i). Let $T_l = (n_1, n_2, \ldots, n_l)$. By Theorem 2.2.4, it is clear that $k = \sum_{i=1}^{l} n_i$. Now L is the adjunct sum of $k+1$ chains and hence contains k adjunct pairs (repetition is allowed, if any). Let $\mathscr{L}'(n, k, r, T_l) \subseteq \mathscr{L}'(n, k, r)$ be the class of lattices wherein T_l represents a fixed $l-$tuple as described above.

In the following Proposition 2.2.9, we prove that, if all the adjunct pairs in the adjunct representation of a lattice, in which all the r reducible elements are comparable, are distinct then the nullity of that lattice can not exceed $\binom{r}{2}$.

Proposition 2.2.9. *Let* $T_l = 1^l = (1, 1, \ldots, 1)$. *Then for any lattice* $L \in \mathscr{L}'(n, k, r, T_l)$, $[\frac{r+1}{2}] \leq k = l \leq \binom{r}{2}$.

Proof. By Proposition 2.2.7, $[\frac{r+1}{2}] \leq k$. Also, if $T_l = (n_1, n_2, \ldots, n_l)$ then by Theorem 2.2.4, $k = \sum_{i=1}^{l} n_i$. Therefore for $T_l = 1^l = (1, 1, \ldots, 1)$, $k = l$. Now let $L \in \mathscr{L}'(n, k, r, 1^l)$. Then the multiplicity of each adjunct pair in an adjunct representation of L is one. Therefore the number of adjunct pairs is l. But L contains r reducible elements and one adjunct pair corresponds to two reducible elements. Therefore $l \leq \binom{r}{2}$. Thus $[\frac{r+1}{2}] \leq k = l \leq \binom{r}{2}$. □

Thus, it follows that, if $L \in \mathscr{L}'(n, k, r)$ and all the adjunct pairs in the adjunct representation of L are distinct then $[\frac{r+1}{2}] \leq k = l \leq \binom{r}{2}$.

2.3 Orientability of a graph

2.3.1 Introduction

Much of the combinatorial interest in finite ordered sets is linked to the properties of two types of undirected graphs commonly used to represent them : the comparability graph and the covering graph. Note that, a pair $\{a, b\}$ of elements of a poset P is an edge of the comparability graph of P if $a < b$ in P.

Definition 2.3.1. A graph G is said to be *orientable* as an ordered set P if G and $C(P)$ are isomorphic as graphs.

The following open problem is posed by O. Ore [20].
Ore's Open Problem.
Characterize graphs which are cover graphs. That is, characterize those graphs which are orientable as an ordered set.
The problem is still open.
Orientability of graphs is already studied (see [47]) in terms of the girth and the chromatic number of a graph.

Definition 2.3.2. The *girth* $g(G)$ of a graph G with a cycle is the length of its shortest cycle. A graph with no cycle has infinite girth. The *chromatic number* of a graph G is the smallest number of colors $\chi(G)$ needed to color the vertices of G so that no two adjacent vertices share the same color.

Theorem 2.3.1. *[47]. If $\chi(G) < g(G)$ then the graph G is orientable.*

It is known that there are graphs of arbitrarily large girth that are not covering graphs (see [47]). This was the conjucture of Bollobas proved by Nešetřil and Rödl (see [56] and [57]) using probabilistic methods.
It is also well known that a graph G is the comparability graph of an ordered set if and only if each odd cycle of G has a triangular chord (see Ghouila-Houri [51] and Gilmore and Hoffman [52]). In contrast little is known about this question (see [20]) : when is a graph the covering graph of an ordered set? Also, it is NP-complete to test whether a graph is a cover graph (see [57] and [60]). This question is already solved for finite distributive lattices. We settle this question for posets dismantlable by doubly irreducibles in this section.

Theorem 2.3.2. *[42]. A finite graph G is the covering graph of a distributive lattice of length n if and only if G is a retract of $C(Q_n)$ and $diam(G) = n$.*

Similar types of characterizations are obtained for Modular Lattices (see J.Jakubik [43]) and Geometric Lattices (see Duffus and Rival [40]).
H. Grötzsch [46] has shown that triangle-free planar graphs are 3-chromatic; consequently, they are orientable.
We give a partial solution to the open problem of orientability by characterizing covering graphs of posets dismantlable by doubly irreducibles and dismantlable lattices. For this, we introduce the concept of an adjunct of ears in graphs in the next subsection.

2.3.2 Adjunct of ears

We introduce here the concept of an adjunct of ears in graphs.

Definition 2.3.3. Let G be any directed graph and P be any directed path (ear) from c to d with $V(G) \cap V(P) = \emptyset$. Let a be a vertex of G. We define the *u-adjunct* of P to G at a to be a directed graph, denoted by $G]_a P$, having vertex set $V(G) \cup V(P)$ and arc set $A(G) \cup A(P) \cup \{(a, c)\}$.
We define the *d-adjunct* of P to G at a to be a directed graph, denoted by $G]^a P$, having vertex set $V(G) \cup V(P)$ and arc set $A(G) \cup A(P) \cup \{(d, a)\}$.
We define the *ud-adjunct* of P to G at (a, b), where (a, b) is a pair of vertices in G such that there is a directed path from a to b in G of length at least 2, to be a directed graph, denoted by $G]_a^b P$, having vertex set $V(G) \cup V(P)$ and arc set $A(G) \cup A(P) \cup \{(a, c), (d, b)\}$.

We say that a directed graph G is *adjunct of directed ears* if it can be obtained by u-adjunction or d-adjunction or ud-adjunction of directed ears starting with a directed path.
An underlying graph of a directed graph which is adjunct of directed ears is called simply *adjunct of ears*.
The u-adjunct (or d-adjunct) of P to G at a is *trivial* if a is pendant; otherwise, it is *non-trivial*. We say adjunct of ears is *non-trivial* if all u-adjunct and d-adjunct are non-trivial.

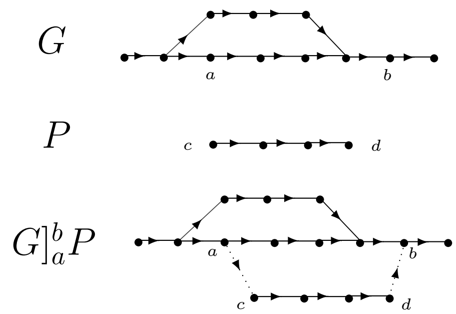


Fig.4 The ud-adjunct of an ear P to a graph G at (a, b)

Note that, the ud-adjunct of P to G at (a, b) is nothing but the u-adjunct of P to G at a and the d-adjunct of P to G at b, simultaneously.

2.3.3 A partial solution to Ore's open problem

As a consequence of the structure theorem (Theorem 2.1.6), we give a partial solution to Ore's open problem in the following Theorem 2.3.3.

Theorem 2.3.3. *A graph is orientable as a poset dismantlable by doubly irreducibles if and only if it is (non-trivial) adjunct of ears.*

Proof. Suppose a graph G is orientable as a poset P dismantlable by doubly irreducibles. Therefore $G \cong C(P)$. By the structure theorem (see Theorem 2.1.6), P can be written as (non-trivial) 1-sum or 2-sum of chains. Suppose $P = C_0]_{\alpha_1} C_1]_{\alpha_2} C_2 \cdots]_{\alpha_k} C_k$, where for each i, α_i is an adjunct element or an adjunct pair and C_i is a chain. Choose $E_0 = C_0$. For $1 \leq i \leq k$, let C_i be a chain $x_1^i \prec x_2^i \prec \cdots \prec x_{m_i}^i$.
Case I : Suppose α_i is an adjunct element.
If α_i corresponds to up 1-sum then choose E_i as an ear $\alpha_i - x_1^i - x_2^i - \cdots - x_{m_i}^i$. And if α_i corresponds to down 1-sum then choose E_i as an

ear $x_1^i - x_2^i - \cdots - x_{m_i}^i - \alpha_i$.

Case II : Suppose α_i is an adjunct pair, say (a, b). Then choose E_i as an ear $a - x_1^i - x_2^i - \cdots - x_{m_i}^i - b$.

Let $G' = E_0]_{\alpha_1} E_1]_{\alpha_2} E_2 \cdots]_{\alpha_k} E_k$. Then G' is a (non-trivial) adjunct of ears, $E_0, E_1, \ldots, E_k$. Note that, there is a directed path from a to b in G of length at least 2, since in P, $a < b$ but $a \not\prec b$. But then $C(P) \cong G'$ under identity vertex map. Hence $G \cong G'$. Thus G is a (non-trivial) adjunct of ears.

Conversely, suppose a graph G is a (non-trivial) adjunct of ears. Let D be a directed graph whose underlying graph is G. Therefore D is adjunct of directed ears, say $F_0, F_1, \ldots, F_l$. For each j, $0 \leq j \leq l$, let F_j be the directed ear $y_1^j - y_2^j - \cdots - y_{n_j}^j$.

Let C_0 be the chain $y_1^0 \prec y_2^0 \prec \cdots \prec y_{n_0}^0$. For $1 \leq j \leq l$,

Case I : If F_j corresponds to u-adjunct of ear then choose $\beta_j = y_1^j$ and C_j as the chain $y_2^j \prec y_3^j \prec \cdots \prec y_{n_j}^j$.

Case II : If F_j corresponds to d-adjunct of ear then choose $\beta_j = y_{n_j}^j$ and C_j as the chain $y_1^j \prec y_2^j \prec \cdots \prec y_{n_j - 1}^j$.

Case III : If F_j corresponds to ud-adjunct of ear then choose $\beta_j = (y_1^j, y_{n_j}^j)$ and C_j as the chain $y_2^j \prec y_3^j \prec \cdots \prec y_{n_j - 1}^j$.

Let $Q = C_0]_{\beta_1} C_1]_{\beta_2} C_2 \cdots]_{\beta_l} C_l$. In Case I and Case II, β_j becomes an adjunct element and in Case III, β_j becomes an adjunct pair, since there is a path in the subgraph $F_0 \cup F_1 \cup \cdots \cup F_{j-1}$ of D, joining y_1^j and $y_{n_j}^j$ whose length is at least two.

As u-adjunct and d-adjunct are non-trivial, 1-sums in Q are also non-trivial. Thus, Q is obtained by (non-trivial) 1-sum or 2-sum of chains.

By Theorem 2.1.6, Q is a poset dismantlable by doubly irreducibles.
Claim : (c, d) is an arc in D if and only if (c, d) is an edge in Q.
Now (c, d) is an arc in D if and only if (c, d) is an arc in F_j, for some $0 \leq j \leq l$ if and only if either (c, d) is an edge in C_j, for some $0 \leq j \leq l$ or $c = y_1^j$ and $d = y_2^j$ for some $1 \leq j \leq l$ or $c = y_{n_j-1}^j$ and $d = y_{n_j}^j$ for some $1 \leq j \leq l$ if and only if either (c, d) is an edge in C_j, for some $0 \leq j \leq l$ or c (but not d) is an adjunct element in Q or d (but not c) is an adjunct element in Q if and only if (c, d) is an edge in Q.
Therefore $G \cong C(Q)$. Thus, G is orientable as an ordered set Q. □

As a consequence of Theorem 1.3.6, we have the following result.

Corollary 2.3.4. *A graph is orientable as a dismantlable lattice if and only if it is ud-adjunct of ears.*

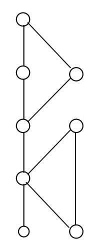
Proof. Dismantlable lattice is dismantlable poset by doubly irreducibles. Also, by Theorem 1.3.6, it can be written as (only) 2-sum of chains. Hence the result follows by Theorem 2.3.3. □

Theorem 2.3.5. *If a graph G is orientable as a lattice in which all the reducible elements are comparable then G is connected and contains a chordless path passing through all the higher degree (≥ 3) vertices.*

Proof. Let L be a lattice in which all the reducible elements are comparable. By Corollary 2.1.22, L is dismantlable. Let G be a graph such that $G \cong C(L)$. That is, G is orientable as lattice L. Clearly G is connected, since $C(L)$ is connected as L is connected. By Theorem 2.1.7, if C_0 is a maximal chain containing all the reducible elements of L then $L = C_0]_{\alpha_1} C_1]_{\alpha_2} C_2 \cdots]_{\alpha_k} C_k$ where $C_1, C_2, \ldots, C_k$ are chains.

Let P be a path in G isomorphic to C_0 in $C(L)$. Then P is chordless path passing through all the higher degree (≥ 3) vertices, since C_0 is chordless path in $C(L)$ as L is an adjunct of chains. □

However, the converse of Theorem 2.3.5 is not true, as it can be easily seen in the following figure.



2.4 An ear decomposition of a graph

2.4.1 Introduction

Ear decompositions have a number of uses, in particular in computing the connectivity of a graph. For instance, Theorem 2.4.1 is well known. The following problems (of finding algorithms) are posed by Y Maon, B. Schieber and U. Vishkin (see [48]).

The ear decomposition problem :

Find an ear decomposition starting with E_0.

The open ear decomposition problem :

Find an open ear decomposition starting with E_0.

Ear decomposition has the flavor of a *general* search technique in graphs. It arranges the vertices of the graph by partitioning them into paths. This enables further exploration of the graph in an "orderly" manner. Such a search technique is called an *Ear-Decomposition Search* (EDS).

It is known that Depth-First Search (DFS) and Breadth-First Search (BFS) are main techniques for searching graphs.

Recall that, an *ear* of a loopless connected graph G is a subgraph of G such that it is a maximal path in which all internal vertices are of degree 2 in G or it is a cycle in which all but one vertex have degree 2 in G. If G is a cycle (or path) itself then that cycle (or path) is the only ear of G.

Definition 2.4.1. An ear of a graph G is called an *open ear* if the two end points do not coincide in G.

Let G be a connected loopless graph and E be an ear in G. By $G - E$ we mean a subgraph of G obtained from G by removing all the internal edges of E and then all the isolated vertices. We now define an ear decomposition of a graph in the following.

Definition 2.4.2. Let G be a loopless, connected graph. An *ear decomposition* of G is a partition of its set of edges into a sequence of ears $E_0, E_1, E_2, \cdots, E_k$, such that (i) for each i, E_i is a cycle or a path of G and (ii) $E_0 \cup E_1 \cup \cdots \cup E_i$ is connected and having E_i as an ear of $E_0 \cup E_1 \cup \cdots \cup E_i$, for all $i = 1, 2, \ldots, k$.

If E_i is a cycle then it is attached to $E_0 \cup E_1 \cup \cdots \cup E_{i-1}$ by exactly one vertex. If E_i is a path then it is attached to $E_0 \cup E_1 \cup \cdots \cup E_{i-1}$ by at least one end vertex. Clearly $G = \bigcup_{i=0}^{k} E_i$.

An *open ear decomposition* of a loopless, connected graph G is an ear decomposition of G in which all the ears (except the first) are open.

In the Appendix, we have depicted in all 75 (cover) graphs. It can be easily observed that each one of them has an (open) ear decomposition starting with a maximal path or a cycle.

There are also some other kinds of ear decompositions, viz, nested ear decomposition (see Eppstein[50]) and tree ear decomposition (see Khuller[49]). Borse and Waphare[45] studied the critically 2-connected graphs using nested ear decomposition. It can be observed that every Hamiltonian graph has an ear decomposition starting with a Hamiltonian cycle. Thus by Tutte's theorem, every planar 4-connected graph (being Hamiltonian) has an ear decomposition.

Ear decompositions have a number of uses, in particular in computing the *connectivity* of a graph.

For instance, the following Theorem 2.4.1 is well known.

Theorem 2.4.1. *(H. Whitney[44])*
A graph is biconnected (2-vertex connected) if and only if it has an open ear decomposition starting with a cycle.

2.4.2 Whitney type characterization

Lemma 2.4.2. *Let G be a tree. Let P be a maximal path in G. Then G has an ear decomposition $E_0, E_1, \dots, E_k$ such that $E_0 = P$ and $G = E_0 \cup E_1 \cup \cdots \cup E_k$.*

Proof. Suppose G is a tree and P is a maximal path in G. Using induction on $n = |E(G)| \geq 1$. If $n \leq 3$ then we are done. Now suppose $n > 3$ and the result is true for all graphs containing the number of edges strictly less than n. If $G = P$ then we are done. Otherwise, G contains at least one vertex of degree at least 3. That means, G

contains at least three pendant vertices. Let $x \notin P$ be a pendant vertex of G. Consider an ear $E = x_1 - x_2 - \cdots - x_t = x$ such that the degree of x_1 is at least three in G. Let $G' = G - E$. Then G' is also a tree containing P with $|E(G')| \leq n-1 < n$. Therefore, by induction hypothesis, G' has an ear decomposition $E_0, E_1, \ldots, E_l$ such that $E_0 = P$ and $G' = E_0 \cup E_1 \cup \cdots \cup E_l$. Now degree of x_1 in G' is at least two. As one end x_1 of E is attached to G', E is an ear of $G = E_0 \cup E_1 \cup \cdots \cup E_l \cup E$. Hence G has an ear decomposition $E_0, E_1, \ldots, E_l, E$ such that $E_0 = P$ and $G = E_0 \cup E_1 \cup \cdots \cup E_l \cup E$. □

Lemma 2.4.3. *Let G be a connected loopless graph containing a cycle C. Then G has an ear decomposition $E_0, E_1, \ldots, E_k$ such that $E_0 = C$ and $G = E_0 \cup E_1 \cup \cdots \cup E_k$.*

Proof. Suppose a loopless graph G is connected and contains a cycle C. Using induction on $n = |E(G)| \geq 2$. If $n = 2$ then G is a cycle of length two and we are done. Now suppose $n > 2$ and the result is true for all graphs containing the number of edges strictly less than n. If G is a block itself then it is 2-connected and the proof follows from Theorem 2.4.1. If G is not a block then it has at least two pendant blocks. Without loss, suppose B is a pendant block in G not containing C. Note that B shares exactly one vetrex(cut-vertex), say a with $G-(B-\{a\})$. As G is connected, B is either an edge(cut-edge) or a maximal 2-connected subgraph of G.

Case I : If B is an edge $E = \{a, b\}$ then consider an ear $E' = x_1 - x_2 - \cdots - a - x_t = b$ such that the degree of x_1 is at least three in G. Let $G' = G - E'$. Then G' is a connected loopless graph containing C

and $|E(G')| \leq n-1 < n$. Therefore by induction hypothesis, G' has an ear decomposition $E_0, E_1, \ldots, E_m$ such that $E_0 = C$ and $G' = E_0 \cup E_1 \cup \cdots \cup E_m$. Now E' shares exactly one vetrex x_1 with G'. Therefore $G = \left(\bigcup_{i=0}^{m} E_i\right) \cup E'$ and hence G has the required ear decomposition.

Case II : Suppose B is a maximal 2-connected subgraph of G. Let C' be a cycle containing a in B. Then by Theorem 2.4.1, B has an ear decomposition $E'_0, E'_1, \ldots, E'_s$ such that $E'_0 = C'$ and $B = E'_0 \cup E'_1 \cup \cdots \cup E'_s$. Now $G'' = G - (B - \{a\})$ is a connected loopless graph containing a cycle C and $|E(G')| \leq n-2 < n$. Therefore by induction hypothesis, G'' has an ear decomposition $E''_0, E''_1, \ldots, E''_t$ such that $E''_0 = C$ and $G'' = E''_0 \cup E''_1 \cup \cdots \cup E''_t$. Now B shares exactly one vetrex a with G'' and $a \in E'_0$. Therefore E'_0 is an ear of G''. Therefore G has an ear decomposition $E''_0, E''_1, \ldots, E''_t, E'_0, E'_1, \ldots, E'_s$ such that $E''_0 = C$ and $G = E''_0 \cup E''_1 \cup \cdots \cup E''_t \cup E'_0 \cup E'_1 \cup \cdots \cup E'_s$. Hence the proof. □

As a consequence of Lemma 2.4.2 and Lemma 2.4.3 and using definition of an ear decomposition of a graph we get a Whitney type theorem as given below.

Theorem 2.4.4. *Let G be a loopless graph. Then G is connected if and only if it has an ear decomposition starting with a maximal path or a cycle.*

Corollary 2.4.5. *If P is a connected poset dismantlable by doubly irreducibles then $C(P)$ has an ear decomposition starting with a maximal path or a cycle.*

Proof. By Theorem 2.1.6, P is obtained by (non-trivial) 1-sum or 2-sum of chains. Therefore $C(P)$ is simple. Now $C(P)$ is connected. Hence by Theorem 2.4.4, $C(P)$ has an ear decomposition starting with a maximal path or a cycle. □

However, the converse is not true. For example, the covering graph of a crown is a cycle but it is not a poset dismantlable by doubly irreducibles.

In fact, as a consequence of Theorem 2.1.7, it follows that, if P is a connected poset dismantlable by doubly irreducibles then $C(P)$ has an open ear decomposition starting with a maximal path.

Chapter 3

Basic blocks

In this Chapter, we study basic blocks associated to posets/lattices. In the first section, we introduce the concept of a basic block (which depends on the concept of nullity) for posets. The second section deals with various properties of basic blocks associated to the posets/lattices. We prove that, a basic block associated to a poset is a retract of that poset. We also obtain a characterization of a basic block in which all the reducible elements are comparable. In the third section, we prove the result, namely, if two basic blocks are non-isomorphic then the posets associated by these basic blocks are also non-isomorphic. As a consequence, we obtain the result, namely, there is a *unique* basic block associated to any poset. In the last section, we introduce the concept of a fundamental basic block and obtain various properties of fundamental basic blocks associated to dismantlable lattices. Using these two concepts, we enumerate certain classes of non-isomorphic lattices on n elements in the subsequent chapters.

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[0]The paper based on partial content of this Chapter has been presented at the 3^{rd} International Conference on Discrete Mathematics (ICDM-2013) held at Karnataka University, Dharwad (India) during 10^{th} to 14^{th} June, 2013.

Recall that, the nullity of a poset P is the nullity of its cover graph $C(P)$. Therefore, the nullity of a poset P is given by $|E(P)| - |P| + c$, where c is the number of components of $C(P)$. Note that, if P is a lattice then $c = 1$. We now introduce the concept of a basic block.

Definition 3.0.3. A poset P is a *basic block* if it is one element or $Irr(P) = \emptyset$ or removal of any doubly irreducible element reduces nullity by one.

For example, a cube $\mathbf{2}^3$ is a basic block. Note that, by Proposition 1.3.2, if P is a dismantlable lattice then $Irr(P) \neq \emptyset$. Therefore, a dismantlable lattice L is a basic block if it is one element or removal of any doubly irreducible element reduces nullity by one. For example, M_2 (see Fig.5) is a basic block.

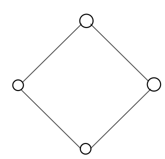


Fig.5 (M_2)

3.1 Basic block associated to posets

In sequel, we introduce the concept of a basic block associated to a poset.

Definition 3.1.1. Let P be a poset. Consider a (Hasse) diagram of P. If $Irr(P) = \emptyset$ then we say that P is a basic block associated to itself. If $Irr(P) \neq \emptyset$ and P is chain then replace it by the smallest element in it and call that element a basic block associated to that chain; otherwise, if $C : x_1 \prec x_2 \prec \cdots \prec x_r$ is any maximal chain of doubly irreducible elements of P then

1. remove C from P if either x_1 has no lower cover or x_r has no upper cover or there is no other directed path from a to b in P whenever $a \prec x_1, x_r \prec b, a, b \in Red(P)$ and

2. remove C except x_1 from P if $a \prec x_1, x_r \prec b, a, b \in Red(P)$ and there is another directed path from a to b in P.

Perform the operation of deletion till there does not remain any chain of type C in P. The resultant subgraph of this directed graph (diagram of P) is a subposet of P, called a *basic block associated to* P.

If B is a basic block associated to a poset P then we also say that P is associated by B.

For example, a crown (see Fig.1) is a basic block associated to itself. M_2 (see Fig.5) is (the) basic block associated to any lattice of nullity one (see Fig.7). In Fig.6, we have depicted all the basic blocks (see Proposition 3.2.5) associated to lattices of nullity two.

3.2 Properties of basic blocks

In the following, we give some properties of basic blocks associated to posets.

Theorem 3.2.1. *Let B be a basic block associated to a poset P. Then*
(i) B is a sublattice of P whenever P is a lattice.
(ii) $nullity(B) = nullity(P)$.

(iii) $Red(B) \subseteq Red(P)$. Further, an equality holds if P is a lattice.
(iv) $Irr(B) \subseteq Irr(P)$.
(v) If an ear is trivial (i.e., of length 1) in B associated to a pair (a, b) then there is no other path from a to b in P and hence there is a unique ear associated to (a, b) in P. Conversely, if there is no other path from a to b in P then there is no non-trivial ear associated to (a, b) in B.
(vi) If $x \in Irr(B)$ and x is associated to a pair (a, b) in B then x is associated to the pair (a, b) in P also. Moreover, every ear in B is either of length 1 or 2.
(vii) If $x \in Irr(B)$ and $x^- \prec x \prec x^+$ in B then $x^-, x^+ \notin Irr(B)$; that is, $x^-, x^+ \in Red(B)$. Moreover, nullity$(B - \{x\}) =$ nullity$(B) - 1$.
(viii) The number of trivial ears in B is greater than or equal to that in P.
(ix) A non-trivial ear in P associated to (a, b) if it exists, becomes a trivial ear in B if and only if there is no other path from a to b in P.
(x) If there is a non-trivial ear associated to (a, b) in B then the number of non-trivial ears (or the number of doubly irreducibles) associated to (a, b) in B is equal to the number of non-trivial ears associated to (a, b) in P.
(xi) The number of ears associated to (a, b) in B is equal to the number of ears associated to (a, b) in P.

Proof. (i) By definition of a basic block associated to a poset and by Proposition 1.3.1, B is clearly a sublattice of P whenever P is a lattice.
(ii) By definition of a basic block associated to a poset and by repeated use of Theorem 2.1.23, nullity$(B) =$ nullity(P).

(iii) As B is a subposet of P obtained by removal of some doubly irreducible elements, it is clear that $Red(B) \subseteq Red(P)$. Now, if P is a lattice then by the definition of a basic block associated to a poset and by the repeated use of Theorem 2.2.2, it is clear that $Red(P) \subseteq Red(B)$. Thus, $Red(B) = Red(P)$.

(iv) Follows from the definition of a basic block associated to a poset.

(v) Let $E : a \prec b$ be a trivial ear in B associated to the pair (a, b). Let E' be the ear associated to (a, b) in P containing E. If $E' = E$ then clearly there is no other path from a to b in P. If $E' \neq E$ then E' is non-trivial ear. If there is another path from a to b in P then there is an element say x of E' such that $x \in B$ and x is associated to (a, b) in B. This is not possible, since $E : a \prec b$ is a trivial ear in B associated to the pair (a, b). Therefore, there is no another path from a to b in P. Hence E' is a unique ear associated to the pair (a, b) in P. The converse follows from the definition of a basic block associated to a poset.

(vi) First part is obvious. Now, suppose there is an ear E associated to (a, b) in B of length at least three. Let $x \prec y$ be the elements of E. Then nullity$(B - \{x\})$ = nullity(B), a contradiction. Therefore, every ear in B is of length at most two.

(vii) Suppose $x \in Irr(B)$. Let E be the ear containing x. If either x^- or x^+ or both are in $Irr(B)$ then as $x^- \prec x \prec x^+$ in B, the length of E is at least three, a contradiction by (vi). Hence $x^-, x^+ \in Red(B)$. Now, the ear $E : x^- \prec x \prec x^+$ is non-trivial in B. Therefore, using the converse part of (v), there is an another path from x^- to x^+ in B.

Hence, nullity$(B - \{x\}) = |E(B - \{x\})| - |B - \{x\}| + 1 = (|E(B)| - 2) - (|B| - 1) + 1 = |E(B)| - |B| = \text{nullity}(B) - 1$.

(viii) Note that, if an ear associated to a pair (a, b) is trivial in P then it is also trivial in B and the proof follows from the definition of a basic block associated to a poset.

(ix) Let $E : a \prec x_1 \prec x_2 \prec \cdots \prec x_r \prec b$ be a non-trivial ear associated to the pair (a, b) in P. Using contrapositive method, suppose there is another path from a to b in P. Then by the definition of a basic block associated to a poset, $a \prec x_1 \prec b$ is a non-trivial ear associated to (a, b) in B.

The converse follows from the definition of a basic block, as there is no another path from a to b in L then we can remove each x_i from E to obtain B and hence E becomes a trivial ear associated to (a, b) in B.

(x) Suppose an ear E associated to (a, b) is non-trivial in B. Let $m \geq 1$ be the number of non-trivial ears associated to (a, b) in B. Let n be the number of non-trivial ears associated to (a, b) in P. From the first part of (vi), it is clear that $m \leq n$. Now, as there is a non-trivial ear E associated to (a, b) in B, by the converse part of (v), there is another path from a to b in P. Therefore by (ix), there are n non-trivial ears associated to (a, b) in B. Therefore $n \leq m$. Thus $m = n$.

(xi) Suppose there is a trivial ear in B associated to (a, b). Since there can not be more than one trivial ear in B associated to (a, b), by (v), there is a unique ear associated to (a, b) in P. Now suppose there is a non-trivial ear in B associated to (a, b). But then the proof follows from (x). □

In the following Proposition 3.2.2, we obtain a partition of $Irr(B)$ for a basic block B. Let $S = \{(a,b) \mid$ there is a doubly irreducible element associated to (a,b) in B, where $a, b \in Red(B)\}$. For $(a,b) \in S$, let $S_B(a,b)$ denote the set of all doubly irreducible elements associated to (a,b) in B. It clearly follows from (vi) of Theorem 3.2.1 that, if $(a,b) \in S$ and E is an ear associated to (a,b) in a basic block B then $l(E) = 2$ and hence E is non-trivial.

Proposition 3.2.2. *Let B be a basic block. Then*

1. For each $(a,b) \in S$, $S_B(a,b) \neq \emptyset$.

2. For all $(a,b),(c,d) \in S$ with $(a,b) \neq (c,d)$, $S_B(a,b) \cap S_B(c,d) = \emptyset$.

3. $Irr(B) = \displaystyle\bigcup_{(a,b)\in S} S_B(a,b).$

4. $\{S_B(a,b) | (a,b) \in S\}$ forms a partition of $Irr(B)$.

Proof. Let $(a,b) \in S$. If E is an ear associated to (a,b) in B then it is non-trivial. Therefore $S_B(a,b) \neq \emptyset$.

Claim 1 : For all $(a,b),(c,d) \in S$ with $(a,b) \neq (c,d)$,
$S_B(a,b) \cap S_B(c,d) = \emptyset$.

For if, suppose $S_B(a,b) \cap S_B(c,d) \neq \emptyset$. Let $x \in S_B(a,b) \cap S_B(c,d)$. Therefore $x \in S_B(a,b)$ and $x \in S_B(c,d)$. Therefore x is a doubly irreducible element associated to (a,b) as well as (c,d) in B. Therefore $a \prec x \prec b$ and $c \prec x \prec d$.

Therefore $a = c$ and $b = d$, since $x \in Irr(B)$. Thus, $(a,b) = (c,d)$, a contradiction.

Claim 2 : $Irr(B) = \displaystyle\bigcup_{(a,b)\in S} S_B(a,b)$.

Clearly for each $(a,b) \in S$, $S_B(a,b) \subseteq Irr(B)$.

Therefore $\bigcup_{(a,b)\in S} S_B(a,b) \subseteq Irr(B)$.

Now by (vi) of Theorem 3.2.1, if $x \in Irr(B)$ and $x^- \prec x \prec x^+$ in B then $x^-, x^+ \in Red(B)$ and therefore x is associated to (x^-, x^+) in B, that is, $x \in S_B(x^-, x^+)$ and $(x^-, x^+) \in S$. Therefore, $Irr(B) \subseteq \bigcup_{(a,b)\in S} S_B(a,b)$.

Hence $Irr(B) = \bigcup_{(a,b)\in S} S_B(a,b)$.

Thus $\{S_B(a,b) | (a,b) \in S\}$ forms a partition of $Irr(B)$. □

In the following Theorem 3.2.3, we prove that, a basic block associated to a lattice is a retract of that lattice.

Theorem 3.2.3. *If B is a basic block associated to a lattice L then B is a retract of L.*

Proof. Consider the diagram of a lattice L as a digraph. Suppose B is a basic block associated to L. Let $C : x_1 \prec x_2 \prec \cdots \prec x_r$ be a maximal chain in L, where $x_i \in Irr(L)$, for all i.

Define a map $\phi : L \to B$ as follows.

If $L = C$ then define $\phi(x) = x_1$, for all $x \in C$.

If $L \neq C$ then

1. For all $x \in Red(L)$, define $\phi(x) = x$.
2. If $a, b \in Red(L)$ are such that $a \prec x_1$ and $x_r \prec b$ and there is no other directed path from a to b then define $\phi(x) = a$, for all $x \in C$.
3. If $a, b \in Red(L)$ are such that $a \prec x_1$ and $x_r \prec b$ and there is another directed path from a to b then define $\phi(x) = x_1$, for all $x \in C$.
4. If x_1 has no lower cover but x_r has an upper cover, say y (in fact, $y \in Red(L)$), then define $\phi(x) = y$, for all $x \in C$.

5. If x_r has no upper cover but x_1 has a lower cover, say z (in fact, $z \in Red(L)$), then define $\phi(x) = z$, for all $x \in C$.

It is clear from the definition of a basic block associated to a lattice and the definition of ϕ (see 1, 2 and 3) that ϕ is the identity map on B.

Claim : ϕ is an order-preserving map.

Let $x \leq y$ in L. Then we have the following four cases.

Case : 1. If $x, y \in Red(L)$ then clearly $\phi(x) \leq \phi(y)$, since ϕ is the identity map on $Red(L)$.

Case : 2. If $x, y \in Irr(L)$ then we have the following two subcases.

i) If $x, y \in C$, a maximal chain such that $C \subseteq Irr(L)$ then $\phi(x) = \phi(y)$.

ii) If $x \in C_1$ and $y \in C_2$, where C_1, C_2 are some maximal chains such that $C_1 \subseteq Irr(L)$ and $C_2 \subseteq Irr(L)$ then there exists at least one $z \in Red(L)$ such that $x < z < y$. Therefore, by the definition of ϕ, $\phi(x) \leq \phi(y)$.

3. If $x \in Red(L)$ and $y \in Irr(L)$ then $\phi(x) = x$ and if $y \in C$, a maximal chain such that $C \subseteq Irr(L)$ then either $\phi(y) = x$ or $x < \phi(y)$, that is, $x \leq \phi(y)$, that is, $\phi(x) \leq \phi(y)$.

4. If $x \in Irr(L)$ and $y \in Red(L)$ then $\phi(y) = y$ and if $x \in C$, a maximal chain such that $C \subseteq Irr(L)$ then either $\phi(x) = y$ or $\phi(x) < y$, that is, $\phi(x) \leq y$. Therefore, $\phi(x) \leq \phi(y)$.

Thus ϕ preserves the order. Hence ϕ is a retraction map.

Now, it follows from the definition of a basic block associated to a lattice and the definition of ϕ, that ϕ is onto, since for all $y \in B$,

1. if $y \in Red(B)$ then by (iii) of Theorem 3.2.1, $y \in Red(L)$ and hence

$\phi(y) = y$, and

2. if $y \in Irr(B)$ then by Proposition 3.2.2, suppose y is associated to (a, b) in B. By (vi) of Theorem 3.2.1, y is associated to (a, b) in L. Also, by the converse part of (v) of Theorem 3.2.1, there must be another path from a to b in L. Hence, by the definition of ϕ (see 3), $\phi(x) = y$ for all $x \in E$, the ear (containing y) associated to (a, b) in L. Thus ϕ is onto. Hence B is a retract of L. □

Definition 3.2.1. A dismantlable lattice which is a block is said to be a *dismantlable block*. Let $\mathscr{B}(n, k)$ be the class of all non-isomorphic dismantlable blocks on n elements such that each block in it has nullity k. Let $\mathscr{B}'(n, k)$ be the subclass of $\mathscr{B}(n, k)$ such that the reducible elements in each block in it are all comparable.

By (vii) of Theorem 3.2.1, it follows that, if B is a basic block in $\mathscr{B}(n, k)$ and $x \in Irr(B)$ then $B - \{x\} \in \mathscr{L}(n - 1, k - 1)$.
In the following, we give the characterization of basic blocks in which the reducible elements are all comparable.

Theorem 3.2.4. *A block $B \in \mathscr{B}'(n, k)$ is a basic block if and only if $B = C_0]_{a_1}^{b_1} C_1]_{a_2}^{b_2} C_2 \cdots]_{a_k}^{b_k} C_k$ with $a_i, b_i \in C_0$, satisfying (i) $|C_i| = 1$, for all i, $1 \leq i \leq k$, (ii) $|C_0| = |Red(B)| + m$, where m is the number of distinct adjunct pairs (a_i, b_i) such that the interval $(a_i, b_i) \subseteq Irr(B)$ and (iii) $n = |C_0| + k$.*

Proof. Suppose a block $B \in \mathscr{B}'(n, k)$ is a basic block. Therefore by Theorem 2.2.4, it is adjunct of $k + 1$ chains. Now all the reducible elements in B are comparable. Therefore by Theorem 2.1.7,

$B = C_0]_{a_1}^{b_1} C_1]_{a_2}^{b_2} C_2 \cdots]_{a_k}^{b_k} C_k$, where C_0 is a maximal chain with $a_i, b_i \in C_0$, for all $i,\ 1 \leq i \leq k$.

Suppose for some $i,\ 1 \leq i \leq k$, that $|C_i| > 1$. Then there exist $x, y \in C_i \cap Irr(B)$, since $Red(B) \subseteq C_1$. But then $B - \{x\} \in \mathscr{L}'(n-1, k)$, a contradiction, since B is a basic block. Therefore $|C_i| = 1$ for all $i,\ 1 \leq i \leq k$. Therefore $n = |B| = |C_0| + k$. Suppose $(a_{i_1}, b_{i_1}), (a_{i_2}, b_{i_2}), \cdots, (a_{i_m}, b_{i_m})$ are the adjunct pairs such that for each $j,\ 1 \leq j \leq m$, the interval $(a_{i_j}, b_{i_j}) \subseteq Irr(B)$. Therefore for each $j,\ 1 \leq j \leq m$, $|(a_{i_j}, b_{i_j}) \cap C_0| = 1$, since B is a basic block. Also, there is no x such that $x \in C_0 \cap Irr(B)$ but $x \notin (a_i, b_i)$ for all $i,\ 1 \leq i \leq k$. For if, suppose there is $x \in C_0 \cap Irr(B)$ but $x \notin (a_i, b_i)$ for all $i,\ 1 \leq i \leq k$, then $B - \{x\} \in \mathscr{L}'(n-1, k)$, a contradiction, since B is a basic block. Therefore $|Irr(B)| = k + m$, since $|C_i| = 1$ and $C_i \subseteq Irr(B)$ for all $i,\ 1 \leq i \leq k$. Also $|B| = |Red(B)| + |Irr(B)|$. But $|B| = |C_0| + k$ implies that $|C_0| + k = |Red(B)| + |Irr(B)|$. Hence $|C_0| = |Red(B)| + m$, since $|Irr(B)| = k + m$.

Conversely, suppose $B = C_0]_{a_1}^{b_1} C_1]_{a_2}^{b_2} C_2 \cdots]_{a_k}^{b_k} C_k$ is a block with $a_i, b_i \in C_0$, satisfying (i) $|C_i| = 1$, for all $i,\ 1 \leq i \leq k$, (ii) $|C_0| = |Red(B)| + m$, where m is the number of distinct adjunct pairs (a_i, b_i) such that the interval $(a_i, b_i) \subseteq Irr(B)$ and (iii) $n = |C_0| + k$. Now, by Theorem 2.2.4, the nullity of B is k. Therefore by assumption $B \in \mathscr{B}'(n, k)$. Let $C_i = \{y_i\}$, for all $i,\ 1 \leq i \leq k$. Let $|Red(B)| = l$ and C_0 be the chain $x_1 \leq x_2 \cdots \leq x_{l+m}$. Suppose $Red(B) = \{x_{i_j} | 1 \leq j \leq l\}$. Now if for some $r,\ 1 \leq r \leq k$, $(a_r, b_r) \subseteq Irr(B)$ then $|(a_r, b_r) \cap C_0| = 1$, since otherwise $|C_0| - |Red(B)| > m$. Therefore let $(a_r, b_r) \cap C_0 = \{x_{i_j}\}$, for some j, $l+1 \leq j \leq l+m$. But then $a_r = x_{i_j-1}$ and $b_r = x_{i_j+1}$. Now $Irr(B) = \{y_1, \cdots, y_k, x_{i_{l+1}}, \cdots, x_{i_{l+m}}\}$. For any $z \in Irr(B)$, $B - \{z\} \in \mathscr{L}'(n-1, k-1)$, where $n = |C_0| + k$. Thus, removal of doubly irreducible element from B decreases its nullity by one. Therefore B is a basic block. $\square$

***Remark* 3.2.1.** By (i) and (ii) of Theorem 3.2.1, a minimal sublattice B is a basic block associated to a lattice L, if the repeated application of the operation of deleting doubly irreducible elements, whose removal causes the nullity unaltered, ends up with B.

Note that, if B is a basic block associated to a dismantlable lattice L then by Proposition 1.3.3, B is dismantlable. Therefore, by Theorem 2.2.3, a basic block associated to any lattice of nullity at most four is dismantlable.

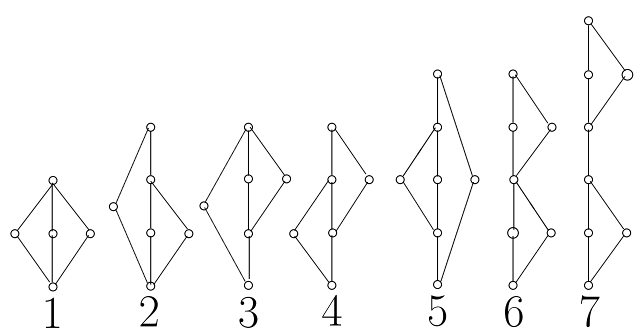


Fig.6

Proposition 3.2.5. *There are exactly seven non-isomorphic basic blocks (given in Fig.6) associated to lattices of nullity two.*

Proof. Let B be a basic block associated to a lattice L of nullity two. By (ii) of Theorem 3.2.1, B is also of nullity two. By Theorem 2.2.4, B must be an adjunct of three chains. By Lemma 2.2.6, $2 \leq |Red(B)| \leq 4$. If

$|Red(B)| = 2$ then B is given by Fig.6(1). If $|Red(B)| = 3$ and suppose the reducible elements in B are $0 < a < 1$ then B is given by Fig.6(2) whenever a is join reducible, B is given by Fig.6(3) whenever a is meet reducible and B is given by Fig.6(6) whenever a is join as well as meet reducible. If $|Red(B)| = 4$ and suppose the reducible elements in B are $0 < a \neq b < 1$ then a and b must be comparable, since otherwise, B would be an adjunct of at least four chains, a contradiction. Without loss, say $a < b$. If both a and b are meet (or join) reducible elements then again B would be an adjunct of at least four chains, again a contradiction. If a is meet reducible and b is join reducible then B is either given by Fig.6(4) or given by Fig.6(5); otherwise, B is given by Fig.6(7). □

3.3 Uniqueness of a basic block

In the following Theorem 3.3.1, we prove that, if two basic blocks are non-isomorphic then the posets associated by these basic blocks are also non-isomorphic.

Theorem 3.3.1. *If B_1, B_2 are basic blocks associated to the posets P_1, P_2 respectively and $P_1 \cong P_2$ then $B_1 \cong B_2$.*

Proof. We give the proof using induction on $n = |P_1| = |P_2| \geq 1$. If $n \leq 4$ then we are done. Now suppose $n \geq 5$ and the result is true for any two isomorphic posets containing $< n$ elements.

If P_1 contains no doubly irreducible element x such that P_1 and $P_1 \setminus \{x\}$ have same nullity then P_1 itself is a basic block associated to it and we

get that $B_1 = P_1 \cong P_2 = B_2$.
Now, suppose there is a doubly irreducible element $x \in P_1$ such that the nullity of P_1 is same as the nullity of $P_1 \setminus \{x\}$.
Let $\phi : P_1 \to P_2$ be an isomorphism.
If $x \notin B_1$ and $\phi(x) \notin B_2$ then B_1 and B_2 are also basic blocks associated to $P_1 \setminus \{x\}$ and $P_2 \setminus \{\phi(x)\}$ respectively.
Now $|P_1 \setminus \{x\}| = |P_2 \setminus \{\phi(x)\}| = n-1 < n$. Therefore by the induction hypothesis $B_1 \cong B_2$.

Without loss, assume that $x \in B_1$. It follows that the ear E containing x in P_1 contains one more element, say y, since B_1 is a basic block and the nullity of P_1 is same as the nullity of $P_1 \setminus \{x\}$. If there is an element $z \in E$ such that $z \notin B_1$ and $\phi(z) \notin B_2$ then B_1 and B_2 are also basic blocks associated to $P_1 \setminus \{z\}$ and $P_2 \setminus \{\phi(z)\}$ respectively. Now $|P_1 \setminus \{z\}| = |P_2 \setminus \{\phi(z)\}| = n-1 < n$. Therefore by the induction hypothesis $B_1 \cong B_2$.
Hence assume that $E = \{x, y\}$ and $\phi(y) \in B_2$.

Define a map $\psi : P_1 \setminus \{x\} \to P_2 \setminus \{\phi(y)\}$ as
$\psi(z) = \phi(z)$, if $z \neq y$ and $\psi(z) = \phi(x)$, if $z = y$.
We prove that ψ is an isomorphism.
(I) We prove ψ is injective. Let $a, b \in P_1 \setminus \{x\}$.
Case 1 : If $a \neq y$ and $b \neq y$ then $\psi(a) = \phi(a)$ and $\psi(b) = \phi(b)$. Therefore $\psi(a) = \psi(b)$ implies that $\phi(a) = \phi(b)$ and hence $a = b$, since ϕ is injective.
Case 2 : If without loss, $a \neq y$ and $b = y$ then $\psi(a) = \phi(a)$ and $\psi(b) = \phi(x)$. If $\phi(a) = \phi(x)$ then $a = x$, since ϕ is injective. This is

not possible. Therefore $\phi(a) \neq \phi(x)$ and hence $\psi(a) \neq \psi(b)$.

(II) We prove ψ is surjective.

Let $w \in P_2 \setminus \{\phi(y)\}$. Therefore $w \neq \phi(y)$. As $w \in P_2$, $w = \phi(c)$ for some $c \in P_1$, since ϕ is surjective. Clearly, $c \neq y$. Therefore $\psi(c) = \phi(c)$. If $c = x$ then $\phi(c) = \phi(x)$ and hence $\psi(c) = \psi(y)$. As ψ is injective, we get $c = y$, a contradiction. Thus, for any $w \in P_2 \setminus \{\phi(y)\}$, there exists $c \in P_1 \setminus \{x\}$ such that $\psi(c) = w$.

(III) We prove ψ is order-embedding.

Now $a \prec b$ in $P_1 \setminus \{x\}$ if and only if $a \prec b$ in P_1 or $a \prec x$ and $x \prec b$ in P_1 if and only if $\phi(a) \prec \phi(b)$ in P_2 or $\phi(a) \prec \phi(x)$ and $\phi(x) \prec \phi(b)$ in P_2 if and only if $\psi(a) \prec \psi(b)$ in P_2 or $\psi(a) \prec \psi(y)$ and $\psi(y) \prec \psi(b)$ in P_2 if and only if $\psi(a) \prec \psi(b)$ in $P_2 \setminus \{\psi(y)\}$.

Therefore $P_1 \setminus \{x\} \cong P_2 \setminus \{\phi(y)\}$, since $P_2 \setminus \{\psi(y)\} = P_2 \setminus \{\phi(x)\} = P_2 \setminus \{\phi(y)\}$, as $\phi(E) = \{\phi(x), \phi(y)\}$.

Now $|P_1 \setminus \{x\}| = |P_2 \setminus \{\phi(y)\}| = n - 1 < n$. Therefore, by the induction hypothesis, the basic blocks associated to $P_1 \setminus \{x\}$ and $P_2 \setminus \{\phi(y)\}$ are isomorphic.

Note that, $(B_1 \setminus \{x\}) \cup \{y\}$ and $(B_2 \setminus \{\phi(y)\}) \cup \{\phi(x)\}$ are basic blocks associated to $P_1 \setminus \{x\}$ and $P_2 \setminus \{\phi(y)\}$ respectively. Therefore by the induction hypothesis, $(B_1 \setminus \{x\}) \cup \{y\} \cong (B_2 \setminus \{\phi(y)\}) \cup \{\phi(x)\}$. As $B_1 \cong (B_1 \setminus \{x\}) \cup \{y\}$ and $B_2 \cong (B_2 \setminus \{\phi(y)\}) \cup \{\phi(x)\}$, we have $B_1 \cong B_2$. Hence the proof is complete. $\square$

However, the converse of the above Theorem 3.3.1 is not true. Since the posets given in the following figure (Fig.7) are not isomorphic to each other but the basic blocks associated to them are isomorphic (In fact, those basic blocks are each isomorphic to M_2 (see Fig.5)).

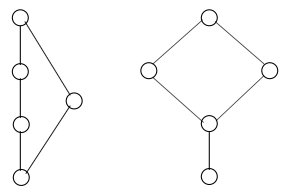


Fig.7

In the following, we prove that, there is a *unique* basic block associated to any poset.

Corollary 3.3.2. *If B_1 and B_2 are basic blocks associated to a poset P then $B_1 \cong B_2$.*

Proof. Consider an identity map $\psi : P \to P$. Then ψ is an isomorphism. Therefore using the above Theorem 3.3.1, $B_1 \cong B_2$. □

3.4 Fundamental basic blocks

In the previous section, we have studied the concept of a basic block associated to a poset. Also, it can be observed that a basic block is the minimal form of a poset with respect to the nullity. In the following, we introduce the concept of a fundamental basic block associated to a dismantlable lattice. Using fundamental basic blocks, it is possible to enumerate the number of non-isomorphic dismantlable lattices with the help of partition theory of numbers.

Definition 3.4.1. A dismantlable lattice B is said to be a *fundamental basic block* if it is a basic block and all the adjunct pairs in the adjunct representation of B are distinct.

For example, M_2 (see Fig.5) is fundamental basic block, whereas M_3 (see Fig.6 (1)) is not a fundamental basic block. In fact, using the above Proposition 3.2.5, we get by observation the following.

Corollary 3.4.1. *There are exactly six non-isomorphic fundamental basic blocks (see Fig.6) of nullity two.*

Proposition 3.4.2. *Let $L \in \mathscr{L}'(n,k,r)$, where $k = [\frac{r+1}{2}]$. Then the multiplicity of each adjunct pair in an adjunct representation of L is one.*

Proof. As L in a lattice of nullity k, by Theorem 2.2.4, L is adjunct of $k+1$ chains. Now the reducible elements in L are all comparable. Therefore by Theorem 2.1.7, $L = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2\cdots]_{\alpha_k}C_k$, where C_0 is a maximal chain containing all the r reducible elements and for each i, C_i is a chain and α_i is an adjunct pair. Suppose for some i, the adjunct pair α_i has multiplicity more than one. Then by Corollary 2.1.20,
$L = C_0]_{\alpha_1}C_1]_{\alpha_2}C_2\cdots]_{\alpha_{i-1}}C_{i-1}]_{\alpha_k}C_k]_{\alpha_{i+1}}C_{i+1}\cdots]_{\alpha_{k-1}}C_{k-1}]_{\alpha_i}C_i$.
But then by Proposition 1.3.1, $L - C_i$ is a sublattice of P. Moreover, $L - C_i$ contains r reducible elements. Also by Theorem 2.2.4, nullity of $L - C_i$ is $k-1$. Thus, $L - C_i \in \mathscr{L}'(n-|C_i|, k-1, r)$. Therefore by Proposition 2.2.7, $k - 1 \geq [\frac{r+1}{2}]$, a contradiction, since $k = [\frac{r+1}{2}]$. □

Using the above Proposition 3.4.2 and using the definition of a fundamental basic block, we have the following.

Corollary 3.4.3. *If $k = [\frac{r+1}{2}]$ then every basic block associated to a lattice in $\mathscr{L}'(n,k,r)$ is a fundamental basic block.*

Proposition 3.4.4. *For any fundamental basic block of nullity k containing r reducible elements which are all comparable, $[\frac{r+1}{2}] \leq k \leq \binom{r}{2}$.*

Proof. By Proposition 2.2.7, $[\frac{r+1}{2}] \leq k$. By the definition of a fundamental basic block, all the adjunct pairs in an adjunct representation of it are distinct. That is, the multiplicity of each adjunct pair in an adjunct representation of a fundamental basic block is one. As the nullity of a fundamental basic block is k, by Theorem 2.2.4, it is adjunct of $k+1$ chains and hence the number of distinct adjunct pairs in its adjunct representation is also k. But one adjunct pair correspond to two reducible elements. Therefore $k \leq \binom{r}{2}$. Thus $[\frac{r+1}{2}] \leq k = l \leq \binom{r}{2}$. □

The above Proposition 3.4.4 can also be proved using Proposition 2.2.9 and using the definition of a fundamental basic block.

Definition 3.4.2. Let L be a dismantlable lattice. Let B be a basic block associated to L. If B itself is a fundamental basic block then we say that B is a fundamental basic block associated to itself. Let (a, b) be an adjunct pair in an adjunct representation of B. If the interval $(a, b) \subseteq Irr(B)$ then remove all but two (non-trivial) ears associated to (a, b) in B; otherwise, remove all but one (non-trivial) ear (if any) associated to (a, b) in B.
Perform the operation of removal of (non-trivial) ears associated to (a, b), for each adjunct pair (a, b) in an adjunct representation of B. The resultant *sublattice* of B is called a *fundamental basic block associated to L*.

For example, M_2 (see Fig.5) is a fundamental basic block associated to M_3 (see Fig.6 (1)). Observe that, if F is a fundamental basic block associated to a dismantlable lattice L then $Red(B) = Red(L)$. Also it can be observed that, the fundamental basic block associated to a dismantlable lattice is having a smaller diagram as compared to the basic block associated to that lattice.

As a consequence of the Theorem 3.3.1, we have the following result.

Theorem 3.4.5. *If F_1 and F_2 are fundamental basic blocks associated to the dismantlable lattices L_1 and L_2 respectively and $L_1 \cong L_2$ then $F_1 \cong F_2$.*

Proof. Let B_1 and B_2 be basic blocks associated to the dismantlable lattices L_1 and L_2 respectively. As $L_1 \cong L_2$, by Theorem 3.3.1, $B_1 \cong B_2$. If B_1 itself is a fundamental basic block then we are done. Suppose B_1 is not a fundamental basic block. Let F_1 and F_2 be fundamental basic blocks associated to L_1 and L_2 respectively. Then F_1 and F_2 are also fundamental basic blocks associated to B_1 and B_2 respectively. Let $x \in S = Irr(B_1) \setminus Irr(F_1)$. Suppose $a \prec x \prec b$ in B_1. Then $a, b \in Red(B_1) = Red(F_1)$ and (a, b) is an adjunct pair in an adjunct representation of B_1. Let $\phi : B_1 \to B_2$ be an order-isomorphism. Then $\phi(a) \prec \phi(x) \prec \phi(b)$ in B_2 and $(\phi(a), \phi(b))$ is an adjunct pair in an adjunct representation of B_2. Moreover, $B_1 \setminus \{x\} \cong B_2 \setminus \{\phi(x)\}$. Thus $F_1 = B_1 \setminus S \cong B_2 \setminus \{\phi(S)\} = F_2$. □

Thus it follows that, if two fundamental basic blocks are non-isomorphic then the lattices associated by these fundamental basic blocks are also non-isomorphic. As a consequence of the Theorem 3.4.5, we prove in the following that there is a *unique* fundamental basic block associated to any dismantlable lattice.

Corollary 3.4.6. *If F_1 and F_2 are basic blocks associated to a dismantlable lattice L then $F_1 \cong F_2$.*

Proof. Consider an identity map $\psi : L \to L$. Clearly ψ is an isomorphism. Therefore using the above Theorem 3.4.5, $F_1 \cong F_2$. □

Chapter 4

Enumeration of lattices

In this Chapter, we mainly deal with the enumeration of the class of all non-isomorphic lattices in which all the reducible elements are comparable. In the first section, we give recursive formulae for obtaining the number of all non-isomorphic fundamental basic blocks containing reducible elements which are all comparable. In the second section, we obtain three sequences. The first is the sequence of fundamental basic blocks containing $r \geq 0$ reducible elements which are all comparable. The second is the sequence of fundamental basic blocks of nullity $l \geq 0$ in which all the reducible elements are comparable. The third is the sequence of basic blocks of nullity $l \geq 0$ in which all the reducible elements are comparable. In the last section, we enumerate the class of all non-isomorphic lattices on n elements in which the reducible elements are all comparable.

4.1 Counting fundamental basic blocks

Definition 4.1.1. Let $FB(r)$ be the class of all non-isomorphic fundamental basic blocks such that each one of them has r reducible elements. Let $FB'(r)$ be the subclass of $FB(r)$ such that all the r reducible elements in each fundamental basic block in it are comparable.

Note that, if the nullity of a fundamental basic block in $FB(r)$ is l then by Theorem 2.2.4, it is adjunct of $l+1$ chains and hence the number of distinct adjunct pairs in its adjunct representation is also l.
Also, if the nullity of a fundamental basic block in $FB'(r)$ is l then by Proposition 3.4.4, $[\frac{r+1}{2}] \leq l \leq \binom{r}{2}$.

Let $a_r = |FB'(r)|$, for all $r \geq 0$. Then
$a_0 = 1$, since $FB'(0)$ consists of a chain only.
$a_1 = 0$, since $FB'(1)$ is an empty class.
$a_2 = 1$, since $FB'(2)$ consists of M_2 (see Fig.5) only.
We now obtain a recursive formula which produces a_r in the following.

Theorem 4.1.1. *For $r \geq 0$, $a_{r+1} = \left(\sum_{j=0}^{r} \binom{r}{j} 2^j a_j\right) - a_r$ with $a_0 = 1$.*

Proof. Let $B \in FB'(r+1)$. Consider the poset P_B obtained from B by deleting 1. Then the basic block B' associated to P_B is in $FB'(j)$ for some $j = 0, 1, 2, \ldots, r$, since all the reducible elements of B are comparable. Clearly $Red(B') \subset Red(B)$. Therefore every element of $FB'(r+1)$ can be obtained from a member B' of $FB'(j)$ by a linear sum with a chain of at most two elements and then using at least

$m = max\{r-j, 1\}$ 2-sums, where all the corresponding adjunct pairs are distinct and of the type $(a, 1)$, where 1 is the largest element of the linear sum and $a \notin Red(B')$ for exactly $r-j$ 2-sums. Further, by Theorem 3.3.1, for any $B_1, B_2 \in FB'(r+1)$, if the basic blocks B_1' and B_2' associated to $P_{B_1} = B_1 - \{1\}$ and $P_{B_2} = B_2 - \{1\}$ respectively are not isomorphic then $P_{B_1} \ncong P_{B_2}$, consequently $B_1 \ncong B_2$.

We now see the procedure for obtaining fundamental basic blocks of $FB'(r+1)$ from that of $FB'(j)$.

Let $B' \in FB'(j)$, for some $j = 0, 1, 2, \ldots, r$. Let $y_1, y_2, \ldots, y_j \in Red(B')$. Let C be a maximal chain containing all the reducible elements of B'. We consider the following two cases.

Case I. Suppose $j < r$. Therefore $r - j \geq 1$. Clearly there are $j-1$ places on C which are separated by the reducible elements of B'. Consider two more places, one is below C and the other is above C. Thus the total number of places separated by all the reducible elements of B is $j+1$. Insert now $r-j$ doubly irreducible elements, say $x_1, x_2, \ldots, x_{r-j}$ in those $j+1$ places. This can be done in $\binom{(r-j)+(j+1)-1}{(j+1)-1} = \binom{r}{j}$ ways.

Let C' be the chain consisting of C alongwith those $r-j$ doubly irreducible elements and let L be the resultant lattice.

Let $B = (L \oplus C'')]_{x_1}^{1}\{c_1\}]_{x_2}^{1}\{c_2\} \cdots]_{x_{r-j}}^{1}\{c_{r-j}\}$, where C'' is a chain of at most two elements. Then (a basic block associated to) $B \in FB'(r+1)$. Also if we choose m of the j reducible elements of B', say $y_{i_1}, y_{i_2}, \ldots, y_{i_m}$ then (a basic block associated to) $B]_{y_{i_1}}^{1}\{d_{i_1}\}]_{y_{i_2}}^{1}\{d_{i_2}\} \cdots]_{y_{i_m}}^{1}\{d_{i_m}\} \in FB'(r+1)$. Note that $0 \leq m \leq j$. Therefore $\sum_{m=0}^{j} \binom{j}{m} = 2^j$ (non-isomorphic) fundamental basic blocks of $FB'(r+1)$ can be constructed using $B' \in$

$FB'(j)$. Hence for fixed j, in all $\binom{r}{j} \times 2^j$ (non-isomorphic) fundamental basic blocks of $FB'(r+1)$ can be constructed using $B' \in FB'(j)$.

Case II. Suppose $j = r$. In this case there is no need to insert any doubly irreducible element in the chain C. Again if we choose m of the j reducible elements of B', say $y_{i_1}, y_{i_2}, \ldots, y_{i_m}$ then (a basic block associated to) $(B' \oplus C'')]^1_{y_{i_1}}\{d_{i_1}\}]^1_{y_{i_2}}\{d_{i_2}\}\cdots]^1_{y_{i_m}}\{d_{i_m}\} \in FB'(r+1)$, where C'' is a chain of at most two elements. Note that $1 \leq m \leq j$, since if $m = 0$ then 1 of $B' \oplus C''$ will not become reducible, consequently, we will not get a fundamental basic block of $FB'(r+1)$. Therefore $\displaystyle\sum_{m=1}^{j} \binom{j}{m} = 2^j - 1$ (non-isomorphic) fundamental basic blocks of $FB'(r+1)$ can be constructed using $B' \in FB'(j)$. Hence in this case, in all $\binom{r}{j} \times (2^j - 1)$, that is, $2^r - 1$ (non-isomorphic) fundamental basic blocks of $FB'(r+1)$ can be constructed using $B' \in FB'(j)$.

Thus $a_{r+1} = |FB'(r+1)| = \displaystyle\sum_{j=0}^{r-1} \sum_{B' \in FB'(j)} \binom{r}{j} 2^j + \sum_{B' \in FB'(r)} (2^r - 1) = \left(\sum_{j=0}^{r-1} \binom{r}{j} 2^j |FB'(j)|\right) + (2^r - 1)|FB'(r)| = \left(\sum_{j=0}^{r} \binom{r}{j} 2^j a_j\right) - a_r.$ □

In the following Theorem 4.1.2, we obtain another form of a recursive formula for a_r which is obtained in Theorem 4.1.1.

Theorem 4.1.2. *For* $r \geq 1$, $a_{r+1} = \displaystyle\sum_{k=1}^{r} \sum_{j=0}^{k} \binom{r}{j} \binom{r-j}{k-j} a_{r-j}$ *with* $a_0 = 1$ *and* $a_1 = 0$.

Proof. By Theorem 4.1.1, $a_{r+1} = \left(\displaystyle\sum_{j=0}^{r} \binom{r}{j} 2^j a_j\right) - a_r$

$= \binom{r}{r} 2^r a_r + \binom{r}{r-1} 2^{r-1} a_{r-1} + \binom{r}{r-2} 2^{r-2} a_{r-2} + \cdots + \binom{r}{r-(r-1)} 2^{r-(r-1)} a_{r-(r-1)}$
$+ \binom{r}{r-r} 2^{r-r} a_{r-r} - a_r$

$= \binom{r}{r}(2^r - 1)a_r + \binom{r}{r-1}2^{r-1}a_{r-1} + \binom{r}{r-2}2^{r-2}a_{r-2}$
$+ \cdots + \binom{r}{r-(r-1)}2^{r-(r-1)}a_{r-(r-1)} + \binom{r}{r-r}2^{r-r}a_{r-r}$
$= \binom{r}{0}(2^r - 1)a_r + \binom{r}{1}(2^{r-1})a_{r-1} + \binom{r}{2}(2^{r-2})a_{r-2}$
$+ \cdots + \binom{r}{r-1}(2^{r-(r-1)})a_{r-(r-1)} + \binom{r}{r}(2^{r-r})a_{r-r}$
$= \binom{r}{0}\left(\binom{r}{1} + \binom{r}{2} + \cdots + \binom{r}{r}\right) a_r + \binom{r}{1}\left(\binom{r-1}{0} + \binom{r-1}{1} + \cdots + \binom{r-1}{r-1}\right) a_{r-1}$
$+ \binom{r}{2}\left(\binom{r-2}{0} + \binom{r-2}{1} + \cdots + \binom{r-2}{r-2}\right) a_{r-2} + \cdots$
$+ \binom{r}{r-1}\left(\binom{r-(r-1)}{0} + \binom{r-(r-1)}{1}\right) a_{r-(r-1)} + \binom{r}{r}\left(\binom{r-r}{0}\right) a_{r-r}$
$= \left(\binom{r}{0}\binom{r}{1}a_r + \binom{r}{1}\binom{r-1}{0}a_{r-1}\right) + \left(\binom{r}{0}\binom{r}{2}a_r + \binom{r}{1}\binom{r-1}{1}a_{r-1} + \binom{r}{2}\binom{r-2}{0}a_{r-2}\right)$
$+ \cdots + \left(\binom{r}{0}\binom{r}{r}a_r + \binom{r}{1}\binom{r-1}{r-1}a_{r-1} + \cdots + \binom{r}{r}\binom{r-r}{0}a_{r-r}\right)$
$= \left(\sum_{j=0}^{1}\binom{r}{j}\binom{r-j}{1-j}a_{r-j}\right) + \left(\sum_{j=0}^{2}\binom{r}{j}\binom{r-j}{2-j}a_{r-j}\right) + \cdots$
$+ \left(\sum_{j=0}^{r}\binom{r}{j}\binom{r-j}{r-j}a_{r-j}\right) = \sum_{k=1}^{r}\sum_{j=0}^{k}\binom{r}{j}\binom{r-j}{k-j}a_{r-j}.$

Thus, $a_{r+1} = \sum_{k=1}^{r}\sum_{j=0}^{k}\binom{r}{j}\binom{r-j}{k-j}a_{r-j}$. □

In Corollary 4.1.5, we obtain one more formula for finding a_r. For this purpose, we define the following.

Definition 4.1.2. Let $FB(l, r)$ be the subclass of $FB(r)$ such that each fundamental basic block in it is of nullity l.

Let $FB'(l, r)$ be the subclass of $FB'(r)$ such that each fundamental basic block in it is of nullity l.

Let $r \geq 1$. For $1 \leq k \leq r$ and for $[\frac{r+2}{2}] \leq l \leq \binom{r+1}{2}$,

let $C_k^l = \{B \in FB'(l, r+1) : \text{Indegree } (d) \text{ of } 1 \text{ in } B \text{ is } k+1\}$.

Note that, $FB(0, 0)$ consists of 1-chain only and $FB(1, 2)$ consists of M_2 (see Fig.5) only.

***Remark* 4.1.1.** Observe that,

1. For $r \geq 1$, $FB(0,r)$ is an empty set.

2. If either $l < [\frac{r+2}{2}]$ or $l > \binom{r+1}{2}$ or $r > 2l$ then $FB'(l,r) = \phi$.

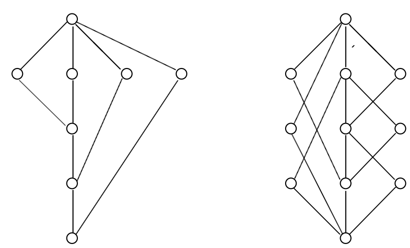
3. The collection $\{FB'(l,r+1) : [\frac{r+2}{2}] \leq l \leq \binom{r+1}{2}\}$ forms a partition of $FB'(r+1)$.

4. If either $k = l = r-1$ or $l < k$ or $\binom{r}{2} < l-k$ then $C_k^l = \phi$.

5. For fixed $[\frac{r+2}{2}] \leq l \leq \binom{r+1}{2}$, the collection $\{C_k^l : 1 \leq k \leq r\}$ forms a partition of $FB'(l,r+1)$.

6. If $l = \binom{r+1}{2}$ then $|C_r^l| = 1$ (see Fig.8(II) for $r = 3$).

7. $|C_r^r| = 1$ (see Fig.8(I) for $r = 3$).



(I) C_3^3 (II) C_3^6

Fig.8

In the following Theorem 4.1.3, we obtain the formula for C_k^l in terms of the number of fundamental basic blocks of nullity $l-k$, in which the reducible elements are all comparable.

Theorem 4.1.3. *For fixed $r \geq 1$, $1 \leq k \leq r$ and $[\frac{r+2}{2}] \leq l \leq \binom{r+1}{2}$,*
$$|C_k^l| = \sum_{j=0}^{k} \binom{r}{j}\binom{r-j}{k-j}|FB'(l-k, r-j)|.$$

Proof. Let $r \geq 1$. Let $B \in C_k^l$, for some $1 \leq k \leq r$ and for some $[\frac{r+2}{2}] \leq l \leq \binom{r+1}{2}$. Consider the poset P_B obtained from B by deleting

1. Then the basic block B' associated to P_B is in $FB'(l-k, r-j)$ for some $j = 0, 1, 2, \ldots, k$, since all the reducible elements of B are comparable and indegree of 1 is $k+1$. Note that, removal of 1 from B result in the removal of k chains (singletons) corresponding to k adjunct pairs of the type $(a, 1)$, where $a \neq 1 \in Red(B)$, since B is a fundamental basic block, and therefore, at most k out of r (other than 1) reducible elements may become irreducible. Clearly $Red(B') \subset Red(B)$.

Therefore, every element of C_k^l can be obtained from a member B' of $FB'(l-k, r-j)$ by a linear sum with a chain of at most two elements and then using at least $max\{j, 1\}$ 2-sums, where all the corresponding adjunct pairs are distinct and of the type $(b, 1)$, where 1 is the largest element of the linear sum and $b \notin Red(B')$ for exactly j 2-sums.

Further, by Theorem 3.3.1, for any $B_1, B_2 \in C_k^l$, if the basic blocks B_1' and B_2' associated to $P_{B_1} = B_1 - \{1\}$ and $P_{B_2} = B_2 - \{1\}$ respectively are not isomorphic then $P_{B_1} \ncong P_{B_2}$, consequently $B_1 \ncong B_2$.

We now see the procedure for obtaining fundamental basic blocks of C_k^l from that of $FB'(l-k, r-j)$.

Let $B \in FB'(l-k, r-j)$ for some $j = 0, 1, 2, \ldots, k$. Let C be a maximal chain containing all $r-j$ reducible elements of B. Clearly there are $r-j-1$ places on C which are separated by the reducible elements of B. Consider two more places, one is below C and the other is above C. Thus the total number of places separated by all the reducible elements of B is $r-j+1$. Insert now j doubly irreducible elements, say $x_1, x_2, \ldots, x_j$ in those $r-j+1$ places. This can be done in $\binom{j+(r-j+1)-1}{(r-j+1)-1} = \binom{r}{r-j} = \binom{r}{j}$ ways.

Let C' be the chain consisting of C alongwith those j doubly irreducible elements and let L be the resultant lattice.

Let $B' = (L \oplus C'')]_{x_1}^{1}\{c_1\}]_{x_2}^{1}\{c_2\}\cdots]_{x_j}^{1}\{c_j\}$, where C'' is a chain of at most two elements. Then (a basic block associated to) B' becomes a fundamental basic block of nullity $l-k+j$ containing $(r-j)+j+1 = r+1$ reducible elements which are all comparable. Note that indegree of 1 of B' is $j+1$.

Let $y_1, y_2, \ldots, y_{r-j} \in Red(B)$. If we choose $k-j$ reducible elements, say $y_{i_1}, y_{i_2}, \ldots, y_{i_{k-j}}$ out of $r-j$ reducible elements of B, which can be done in $\binom{r-j}{k-j}$ ways, then (a basic block associated to)

$B'' = B']_{y_{i_1}}^{1}\{d_1\}]_{y_{i_2}}^{1}\{d_2\}\cdots]_{y_{i_{k-j}}}^{1}\{d_{k-j}\} \in C_k^l$. Thus given $B \in FB'(l-k, r-j)$, we can obtain in all $\binom{r}{j}\binom{r-j}{k-j}$ (non-isomorphic) fundamental basic blocks of C_k^l. Hence $|C_k^l| = \sum_{j=0}^{k}\binom{r}{j}\binom{r-j}{k-j}|FB'(l-k, r-j)|$. □

In the following Corollary 4.1.4, we obtain the recursive formula for the number of fundamental basic blocks of nullity l, containing $r+1$ reducible elements which are all comparable.

Corollary 4.1.4. *For fixed $r \geq 1$ and $[\frac{r+2}{2}] \leq l \leq \binom{r+1}{2}$,*

$$|FB'(l, r+1)| = \sum_{k=1}^{r}\sum_{j=0}^{k}\binom{r}{j}\binom{r-j}{k-j}|FB'(l-k, r-j)|.$$

Proof. For fixed l, $\{C_k^l : 1 \leq k \leq r\}$ forms a partition of $FB'(l, r+1)$. Therefore for each $[\frac{r+2}{2}] \leq l \leq \binom{r+1}{2}$, $FB'(l, r+1) = \bigcup_{k=1}^{r} C_k^l$. Hence

$$|FB'(l, r+1)| = \sum_{k=1}^{r}|C_k^l|.$$

Therefore the proof follows from the above Theorem 4.1.3. □

In the following Corollary 4.1.5, we now obtain one more formula for finding a_r in terms of the number of fundamental basic blocks in which the reducible elements are all comparable.

Corollary 4.1.5. *For $r \geq 1$,*

$$a_{r+1} = \sum_{l=[\frac{r+2}{2}]}^{\binom{r+1}{2}} \sum_{k=1}^{r} \sum_{j=0}^{k} \binom{r}{j}\binom{r-j}{k-j} |FB'(l-k, r-j)|.$$

Proof. As $\{FB'(l, r+1) : [\frac{r+2}{2}] \leq l \leq \binom{r+1}{2}\}$ forms a partition of $FB'(r+1)$, we have $FB'(r+1) = \bigcup_{l=[\frac{r+2}{2}]}^{\binom{r+1}{2}} FB'(l, r+1)$. Hence $a_{r+1} =$

$$|FB'(r+1)| = \sum_{l=[\frac{r+2}{2}]}^{\binom{r+1}{2}} |FB'(l, r+1)|.$$

Therefore the proof follows from the above Corollary 4.1.4. □

Definition 4.1.3. For $1 \leq k \leq r$,
let $FB_k(r+1) = \{B \in FB'(r+1) :$ Indegree (d) of 1 in B is $k+1\}$.

It is clear that $2 \leq d \leq r+1$, for any $B \in FB'(r+1)$.
Also, for fixed $1 \leq k \leq r$, the collection $\{C_k^l : [\frac{r+2}{2}] \leq l \leq \binom{r+1}{2}\}$ forms a partition of $FB_k(r+1)$. Therefore we have the following.

Corollary 4.1.6. *For fixed $r \geq 1$ and for fixed $1 \leq k \leq r$,*

$$|FB_k(r+1)| = \sum_{l=[\frac{r+2}{2}]}^{\binom{r+1}{2}} \sum_{j=0}^{k} \binom{r}{j}\binom{r-j}{k-j} |FB'(l-k, r-j)|.$$

Proof. For fixed $1 \leq k \leq r$, the collection $\{C_k^l : [\frac{r+2}{2}] \leq l \leq \binom{r+1}{2}\}$ forms a partition of $FB_k(r+1)$. Therefore for each $1 \leq k \leq r$, $FB_k(r+1) =$

$\bigcup_{l=[\frac{r+2}{2}]}^{\binom{r+1}{2}} C_k^l$. Hence $|FB_k(r+1)| = \sum_{l=[\frac{r+2}{2}]}^{\binom{r+1}{2}} |C_k^l|$. Therefore the proof follows from the above Theorem 4.1.3. □

The collection $\{FB_k(r+1) : 1 \leq k \leq r\}$ also forms a partition of $FB'(r+1)$. Therefore the above Corollary 4.1.5 can also be obtained using Corollary 4.1.6.

It can be observed that, the fundamental basic block associated to a dismantlable lattice is having a smaller (or same) diagram as compared to the basic block associated to that lattice.

In Theorem 4.1.7, we obtain the formula for obtaining the number of non-isomorphic basic blocks of nullity l, containing reducible elements which are all comparable, using the number of non-isomorphic fundamental basic blocks of nullity $m \leq l$. For this purpose, let us use the following.

Definition 4.1.4. Let $B(r)$ be the class of all non-isomorphic basic blocks such that each basic block in it has r reducible elements. Let $B'(r)$ be the subclass of $B(r)$ such that the reducible elements in each basic block in it are all comparable.

Definition 4.1.5. Let $B(l, r)$ be the subclass of $B(r)$ such that each basic block in it is of nullity l. Let $B'(l, r)$ be the subclass of $B'(r)$ such that each basic block in it is of nullity l.

By Corollary 3.4.3, if $l = m = [\frac{r+1}{2}]$ then $B'(l, r) = FB'(m, r)$.

In general, if $l \geq m$ then $|B'(l, r)| \geq |FB'(m, r)|$.

Let p_n^r denote the number of (weak) compositions of n into r (non-negative) parts. Then p_n^r is the number of non-negative integer solutions to the equation $n = x_1 + x_2 + \cdots + x_r$. The number of solutions is actually the number of distinct r-tuples, $(x_1, x_2, \ldots, x_r)$ satisfying the equation $n = x_1 + x_2 + \cdots + x_r$, where for each i, $x_i \geq 0$. It is known that $p_n^r = \binom{n+r-1}{r-1}$.

We now obtain the formula for the number of non-isomorphic basic blocks of nullity l containing reducible elements which are all comparable.

Theorem 4.1.7. *For $r \geq 2$ and for $[\frac{r+2}{2}] \leq m \leq l \leq \binom{r}{2}$,*

$$|B'(l,r)| = \sum_{m=[\frac{r+1}{2}]}^{l} \binom{l-1}{m-1} |FB'(m,r)|.$$

Proof. Let $B \in B'(l,r)$ for some l. Suppose B' is the fundamental basic block associated to B. Clearly, $Red(B) = Red(B')$. If m is the nullity of B' then it is clear that $m \leq l$. Let $s = l - m$.

Therefore, any $B \in B'(l,r)$ can be obtained from a member B' of $FB'(m,r)$ using exactly s 2-sums, where all the adjunct pairs are the adjunct pairs of B'.

Further, note that, for any $B_1, B_2 \in B'(l,r)$, if the corresponding fundamental basic blocks B_1' and B_2' are not isomorphic then $B_1 \ncong B_2$. Also, for any isomorphism ϕ of $B' \in FB'(m,r)$ to itself, if (a,b) is an adjunct pair then $\phi(a) = a$ and $\phi(b) = b$, since the reducible elements of B' are all comparable.

By Theorem 2.2.4, as nullity of B' is m, it is an adjunct of $m+1$ chains. Suppose C is a maximal chain containing all the r reducible elements of B'. Then by Theorem 2.1.7 and using definition of a fundamental basic block, $B' = C]_{\alpha_1}\{c_1\}]_{\alpha_2}\{c_2\}\cdots]_{\alpha_m}\{c_m\}$ where all the adjunct pairs $\alpha_i = (a_i, b_i)$ are distinct. By Corollary 2.1.20, without loss, we can assume that $(a_1, b_1) < (a_2, b_2) < \ldots < (a_m, b_m)$ with respect to the dictionary order defined on $C \times C$. Let n_i be the multiplicity of an adjunct pair (a_i, b_i) in $B \in B'(l, r)$.

For each $B' \in FB'(m, r)$, let $A_{B'} = \{B \in B'(l, r) : B'$ is the associated fundamental basic block of $B\}$. Then there is a one-to-one correspondence between the set $A_{B'}$ and the set $S = \{(n_1, n_2, \ldots, n_m) : n_1 + n_2 + \cdots + n_m = l,\ n_i \geq 1\}$.

Now S is equivalent to the set $S' = \{(n_1, n_2, \ldots, n_m) : n_1 + n_2 + \cdots + n_m = s,\ n_i \geq 0\}$ and $|S| = p_s^m$. Therefore $|A_{B'}| = p_s^m$. But $p_s^m = \binom{s+m-1}{m-1} = \binom{l-1}{m-1}$. Hence $|A_{B'}| = \binom{l-1}{m-1}$.

Thus, for fixed m, the number of basic blocks in $B'(l, r)$ which can be obtained from all $B' \in FB'(m, r)$ is

$$\sum_{B' \in FB'(m,r)} |A_{B'}| = \sum_{B' \in FB'(m,r)} \binom{l-1}{m-1} = \binom{l-1}{m-1} |FB'(m, r)|.$$

Hence $|B'(l, r)| = \displaystyle\sum_{m=[\frac{r+1}{2}]}^{l} \binom{l-1}{m-1} |FB'(m, r)|.$ □

In order to obtain the number of all non-isomorphic basic blocks as well as fundamental basic blocks of given nullity, we see first the following.

Definition 4.1.6. Let $\mathscr{FB}(l)$ be the class of all non-isomorphic fundamental basic blocks of nullity l. Let $\mathscr{FB}'(l)$ be the subclass of $\mathscr{FB}(l)$ such that the reducible elements in each fundamental basic block in it are all comparable.

Definition 4.1.7. Let $\mathscr{B}(l)$ be the class of all non-isomorphic basic blocks of nullity l. Let $\mathscr{B}'(l)$ be the subclass of $\mathscr{B}(l)$ such that the reducible elements in each basic block in it are all comparable.

It follows from Lemma 2.2.6 that,
$|\mathscr{F}\mathscr{B}(l)| = \sum_{r=2}^{2l} |FB(l,r)|$ and $|\mathscr{B}(l)| = \sum_{r=2}^{2l} |B(l,r)|$.
Hence $|\mathscr{F}\mathscr{B}'(l)| = \sum_{r=2}^{2l} |FB'(l,r)|$ and $|\mathscr{B}'(l)| = \sum_{r=2}^{2l} |B'(l,r)|$.
In the next section, we verify the various formulae which we have seen in this Chapter. We also obtain the first few terms of the sequences giving the values of a_r, $|\mathscr{F}\mathscr{B}'(l)|$ and $|\mathscr{B}'(l)|$.

4.2 Counting basic blocks

Now we will obtain for all $r \geq 1$, for all k, $1 \leq k \leq r$ and for all l, $[\frac{r+2}{2}] \leq l \leq \binom{r+1}{2}$, the cardinalities of $FB_k(r+1)$ and $FB'(l, r+1)$ using that of C_k^l. Using these cardinalities, we also obtain the cardinalities of $B'(l,r)$, $FB'(r)$, $\mathscr{F}\mathscr{B}'(l)$ and $\mathscr{B}'(l)$.

Tables for $|C_k^l|$

1. If $r = 1$ then $r+1 = 2, k = 1, l = 1$. Now M_2 (see Fig.5) is the only fundamental basic block satisfying these values. Also, using Theorem 4.1.3, Corollary 4.1.4, Corollary 4.1.6 and Corollary 4.1.5, we have respectively $|C_k^l| = 1$, $|FB'(l, r+1)| = 1$, $|FB_k(r+1)| = 1$ and $a_{r+1} = 1$. On the similar lines,

2. If $r = 2$ then $r+1 = 3, 1 \le k \le 2, 2 \le l \le 3$ and $|C_k^l|$ is given by the following Table 3.

$k \backslash l$	2	3	$\lvert FB_k(3)\rvert$
1	2	0	2
2	1	1	2
$\lvert FB'(l,3)\rvert$	3	1	$a_3 = 4$

Table 3

3. If $r = 3$ then $r+1 = 4, 1 \le k \le 3, 2 \le l \le 6$ and $|C_k^l|$ is given by the following Table 4.

$k \backslash l$	2	3	4	5	6	$\lvert FB_k(4)\rvert$
1	3	9	3	0	0	15
2	0	6	9	3	0	18
3	0	1	3	3	1	8
$\lvert FB'(l,4)\rvert$	3	16	15	6	1	$a_4 = 41$

Table 4

4. If $r = 4$ then $r+1 = 5, 1 \le k \le 4, 3 \le l \le 10$ and $|C_k^l|$ is given by the following Table 5.

$k \backslash l$	3	4	5	6	7	8	9	10	$\lvert FB_k(5)\rvert$
1	24	68	60	24	4	0	0	0	180
2	6	54	108	90	36	6	0	0	300
3	0	12	48	76	60	24	4	0	224
4	0	1	6	15	20	15	6	1	64
$\lvert FB'(l,5)\rvert$	30	135	222	205	120	45	10	1	$a_5 = 768$

Table 5

5. If $r = 5$ then $r + 1 = 6, 1 \leq k \leq 5, 3 \leq l \leq 15$ and $|C_k^l|$ is given by the following Table 6.

$k \backslash l$	3	4	5	6	7	8	9	10	11	12	13	14	15	$\|FB_k(6)\|$
1	15	230	750	1140	1030	600	225	50	5	0	0	0	0	4045
2	0	90	630	1650	2340	2070	1200	450	100	10	0	0	0	8540
3	0	10	180	810	1800	2400	2080	1200	450	100	10	0	0	9040
4	0	0	20	150	500	975	1230	1045	600	225	50	5	0	4800
5	0	0	1	10	45	120	210	252	210	120	45	10	1	1024
$\|FB'(l,6)\|$	15	330	1581	3760	5715	6165	4945	2997	1365	455	105	15	1	$a_6 = 27449$

Table 6

Using Corollary 4.1.4, we get the following Table 7 containing the values of $|FB'(l,r)|$ for $0 \leq r \leq 10$ and $0 \leq l \leq 5$.

This table also gives the first six terms of the sequence of $|\mathscr{F}\mathscr{B}'(l)|$.

$r \backslash l$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	0	0	0	0	0
2	0	1	0	0	0	0
3	0	0	3	1	0	0
4	0	0	3	16	15	6
5	0	0	0	30	135	222
6	0	0	0	15	330	1581
7	0	0	0	0	315	4275
8	0	0	0	0	105	5880
9	0	0	0	0	0	3780
10	0	0	0	0	0	945
$\|\mathscr{F}\mathscr{B}'(l)\|$	1	1	6	62	900	16689

Table 7

Using Corollary 4.1.4, we get the following Table 8 containing the values of $|FB'(l,r)|$ for $0 \le r \le 6$ and $0 \le l \le 15$.

This table also gives the first seven terms of the sequence of a_r.

$r \backslash l$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	a_r
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
3	0	0	3	1	0	0	0	0	0	0	0	0	0	0	0	0	4
4	0	0	3	16	15	6	1	0	0	0	0	0	0	0	0	0	41
5	0	0	0	30	135	222	205	120	45	10	1	0	0	0	0	0	768
6	0	0	0	15	330	1581	3760	5715	6165	4945	2997	1365	455	105	15	1	27449

Table 8

As $|\mathscr{B}'(l)| = \sum_{r=2}^{2l} |B'(l,r)|$, by Theorem 4.1.7, we get the following Table 9 containing the values of $|B'(l,r)|$ for $0 \leq r \leq 10$ and $0 \leq l \leq 5$. This table also gives the first six terms of the sequence of $|\mathscr{B}'(l)|$.

$r\backslash l$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	0	0	0	0	0
2	0	1	1	1	1	1
3	0	0	3	7	12	18
4	0	0	3	22	72	174
5	0	0	0	30	225	942
6	0	0	0	15	375	2991
7	0	0	0	0	315	5535
8	0	0	0	0	105	6300
9	0	0	0	0	0	3780
10	0	0	0	0	0	945
$\lvert\mathscr{B}'(l)\rvert$	1	1	7	75	1105	20,686

Table 9

Thus, we get in the end three important sequences viz., $< a_r >$, $< \mathscr{F}\mathscr{B}'(l) >$ and $< \mathscr{B}'(l) >$ which are given below.

1. For $r \geq 0$, a_r : 1, 0, 1, 4, 41, 768, 27,449, ...
2. For $l \geq 0$, $\mathscr{F}\mathscr{B}'(l)$: 1, 1, 6, 62, 900, 16,689, ...
3. For $l \geq 0$, $\mathscr{B}'(l)$: 1, 1 (see Fig.5), 7 (see Fig.6), 75 (see the Appendix for actual figures), 1105, 20,686, ...

4.3 Enumeration of lattices in which reducible elements are comparable

In this section, we obtain the number of non-isomorphic lattices on n elements in which the reducible elements are all comparable. For this purpose, let us see the following.

Definition 4.3.1. Let $\mathscr{B}(n,k,r) = \{B \in \mathscr{B}(n,k) : |Red(B)| = r\}$. Let $\mathscr{B}'(n,k,r) = \{B \in \mathscr{B}'(n,k) : |Red(B)| = r\}$.

Let $B \in \mathscr{B}'(n,k,s)$. Then $n \geq k+s$ and by Lemma 2.2.6, $2 \leq s \leq 2k$. Let Bb be a basic block associated to B. Then $Bb \in B'(k,s)$. By Theorem 2.1.7, suppose $Bb = C_0]_{a_1}^{b_1}C_1]_{a_2}^{b_2}C_2 \cdots]_{a_k}^{b_k}C_k$ where C_0 is a maximal chain with $a_i,\ b_i \in C_0$, for all $i,\ 1 \leq i \leq k$.
By Proposition 3.2.4, $|C_i| = 1$, for all $i,\ 1 \leq i \leq k$ and $|C_0| = s + m$, where m is the number of distinct adjunct pairs (a_i, b_i) such that $(a_i, b_i) \subseteq Irr(B)$. Note that $m \geq 0$.
Consider a chain $C : x_1 \prec x_2 \cdots \prec x_s$ of the reducible elements of Bb. Clearly $C \subseteq C_0$. Let $M = \{(x_i, x_{i+1}) \mid (x_i, x_{i+1}) = (a_j, b_j)$ for some $j, 1 \leq j \leq k\}$. Then $m = |M|$ and $|(x_i, x_{i+1}) \cap C_0| = 1$, for

all $(x_i, x_{i+1}) \in M$. Now $Red(B) = Red(Bb)$ and $|Irr(Bb)| = m + k$. Therefore $|Bb| = s + m + k$.

Let us denote these m adjunct pairs, if they exist, in Bb by (a'_j, b'_j), $1 \leq j \leq m$. Let m_j be the multiplicity of the adjunct pair (a'_j, b'_j) in Bb. Let $p \geq 0$ be the number of ordered pairs (x_i, x_j), $j > i+1$ such that (x_i, x_j) is an adjunct pair in Bb and $(x_i, x_j) \cap Red(Bb) \neq \phi$ (that is, the interval (x_i, x_j) contains at least one reducible element). Let us denote these p adjunct pairs, if they exist, in Bb by (a''_j, b''_j), $1 \leq j \leq p$. Let p_j be the multiplicity of the adjunct pair (a''_j, b''_j) in Bb.

Now we have the following.

1. $k = \sum_{j=1}^{m} m_j + \sum_{j=1}^{p} p_j$.

2. Also $|Irr(Bb)| = m + k$. That means there are $m + k$ doubly irreducible elements in Bb. Therefore there are $m + k$ chains of doubly irreducible elements in B which correspond to those $m+k$ doubly irreducible elements in Bb.

3. Now there are $l = s - 1 - m$ edges, if they exist, on a maximal chain containing all the reducible elements in Bb. Therefore corresponding to these edges there are l edges or chains of doubly irreducible elements in B.

4. Note that $m + k = m + \sum_{j=1}^{m} m_j + \sum_{j=1}^{p} p_j = \sum_{j=1}^{m}(m_j + 1) + \sum_{j=1}^{p} p_j$.

5. Now $|B| = n$ and $|Red(B)| = s$. Therefore $|Irr(B)| = n - s$ and these $n - s$ elements can be spread into $m + p + l$ parts, say

u_i, $1 \le i \le m+p+l$ satisfying $u_i \ge m_i+1$, for $1 \le i \le m$, $u_i \ge p_i$, for $m+1 \le i \le m+p$ and $u_i \ge 0$, for $m+p+1 \le i \le m+p+l$.

6. Consider $n-s = u_1+u_2+\cdots+u_{m+p+l}$, where $u_i \ge m_i+1$, for $1 \le i \le m$, $u_i \ge p_i$, for $m+1 \le i \le m+p$ and $u_i \ge 0$, for $m+p+1 \le i \le m+p+l$. $(*)$
   Let N be the number of integer solutions to the above equation given in $(*)$. Then $N = P(n-s, m+p+l)$ is the number of weak compositions of $n-s$ into $m+p+l$ parts satisfying the given restrictions.

Note that, the number of weak compositions of 4 into 3 parts is $\binom{4+3-1}{3-1} = \binom{6}{2} = 15$, viz, $(1,1,2), (1,2,1), (2,1,1), (0,2,2), (2,0,2), (2,2,0), (0,1,3), (1,0,3), (1,3,0), (0,3,1), (3,0,1), (3,1,0), (4,0,0), (0,4,0), (0,0,4)$. Therefore the number of weak compositions of 4 into 3 parts, satisfying the given restrictions is at most 15.

A 'C' program can be prepared to find $P(n-s, m+p+l)$.

Let P_n^k denote the number of partitions of n into k (non-zero) parts. Using the notations as discussed above, in the following Proposition 4.3.1, we obtain the number of non-isomorphic blocks associated by the basic block of nullity k, containing s reducible elements which are all comparable.

Proposition 4.3.1. *For any $k \ge 1$, for any $2 \le s \le 2k$, for any $n \ge k+s$ and for $Bb \in B'(k,s)$, the number N_{Bb}^s of non-isomorphic blocks in $\mathscr{B}'(n,k,s)$ which are associated by the basic block Bb is given by* $N_{Bb}^s = \displaystyle\sum_{n-s=u_1+u_2+\cdots+u_{m+p+l}} \left(\prod_{i=1}^{m} P_{u_i}^{m_i+1}\right) \times \left(\prod_{i=m+1}^{m+p} P_{u_i}^{p_i}\right),$

where $u_i \geq m_i + 1$*, for* $1 \leq i \leq m$*,* $u_i \geq p_i$*, for* $m+1 \leq i \leq m+p$ *and* $u_i \geq 0$*, for* $m+p+1 \leq i \leq m+p+l$*.*

Proof. Let $Bb \in B'(k,s)$. Let B be a block in $\mathscr{B}'(n,k,s)$ which is associated by the basic block Bb. As B contains s reducible elements, the remaining $n-s$ irreducible elements of B can be spread in $m+p+l$ parts (u_i) of Bb satisfying the conditions $u_i \geq m_i + 1$, for $1 \leq i \leq m$, $u_i \geq p_i$, for $m+1 \leq i \leq m+p$ and $u_i \geq 0$, for $m+p+1 \leq i \leq m+p+l$. Consider a solution of the equation $n-s = u_1+u_2+\cdots+u_{m+p+l}$, where $u_i \geq m_i+1$, for $1 \leq i \leq m$, $u_i \geq p_i$, for $m+1 \leq i \leq m+p$ and $u_i \geq 0$, for $m+p+1 \leq i \leq m+p+l$. Now the m parts satisfy $u_i \geq m_i+1$, for $1 \leq i \leq m$. Therefore for fixed i, $1 \leq i \leq m$, u_i is partitioned into m_i+1 parts in $P_{u_i}^{m_i+1}$ ways. Further, the p parts satisfy $u_i \geq p_i$, for $m+1 \leq i \leq m+p$. Therefore for fixed i, $m+1 \leq i \leq m+p$, u_i is partitioned into p_i parts in $P_{u_i}^{p_i}$ ways. Furthermore, the l parts satisfying $u_i \geq 0$, for $m+p+1 \leq i \leq m+p+l$ are assigned in unique way. Thus, the total number of non isomorphic blocks in $\mathscr{B}'(n,k,s)$ which are associated by the basic block Bb is given by

$$N_{Bb}^{s} = \sum_{n-s=u_1+u_2+\cdots+u_{m+p+l}} \left(\prod_{i=1}^{m} P_{u_i}^{m_i+1}\right) \times \left(\prod_{i=m+1}^{m+p} P_{u_i}^{p_i}\right),$$

$u_i \geq m_i+1$, for $1 \leq i \leq m$, $u_i \geq p_i$, for $m+1 \leq i \leq m+p$ and $u_i \geq 0$, for $m+p+1 \leq i \leq m+p+l$. □

In the following Proposition 4.3.2, we obtain the number of non-isomorphic blocks of nullity k, containing s reducible elements which are all comparable.

Proposition 4.3.2. *For any $k \geq 1$, for any $2 \leq s \leq 2k$ and for any $n \geq k+s$, $|\mathscr{B}'(n,k,s)| = \sum_{Bb \in B'(k,s)} N^s_{Bb}$.*

In the following Proposition 4.3.3, we obtain the number of non-isomorphic blocks on n elements, having nullity k, in which reducible elements are all comparable.

Proposition 4.3.3. *For any $k \geq 1$ and for any $n \geq k+3$,*

$$|\mathscr{B}'(n,k)| = \sum_{s=2}^{2k} |\mathscr{B}'(n,k,s)|.$$

Proof. The proof follows from Lemma 2.2.6 and the fact that the collection $\{\mathscr{B}'(n,k,s) : 2 \leq s \leq 2k\}$ forms a partition of $\mathscr{B}'(n,k)$. □

In the following Theorem 4.3.4, we obtain the number of non-isomorphic lattices on n elements, having nullity k, in which reducible elements are all comparable.

Theorem 4.3.4. *For any $k \geq 1$ and for any $n \geq k+3$,*

$$|\mathscr{L}'(n,k)| = \sum_{i=0}^{n-k-3} (i+1)|\mathscr{B}'(n-i,k)|.$$

Proof. It is clear that a lattice $L \in \mathscr{L}'(n,k)$ if and only if $L = C \oplus B \oplus C'$, where $B \in \mathscr{B}'(n-i,k)$ and $C,\ C'$ are chains with $|C|+|C'| = i$. For fixed $i \geq 0$, the i elements can be allocated to the chains C and C' in $i+1$ ways. Let $j = n-i$. Now for any $B \in \mathscr{B}'(j,k)$, $j \geq k+3$. Therefore $i = n-j \leq n-(k+3) = n-k-3$. Thus $0 \leq i \leq n-k-3$. Hence the proof is complete. □

Definition 4.3.2. Let $\mathscr{L}'(n)$ be the class of all non-isomorphic lattices of order n such that the reducible elements in each lattice are all comparable.

Theorem 4.3.5. *For any* $n \geq 1$,
$|\mathscr{L}'(n)| = 1 + \sum_{k=1}^{n-3} |\mathscr{L}'(n, k)|$.

Proof. We know that a chain is the only lattice on n elements of nullity 0. Let $L \in \mathscr{L}'(n)$ be any lattice of nullity $k \geq 1$. Then $n \geq k + 3$. Therefore the proof follows from the Theorem 4.3.4. □

Chapter 5

Lattices of nullity up to three

In this Chapter, we count the number of all non-isomorphic lattices of nullity up to three. In the first section, we discuss the enumerations of all non-isomorphic lattices of nullity up to two. In the second section, we enumerate all the non-isomorphic lattices on n elements and having nullity 3, in which at least two of the reducible elements are incomparable. In regard to this, we prove that, there are in all *seventeen* (see Fig.9, Fig.10 and Fig.11) non-isomorphic basic blocks (in fact, fundamental basic blocks) of nullity 3, in which at least two of the reducible elements are incomparable. In the last section, we enumerate all the non-isomorphic lattices on n elements and having nullity three.

[0]The paper based on partial content of this chapter has been presented in the 71^{st} National Conference of Indian Mathematical Society held at Indian Institute of Technology, Roorkee, during 26^{th} to 29^{th} December, 2005. The IMS prize, 2005 has been awarded for the best paper presentation in Discrete Mathematics.

5.1 Enumeration of lattices of nullity up to two

By Theorem 2.2.3, it follows that the lattices of nullity up to three are dismantlable. Recall that, $\mathscr{L}(n,k)$ denotes the class of all non-isomorphic dismantlable lattices on n elements such that each lattice in it has nullity k. Note that, there is only one lattice, a chain, having nullity zero. Therefore $\mathscr{L}(n,0)$ consists of the chain on n elements. The enumeration of all non-isomorphic lattices on n elements and having nullity up to two was carried out by Thakare, Pawar and Waphare [13].

Theorem 5.1.1. *[8],[13]. For any integer $n \geq 4$,*

$$|\mathscr{L}(n,1)| = \begin{cases} \frac{m(m-1)(4m+1)}{6} & if \quad n = 2m+1 \\ \\ \frac{m(m-1)(4m-5)}{6} & if \quad n = 2m. \end{cases}$$

Let $[x]$ denote the integer part of real number x and let $< x >$ denote the nearest integer of real number x.

Theorem 5.1.2. *[13]. For any integer $n \geq 5$,*
$|\mathscr{L}(n,2)| = \sum_{i=0}^{n-5}(i+1)|\mathscr{B}(n-i,2)|,$ *where*

$$|\mathscr{B}(j,2)| = \begin{cases} \left\langle \frac{14k^4+54k^3+68k^2+36k+9}{12} \right\rangle & if \quad j = 2k+5; \\ \\ \left[\frac{(k+2)(7k^3+27k^2+31k+13)}{6} \right] & if \quad j = 2k+6. \end{cases}$$

In the next section, we enumerate the class of all non-isomorphic lattices on n elements such that each lattice in it has nullity three and at least two of the reducible elements in each lattice in it are incomparable.

5.2 Lattices in which reducible elements are incomparable

It is clear that the reducible elements of a lattice of nullity up to two are all comparable and that for nullity at least three these may be incomparable.

5.2.1 Counting fundamental basic blocks

In this subsection, we count all the non-isomorphic fundamental basic blocks of nullity three such that at least two of the reducible elements in each are incomparable. This counting would help us in enumerating the lattices of nullity three. For this purpose, let us begin with the following.

Definition 5.2.1. Let $B''(k, r)$ be the class of all non-isomorphic basic blocks of nullity k such that at least two of the r reducible elements in each of the basic blocks in it are incomparable.

For the class $B''(k, r)$, it follows from Proposition 2.2.8 that $k \geq 3$ and $r \geq 4$. Also, by Lemma 2.2.6, if $k = 3$ then $2 \leq r \leq 6$. Therefore, if a lattice of nullity three contains r reducible elements such that at least two of them are incomparable then $4 \leq r \leq 6$.
In the following Proposition 5.2.1, we prove that, there are three (see Fig.9) non-isomorphic basic blocks of nullity three, containing four reducible elements such that at least two of them are incomparable.

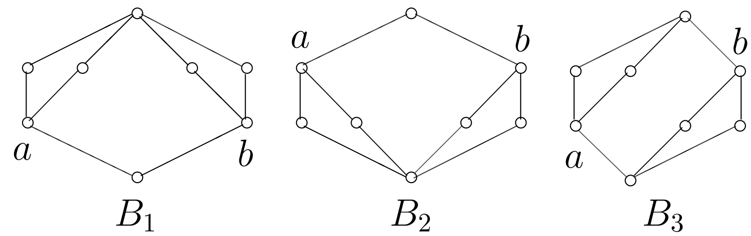


Fig.9

Proposition 5.2.1. $|B''(3,4)| = 3$.

Proof. Let $B \in B''(3,4)$. Let $0, 1, a, b$ be the reducible elements of B. Now at least two of them are incomparable, therefore $a||b$. Also $a \wedge b = 0$ and $a \vee b = 1$. Clearly none of a or b is both meet as well as join reducible, since otherwise nullity of B is greater than 3. Therefore we have the following three cases.

1. If a and b both are meet reducible elements then B is isomorphic to the block given in figure Fig.9(B_1).

2. If a and b both are join reducible elements then B is isomorphic to the block given in figure Fig.9(B_2).

3. If without loss of generality, suppose a is meet reducible element and b is join reducible element then B is isomorphic to the block given in figure Fig.9(B_3). □

In the following Proposition 5.2.2, we prove that, there are eight (see Fig.10) non-isomorphic basic blocks of nullity three, containing five reducible elements such that at least two of them are incomparable.

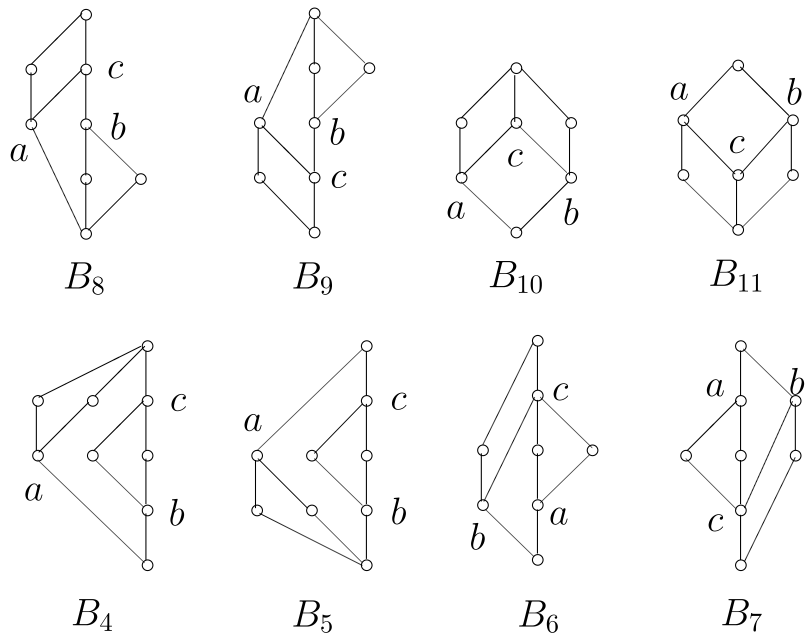


Fig.10

Proposition 5.2.2. $|B''(3,5)| = 8$.

Proof. Let $B \in B''(3,5)$. Therefore not all 5 reducible elements of B are comparable. Let $0, 1, a, b, c$ be the reducible elements of B. Now at least two of them are incomparable. Without loss of generality, suppose $a||b$. Now we have the following three cases.

1. If $c||a$ and $c||b$ then the nullity of B is greater than 3. This is not possible.

2. If without loss of generality, suppose $c||a$ and c is comparable to b. Now we have the following three subcases.

(i) Suppose a is a meet reducible element only. If both b and c are either meet reducible elements or join reducible elements then the nullity of B is greater than 3. This is not possible. If without loss of generality, suppose b is a meet reducible element and c is a join reducible element then nullity of B is 3 implies that (b, c) is an adjunct pair in an adjunct representation of B. In this case B is isomorphic to the block given in figure Fig.10(B_4).

(ii) Suppose a is a join reducible element only. If both b and c are either meet reducible elements or join reducible elements then the nullity of B is greater than 3. This is not possible. If without loss of generality, suppose b is a meet reducible element and c is a join reducible element then nullity of B is 3 implies that (b, c) is an adjunct pair in an adjunct representation of B. In this case B is isomorphic to the block given in figure Fig.10(B_5).

(iii) Suppose a ia meet reducible as well as join reducible element. Then nullity of B is greater than 3. This is not possible.

3. If c is comparable to both a and b. Then we have the following three subcases.

(i) If c is a meet reducible element only then a and b can not both be meet reducible elements, since otherwise nullity of B is greater than 3. If both a and b are join reducible elements then $a \wedge b = c$ and (c, a) and (c, b) can not both be adjunct pairs in an adjunct representation of B, since otherwise the nullity of B is greater than 3. Therefore at the most one of them may be an adjunct pair in an adjunct representation of B. Suppose without loss of generality, (c, a) is an adjunct pair. But then there exists $x \in B$ such that $x \wedge c = 0$ and $x \vee c = b$. In this case B is isomorphic to the block given in figure Fig10(B_7), since the nullity of B is 3. If none of them is an adjunct pair then there exist $x, y \in B$ such that $x \wedge c = 0, x \vee c = a, y \wedge c = 0$ and $y \vee c = b$. In this case B is isomorphic to the block given in figure Fig.10(B_{11}), since the nullity of B is 3. Also, if a is join(meet) reducible and b is meet(join) reducible element then B is isomorphic to the block given in figure Fig.10(B_9), since the nullity of B is 3.

(ii) If c is a join reducible element only then a and b can not both be join reducible elements, since otherwise the nullity of B is greater than 3.If both a and b are meet reducible elements then $a \vee b = c$ and (a, c) and (b, c) can not both be adjunct pairs in an adjunct representation of B, since otherwise the nullity of B is greater than 3. Therefore at the most one of them may be an adjunct pair in an adjunct representation of B. Suppose without loss of generality, (a, c) is an adjunct pair. But then there exists $x \in B$ such that $x \wedge c = b$ and $x \vee c = 1$. In this case B is isomorphic to the block given in figure Fig10(B_6), since the nullity of B is 3. If none of them is an adjunct pair then there exist $x, y \in B$ such that $x \wedge c = a, x \vee c = 1, y \wedge c = b$ and $y \vee c = 1$. In this case B is isomorphic to the block given in figure Fig.10(B_{10}), since the nullity of B is 3. Also, if a is join(meet) reducible and b is meet(join) reducible element then B is isomorphic to the block given in figure Fig.10(B_8), since the nullity of B is 3.

(iii) If c is both meet reducible as well as join reducible element then we have either $a \wedge b = 0$ or c. In any case, the nullity of B is greater than 3. This is not possible. □

In the following Proposition 5.2.3, we prove that, there are six (see Fig.11) non-isomorphic basic blocks of nullity three, containing six reducible elements such that at least two of them are incomparable.

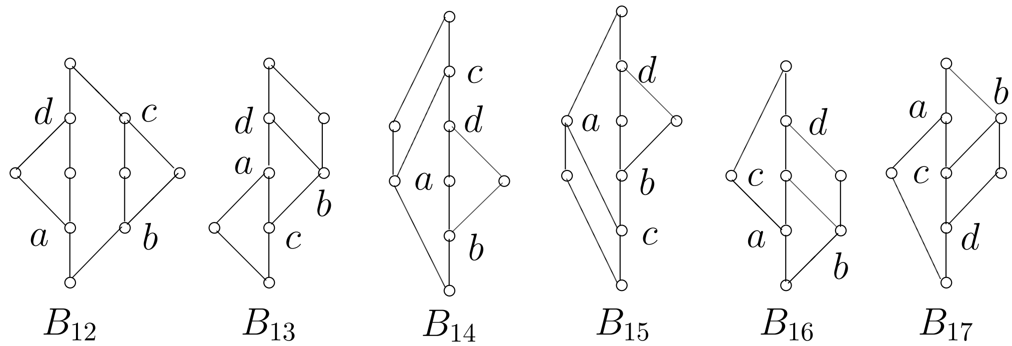


Fig.11

Proposition 5.2.3. $|B''(3,6)| = 6$.

Proof. Let $B \in B''(3,6)$. Therefore not all 6 reducible elements of B are comparable. Let $0, 1, a, b, c, d$ be the reducible elements of B. Now at least two of them are incomparable. Without loss of generality, suppose $a||b$. Now we have the following three cases.

1. Neither c nor d is incomparable to both a and b, since otherwise the nullity of B is greater than 3.

2. If without loss of generality, suppose (among c and d) $c||a$ and c is comparable to b. If $a||d$ then nullity of B is greater than 3. Therefore a and d are comparable. If d is also comparable to either b or c then nullity of B is greater than 3. Hence $d||b$ and $d||c$. But then B is isomorphic to the block given in figure Fig.11(B_{12}).

3. If without loss of generality, suppose (among c and d) c is comparable to both a and b. Then we have the following three subcases.

(i) Suppose c is meet reducible only. Let $x = a \wedge b$.

If $x = 0$ then $a \vee b \neq c$, since c is meet reducible element only. Also $a \vee b \neq 1$, since otherwise we get a contradiction to the fact that c is comparable to both a and b. Therefore $a \vee b = d$. This implies that

$d < c$, since c is comparable to both a and b. But then the nullity of B is greater than 3. This is not possible.

If $x \neq 0$ then either $x = c$ or $x = d$. Without loss of generality, if $x = c$ then either $a \vee b = d$ or $a \vee b = 1$. If $a \vee b = d$ then B is isomorphic to the block given in figure Fig.11(B_{13}), since the nullity of B is 3. If $a \vee b = 1$ then either $c||d$ or c is comparable to d. If $c||d$ then nullity of B is greater than 3. This is not possible. If $d < c$ then B is isomorphic to the block given in figure Fig.11(B_{17}), since nullity of B is 3. If $c < d$ then d is incomparable to either a or b. If $d||a$ and $d||b$ then nullity of B is greater than 3. Therefore, if without loss of generality, suppose (among a and b) $d||a$ and d is comparable to b then B is isomorphic to the block given in figure Fig.11(B_{15}), since the nullity of B is 3.

(ii) Suppose c is join reducible only. Let $x = a \vee b$.

If $x = 1$ then $a \wedge b \neq c$, since c is join reducible only. Also $a \wedge b \neq 0$, since otherwise we get a contradiction to the fact that c is comparable to both a and b. Therefore $a \wedge b = d$. This implies that $c < d$, since c is comparable to both a and b. But then the nullity of B is greater than 3. This is not possible.

If $x \neq 1$ then either $x = c$ or $x = d$. Without loss of generality, if $x = c$ then either $a \wedge b = d$ or $a \wedge b = 0$. If $a \wedge b = d$ then B is isomorphic to the block given in figure Fig.11(B_{13}), since the nullity of B is 3. If $a \wedge b = 0$ then either $c||d$ or c is comparable to d. If $c||d$ then the nullity of B is greater than 3. This is not possible. If $c < d$ then B is isomorphic to the block given in figure Fig.11(B_{16}), since the nullity of B is 3. If $d < c$ then d is incomparable to either a or b. If $d||a$ and

$d||b$ then the nullity of B is greater than 3. Therefore, if without loss of generality, suppose (among a and b) $d||a$ and d is comparable to b then B is isomorphic to the block given in figure Fig.11(B_{14}), since the nullity of B is 3.

(iii) Suppose c is a meet reducible as well as join reducible. Then the nullity of B is greater than 3. This is not possible. □

***Remark* 5.2.1.** From the figures Fig.9, Fig.10 and Fig.11, it follows by observation that all the basic blocks depicted in these figures are fundamental basic blocks. Therefore by Proposition 5.2.1, Proposition 5.2.2 and Proposition 5.2.3, we have

1. $B''(3,4) = \{B_1, B_2, B_3\}$.
2. $B''(3,5) = \{B_4, B_5, B_6, B_7, B_8, B_9, B_{10}, B_{11}\}$.
3. $B''(3,6) = \{B_{12}, B_{13}, B_{14}, B_{15}, B_{16}, B_{17}\}$.

Thus, there are in all seventeen non-isomorphic basic blocks of nullity three such that at least two of the reducible elements in each are incomparable.

Definition 5.2.2. Let $\mathscr{B}''(n,k)$ be the class of all non-isomorphic blocks on n elements such that each block in it has nullity k and at least two of the reducible elements in each block in it are incomparable.

Let $\mathscr{B}''(n,k,r)$ be the subclass of $\mathscr{B}''(n,k)$ such that each block in it contains r reducible elements.

For the class $\mathscr{B}''(n,k,r)$, if $k = 3$ then $4 \leq r \leq 6$. Therefore, $\mathscr{B}''(n,3) = \mathscr{B}''(n,3,4)\dot{\cup}\mathscr{B}''(n,3,5)\dot{\cup}\mathscr{B}''(n,3,6)$. Thus, in order to obtain the cardinality of the class $\mathscr{B}''(n,3)$, we first obtain the cardinalities of the classes $\mathscr{B}''(n,3,4)$, $\mathscr{B}''(n,3,5)$ and $\mathscr{B}''(n,3,6)$. For this purpose, we define in the following seventeen classes corresponding to each (fundamental) basic block of nullity three, in which at least two of the reducible elements are incomparable.

Definition 5.2.3. For each i, $1 \leq i \leq 17$, let
$\mathbb{B}_i = \{B \in \mathscr{B}''(n,3) \mid B_i \text{ is the basic block associated to } B\}$.

By Theorem 3.3.1, it follows that $\{\mathbb{B}_i : 1 \leq i \leq 17\}$ forms a partition of the class $\mathscr{B}''(n,3)$. By observation, again using Theorem 3.3.1, it also follows that $\{\mathbb{B}_i : 1 \leq i \leq 3\}$ forms a partition of the class $\mathscr{B}''(n,3,4)$, $\{\mathbb{B}_i : 4 \leq i \leq 11\}$ forms a partition of the class $\mathscr{B}''(n,3,5)$ and $\{\mathbb{B}_i : 12 \leq i \leq 17\}$ forms a partition of the class $\mathscr{B}''(n,3,6)$.

5.2.2 Enumeration of blocks on four reducible elements

We now consider the problem of enumeration of blocks on four reducible elements; that is, to find $|\mathscr{B}''(n,3,4)|$. Recall that, $\{\mathbb{B}_i : 1 \leq i \leq 3\}$ forms a partition of the class $\mathscr{B}''(n,3,4)$. Therefore, it is required to find the cardinality of the class $\mathbb{B}_i$ for each i, $1 \leq i \leq 3$.
To begin with, we first define the class $\mathscr{L}^1(n,1)$ as the subclass of $\mathscr{L}(n,1)$, containing the lattices in which 1 is a reducible element.
In the following, we obtain the cardinality of the class $\mathscr{L}^1(n,1)$.

Lemma 5.2.4. *For $n \geq 4$, $|\mathscr{L}^1(n,1)| = \displaystyle\sum_{i=0}^{n-4} \left[\frac{n-i-2}{2}\right]$.*

Proof. Let $L \in \mathscr{L}^1(n,1)$. Then $L = C \oplus B$ where C is a chain with $|C| = i \geq 0$ and $B \in \mathscr{B}(j,1)$ with $n = i + j$. Now $j \geq 4$. Therefore $i = n - j \leq n - 4$. The proof follows from the fact that $|\mathscr{B}(j,1)| = P^2_{j-2} = [\frac{j-2}{2}]$ for all $j \geq 4$. □

***Remark* 5.2.2.** For any $j \geq 3$, let S_j be the set of all non-isomorphic posets Y such that $Y = C]_x C'$ and $|Y| = j$, where C, C' are chains. Then $Y \in S_j$ if and only if $Y \oplus \{1\} \in \mathscr{L}^1(j+1, 1)$. Therefore $|S_j| = |\mathscr{L}^1(j+1,1)|$.

If $s_j = |S_j|$ for all j then $s_j = |\mathscr{L}^1(j+1,1)| = \sum_{i=0}^{j-3} \left[\frac{j-i-1}{2}\right]$.

Recall that, $\mathbb{B}_1 = \{B \in \mathscr{B}''(n,3) \mid B_1$ (see Fig.9) is the basic block associated to $B\}$. In the following Proposition 5.2.5, we obtain the cardinality of the class $\mathbb{B}_1$.

Proposition 5.2.5. *For* $n \geq 8$,

$$|\mathbb{B}_1| = \begin{cases} \sum\limits_{n=i+j+2, i>j} s_i s_j, & \textit{if } n \textit{ is odd;} \\ \sum\limits_{n=i+j+2, i>j} s_i s_j + \dfrac{s_{\frac{n-2}{2}}\left(s_{\frac{n-2}{2}}+1\right)}{2}, & \textit{if } n \textit{ is even,} \end{cases}$$

where $s_i = \sum_{k=0}^{i-3} \left[\frac{i-k-1}{2}\right]$.

Proof. Let $B \in \mathbb{B}_1$. Then $B - \{0,1\}$ is the disjoint union of two subposets, say Y_1 and Y_2 of B such that each one of them is an up 1-sum of two chains. By (ii) of Theorem 3.2.1, $Red(B) = Red(B_1)$. Therefore, as $a, b \in Red(B_1)$, let $Y_1 = C_1]_a C_2$ and let $Y_2 = C_3]_b C_4$ with $|Y_1| = i \geq 3$ and $|Y_2| = j \geq 3$, where C_1, C_2, C_3 and C_4 are chains. Suppose without loss of generality, $B = (\{0\} \oplus Y_1 \oplus \{1\})]_0^1 Y_2$ with $|Y_1| = i \geq |Y_2| = j$ and $|B| = n = i+j+2 \geq 8$. It is clear that $Y_1 \in S_i$ and $Y_2 \in S_j$. Let $B' \in \mathbb{B}_1$ be such that $B' = (\{0\} \oplus Y_1' \oplus \{1\})]_0^1 Y_2'$. Then $B \cong B'$ if and only if $Y_1 \cong Y_1'$ and $Y_2 \cong Y_2'$. Therefore, if $i > j$ then there are $\sum\limits_{n=i+j+2} (|S_i| \times |S_j|)$ non-isomorphic blocks in $\mathbb{B}_1$. But if

$i = j$ then n must be even and it seems that there are $|S_i|^2$ blocks(all may not non-isomorphic). In fact, there are $\binom{|S_i|}{2}$ blocks which are counted twice, since $i = j$. Therefore in the case when $i = j$, there are $|S_i|^2 - \binom{|S_i|}{2} = \frac{|S_i|(|S_i|+1)}{2}$ non-isomorphic blocks in $\mathbb{B}_1$. The proof follows from the fact that $s_i = |S_i| = \sum_{k=0}^{i-3} \left[\frac{i-k-1}{2}\right]$. □

Recall that, $\mathbb{B}_2 = \{B \in \mathscr{B}''(n,3) \mid B_2$ (see Fig.9) is the basic block associated to $B\}$. Note that, the dual B_2^* of the basic block B_2 is B_1 (see Fig.9). In the following Corollary 5.2.6, we obtain the cardinality of the class $\mathbb{B}_2$.

Corollary 5.2.6. *For $n \geq 8$,*

$$|\mathbb{B}_2| = \begin{cases} \sum\limits_{n=i+j+2, i>j} s_i s_j, & if\ n\ is\ odd; \\ \sum\limits_{n=i+j+2, i>j} s_i s_j + \dfrac{s_{\frac{n-2}{2}}\left(s_{\frac{n-2}{2}}+1\right)}{2}, & if\ n\ is\ even, \end{cases}$$

where $s_i = \sum_{k=0}^{i-3} \left[\frac{i-k-1}{2}\right]$.

Proof. Clearly $|\mathbb{B}_2| = |\mathbb{B}_1|$, since $B \in \mathbb{B}_2$ if and only if the dual of B, $B^* \in \mathbb{B}_1$. Thus the proof follows by Proposition 5.2.5. □

Recall that, $\mathbb{B}_3 = \{B \in \mathscr{B}''(n,3) \mid B_3$ (see Fig.9) is the basic block associated to $B\}$. In the following Proposition 5.2.7, we obtain the cardinality of the class $\mathbb{B}_3$.

Proposition 5.2.7. *For $n \geq 8$,*

$$|\mathbb{B}_3| = \sum_{n=i+j+2} s_i s_j, \ where\ s_i = \sum_{k=0}^{i-3} \left[\frac{i-k-1}{2}\right].$$

Proof. Let $B \in \mathbb{B}_3$. Then $B - \{0,1\}$ is the disjoint union of two subposets, say Y_1 and Y_2 of B such that one of them is an up 1-sum of two chains and the other is a down 1-sum of two chains. By (ii) of Theorem 3.2.1, $Red(B) = Red(B_3)$. Therefore, as $a, b \in Red(B_3)$, let $Y_1 = C_1]_a C_2$ and $Y_2 = C_3]^b C_4$ with $|Y_1| = i \geq 3$ and $|Y_2| = j \geq 3$, where C_1, C_2, C_3 and C_4 are chains. Then either $B = (\{0\} \oplus Y_1 \oplus \{1\})]_0^1 Y_2$ or $B = (\{0\} \oplus Y_2 \oplus \{1\})]_0^1 Y_1$ with $|B| = n = i + j + 2 \geq 8$. It is clear that $Y_1 \in S_i$ and the dual of Y_2, $Y_2^* \in S_j$. (Note that $Y_1 \oplus \{1\} \in \mathscr{L}^1(i+1, 1)$ and $(\{0\} \oplus Y_2)^* \in \mathscr{L}^1(j+1, 1)$.)
Therefore, $|\mathbb{B}_3| = \sum_{n=i+j+2} (|S_i| \times |S_j|)$. The proof follows from the fact that $s_k = |S_k| = \sum_{i=0}^{k-3} \left[\frac{k-i-1}{2}\right]$. □

In the following Theorem 5.2.8, we obtain the number of non-isomorphic blocks on n elements, having nullity three, and containing four reducible elements such that at least two of them are incomparable.

Theorem 5.2.8. *For $n \geq 8$,*

$$|\mathscr{B}''(n,3,4)| = \begin{cases} \sum\limits_{n=i+j+2, i>j} 4 s_i s_j & \textit{if } n \textit{ is odd;} \\ \sum\limits_{n=i+j+2, i>j} 4 s_i s_j + s_{\frac{n-2}{2}} \left(2 s_{\frac{n-2}{2}} + 1\right) & \textit{if } n \textit{ is even,} \end{cases}$$

where $s_i = \sum_{k=0}^{i-3} \left[\frac{i-k-1}{2}\right]$.

Proof. As $\{\mathbb{B}_i : 1 \leq i \leq 3\}$ forms a partition of the class $\mathscr{B}''(n,3,4)$, we have $|\mathscr{B}''(n,3,4)| = |\mathbb{B}_1| + |\mathbb{B}_2| + |\mathbb{B}_3|$. But by Corollary 5.2.6, $|\mathbb{B}_2| = |\mathbb{B}_1|$. Therefore $\mathscr{B}''(n,3,4) = 2|\mathbb{B}_1| + |\mathbb{B}_3|$. The remaining proof follows from Proposition 5.2.5 and Proposition 5.2.7. □

5.2.3 Enumeration of blocks on five reducible elements

We now consider the problem of enumeration of blocks on five reducible elements; that is, to find $|\mathscr{B}''(n,3,5)|$. Recall that, $\{\mathbb{B}_i : 4 \leq i \leq 11\}$ forms a partition of the class $\mathscr{B}''(n,3,5)$. Therefore, it is required to find the cardinality of the class $\mathbb{B}_i$ for each i, $4 \leq i \leq 11$.
Now recall that, $\mathbb{B}_4 = \{B \in \mathscr{B}''(n,3) \mid B_4$ (see Fig.10) is the basic block associated to $B\}$. In the following Proposition 5.2.9, we obtain the cardinality of the class $\mathbb{B}_4$.

Proposition 5.2.9. *For $n \geq 9$,*

$$|\mathbb{B}_4| = \sum_{n=i+j+2} (|\mathscr{L}(i,1)| \times s_j), \textit{ where } s_j = \sum_{i=0}^{j-3} \left[\frac{j-i-1}{2}\right] \textit{ and}$$

$$|\mathscr{L}(i,1)| = \begin{cases} \frac{m(m-1)(4m+1)}{6} & if \quad i = 2m+1; \\ \frac{m(m-1)(4m-5)}{6} & if \quad i = 2m. \end{cases}$$

Proof. Let $B \in \mathbb{B}_4$. Then $B - \{0,1\}$ is the disjoint union of a sublattice $M \in \mathscr{L}(i,1)$ and a subposet $Y \in S_j$ of B, where $i \geq 4$ and $j \geq 3$ with $|B| = n = i+j+2 \geq 9$. Then either $B = (\{0\} \oplus M \oplus \{1\})]_0^1 Y$ or $B = (\{0\} \oplus Y \oplus \{1\})]_0^1 M$. Therefore, $|\mathbb{B}_4| = \sum_{n=i+j+2} (|\mathscr{L}(i,1)| \times |S_j|)$. Therefore the proof follows from the fact that $s_j = |S_j| = \sum_{i=0}^{j-3} \left[\frac{j-i-1}{2}\right]$ and Theorem 5.1.1. □

Recall that, $\mathbb{B}_5 = \{B \in \mathscr{B}''(n,3) \mid B_5$ (see Fig.10) is the basic block associated to $B\}$. Note that, the dual B_5^* of the basic block B_5 is B_4 (see Fig.10). In the following Corollary 5.2.10, we obtain the cardinality of the class $\mathbb{B}_5$.

Corollary 5.2.10. *For $n \geq 9$,*

$$|\mathbb{B}_5| = \sum_{n=i+j+2} (|\mathscr{L}(i,1)| \times s_j), \text{ where } s_j = \sum_{i=0}^{j-3} \left[\frac{j-i-1}{2}\right] \text{ and}$$

$$|\mathscr{L}(i,1)| = \begin{cases} \frac{m(m-1)(4m+1)}{6} & if \quad i = 2m+1; \\ \frac{m(m-1)(4m-5)}{6} & if \quad i = 2m. \end{cases}$$

Proof. Clearly $|\mathbb{B}_5| = |\mathbb{B}_4|$, since $B \in \mathbb{B}_5$ if and only if the dual of B, $B^* \in \mathbb{B}_4$. Thus the proof follows by Proposition 5.2.9. □

For $n \geq 6$, let $\mathcal{B}^n_{1,2}$ be the class of all non-isomorphic blocks (of nullity two) of the type B, where $B = C_1]^{b_1}_{a_1} C_2]^{b_2}_{a_2} C_3$ and $0 = a_1 < a_2 < b_1 = b_2 = 1$.

Proposition 5.2.11. *[13]. For $n \geq 6$,*

$$|\mathcal{B}^n_{1,2}| = \sum_{r=1}^{\left[\frac{(n-4)}{2}\right]} \sum_{l=1}^{(n-2r-3)} (n-l-2r-2).$$

Recall that, $\mathbb{B}_6 = \{B \in \mathscr{B}''(n,3) \mid B_6$ (see Fig.10) is the basic block associated to $B\}$. In the following Proposition 5.2.12, we obtain the cardinality of the class $\mathbb{B}_6$.

Proposition 5.2.12. *For $n \geq 8$,*

$$|\mathbb{B}_6| = \sum_{j=1}^{(n-7)} \sum_{i=1}^{(n-j-6)} \sum_{r=1}^{\left[\frac{(n-i-j-4)}{2}\right]} \sum_{l=1}^{(n-i-j-2r-3)} (n-i-j-l-2r-2)(l).$$

Proof. Let $B \in \mathbb{B}_6$. As the nullity of B is 3, by Corollary 2.2.4, $B = C_1]^{b_1}_{a_1=0} C_2]^{b_2=b_1}_{a_2} C_3]^{b_3=1}_{a_3} C_4$, where C_1 is a maximal chain containing a_1, a_2, b_1, b_2, b_3 and C_2, C_3, C_4 are chains with $a_1 = 0 < (a_2 = a \mid\mid a_3 = b) < b_1 = b_2 = c < b_3 = 1$ and $a_3 \in C_2$. Note that, by (ii) of

Theorem 3.2.1, $Red(B) = Red(B_6)$ and $a, b, c \in Red(B_6)$. Let $B' = (C_1 \cap [a_1, b_1])]_{a_1}^{b_1} C_2]_{a_2}^{b_2} C_3$, $C_1' = (C_1 \cap (b_1, b_3])$ and $C_2' = C_4$. Then $B = (B' \oplus C_1')]_{a_3}^{1} C_2'$. Let $|B'| = k \geq 6$, $|C_1'| = i \geq 1$ and $|C_2'| = j \geq 1$. Then $B' \in \mathcal{B}_{1,2}^k$ and $|B| = n = i + j + k \geq 8$. By Proposition 5.2.11,

$$|\mathcal{B}_{1,2}^k| = \sum_{r=1}^{\left[\frac{(k-4)}{2}\right]} \sum_{l=1}^{(k-2r-3)} (k - l - 2r - 2),$$

where $l = |C_2|$, $r = |C_3|$ and for fixed l and r, $(k - l - 2r - 2)$ is the number of possible positions of a_2 in the block B' and hence in the block $B \in \mathbb{B}_6$. Now for fixed i and j, $k = n - i - j \geq 6$. Therefore for fixed j, we have $1 \leq i = n - j - k \leq n - j - 6$ and therefore $1 \leq j = n - i - k \leq n - 1 - 6 = n - 7$. Now a_3 takes $|C_2| = l$ number of positions in the block $B' \in \mathcal{B}_{1,2}^k$ and hence in the block $B \in \mathbb{B}_6$. Therefore we have for all $n \geq 8$,

$$|\mathbb{B}_6| = \sum_{j=1}^{(n-7)} \sum_{i=1}^{(n-j-6)} \sum_{r=1}^{\left[\frac{(n-i-j-4)}{2}\right]} \sum_{l=1}^{(n-i-j-2r-3)} (n - i - j - l - 2r - 2) \times (l).$$

□

Recall that, $\mathbb{B}_7 = \{B \in \mathscr{B}''(n, 3) \mid B_7$ (see Fig.10) is the basic block associated to $B\}$. Note that, the dual B_7^* of the basic block B_7 is B_6 (see Fig.10). In the following Corollary 5.2.13, we obtain the cardinality of the class $\mathbb{B}_7$.

Corollary 5.2.13. *For $n \geq 8$,*

$$|\mathbb{B}_7| = \sum_{j=1}^{(n-7)} \sum_{i=1}^{(n-j-6)} \sum_{r=1}^{\left[\frac{(n-i-j-4)}{2}\right]} \sum_{l=1}^{(n-i-j-2r-3)} (n - i - j - l - 2r - 2)(l).$$

Proof. Clearly $|\mathbb{B}_7| = |\mathbb{B}_6|$, since $B \in \mathbb{B}_7$ if and only if the dual of B, $B^* \in \mathbb{B}_6$. Thus the proof follows by Proposition 5.2.12. □

For $n \geq 6$, let $\mathcal{B}^n_{1,3}$ be the class of all non-isomorphic blocks (of nullity two) of the type B, where $B = C_1]^{b_1}_{a_1} C_2]^{b_2}_{a_2} C_3$ and $0 = a_1 = a_2 < b_1 < b_2 = 1$. Note that, $B \in \mathcal{B}^n_{1,3}$ if and only if $B^* \in \mathcal{B}^n_{1,2}$.
Recall that, $\mathbb{B}_8 = \{B \in \mathscr{B}''(n,3) \mid B_8$ (see Fig.10) is the basic block associated to $B\}$. In the following Corollary 5.2.14, we obtain the cardinality of the class $\mathbb{B}_8$.

Corollary 5.2.14. *For $n \geq 8$,*

$$|\mathbb{B}_8| = \sum_{j=1}^{(n-7)} \sum_{i=1}^{(n-j-6)} \sum_{r=1}^{\left[\frac{(n-i-j-4)}{2}\right]} \sum_{l=1}^{(n-i-j-2r-3)} (n-i-j-l-2r-2)(l).$$

Proof. Let $B \in \mathbb{B}_8$. As nullity of B is 3,
$B = C_1]^{b_1}_{a_1=a_2=0} C_2]^{b_2}_{a_2} C_3]^{b_3=1}_{a_3} C_4$, where C_1 is a maximal chain containing a_1, a_2, b_1, b_2, b_3 and C_2, C_3, C_4 are chains with $a_1 = a_2 = 0 < (b_1 = b \mid\mid a_3 = a) < b_2 = c < b_3 = 1$ and $a_3 \in C_3$. Note that, by (ii) of Theorem 3.2.1, $Red(B) = Red(B_8)$ and $a, b, c \in Red(B_8)$. Let $B' = (C_1 \cap [a_1, b_2])]^{b_1}_{a_1} C_2]^{b_2}_{a_2} C_3$, $C'_1 = (C_1 \cap (b_2, b_3])$ and $C'_2 = C_4$. Then $B = (B' \oplus C'_1)]^1_{a_3} C'_2$. Let $|B'| = k \geq 6$, $|C'_1| = i \geq 1$ and $|C'_2| = j \geq 1$. Then $B' \in \mathcal{B}^k_{1,3}$ and $|B| = n = i + j + k \geq 8$. Now $|\mathcal{B}^k_{1,3}| = |\mathcal{B}^k_{1,2}|$ for all $k \geq 6$, since a block $D \in \mathcal{B}^k_{1,3}$ if and only if its dual $D^* \in \mathcal{B}^k_{1,2}$. Therefore $|\mathbb{B}_8| = |\mathbb{B}_6|$ and hence by Proposition 5.2.12,

$$|\mathbb{B}_8| = \sum_{j=1}^{(n-7)} \sum_{i=1}^{(n-j-6)} \sum_{r=1}^{\left[\frac{(n-i-j-4)}{2}\right]} \sum_{l=1}^{(n-i-j-2r-3)} (n-i-j-l-2r-2) \times (l).$$

Note here that $l = |C_3|$, $r = |C_2|$ and for fixed l and r, $(k - l - 2r - 2)$ is the number of possible positions of b_1 in the block B' and hence in the block $B \in \mathbb{B}_8$. □

Recall that, $\mathbb{B}_9 = \{B \in \mathscr{B}''(n,3) \mid B_9$ (see Fig.10) is the basic block associated to $B\}$. Note that, the dual B_9^* of the basic block B_9 is B_8 (see Fig.10). In the following Corollary 5.2.15, we obtain the cardinality of the class $\mathbb{B}_9$.

Corollary 5.2.15. *For $n \geq 8$,*

$$|\mathbb{B}_9| = \sum_{j=1}^{(n-7)} \sum_{i=1}^{(n-j-6)} \sum_{r=1}^{\left[\frac{(n-i-j-4)}{2}\right]} \sum_{l=1}^{(n-i-j-2r-3)} (n-i-j-l-2r-2)(l).$$

Proof. Clearly $|\mathbb{B}_9| = |\mathbb{B}_8|$, since $B \in \mathbb{B}_9$ if and only if the dual of B, $B^* \in \mathbb{B}_8$. Thus the proof follows by Corollary 5.2.14. □

Recall that, $\mathbb{B}_{10} = \{B \in \mathscr{B}''(n,3) \mid B_{10}$ (see Fig.10) is the basic block associated to $B\}$. In the following Proposition 5.2.16, we obtain the cardinality of the class $\mathbb{B}_{10}$.

Proposition 5.2.16. *For $n \geq 7$, $|\mathbb{B}_{10}| = |A_1| + |A_2| + |A_3|$, where*

$$|A_1| = \sum_{t=2}^{(n-6)} \sum_{r=1}^{(n-t-5)} \sum_{l=1}^{(n-r-t-4)} \sum_{r_1=1}^{\left[\frac{n-l-r-t-2}{2}\right]} (e),$$

where $e = (n - r_1 - l - r - t - 1)(r_1)$,

$$|A_2| = \sum_{t=2}^{(n-6)} \sum_{r=1}^{\left[\frac{n-t-4}{2}\right]} \sum_{l=r+1}^{(n-r-t-3)} \left[\frac{n-l-r-t-1}{2}\right]^2 \text{ and}$$

$$|A_3| = \sum_{t=2}^{(n-5)} \sum_{l=1}^{\left[\frac{n-t-3}{2}\right]} \frac{(u)(u+1)}{2}, \text{ where } u = \left[\frac{n-2l-t-1}{2}\right].$$

Proof. Let $A_1 = \{(1, l_1, r_1, l, r, t) : l_1 \geq r_1 + 1, t \geq 2, l_1, r_1, l, r, t \in \mathbb{N}, l_1 + r_1 + l + r + t + 1 = n \geq 8\}$,
$A_2 = \{(1, l_1, r_1, l, r, t) : l_1 = r_1, l \geq r + 1, t \geq 2, l_1, r_1, l, r, t \in \mathbb{N}, l_1 + r_1 + l + r + t + 1 = n \geq 8\}$ and
$A_3 = \{(1, l_1, r_1, l, r, t) : l_1 = r_1, l = r, t \geq 2, l_1, r_1, l, r, t \in \mathbb{N}, l_1 + r_1 + l + r + t + 1 = n \geq 7\}$.
Let $B = C_1]_{a_1=0}^{b_1} C_2]_{a_2}^{b_2=1} C_3]_{a_3}^{b_3=b_2} C_4 \in \mathbb{B}_{10}$, where
$C_1 \equiv x_0 < x_1 < x_2 < \ldots < x_{l_1+t}$, $C_2 \equiv y_1 < y_2 < \ldots < y_{r_1}$,
$C_3 \equiv z_1 < z_2 < \ldots < z_l$ and $C_4 \equiv w_1 < w_2 < \ldots < w_r$ are disjoint chains with $a_1 = x_0, b_1 = x_{l_1+1}, a_2 \in C_1 \cap [x_1, x_{l_1}], a_3 \in C_2$ and $b_3 = b_2 = x_{l_1+t}$. Then $B \in \mathbb{B}_{10}$ if and only if $(1, l_1, r_1, l, r, t) \in A = A_1 \dot{\cup} A_2 \dot{\cup} A_3$. Therefore $|\mathbb{B}_{10}| = |A| = |A_1| + |A_2| + |A_3|$.
We first find $|A_1|$ in the following.
For fixed value of r_1, l, r and t, let
$A_{r_1lrt} = \{(1, l_1, r_1, l, r, t) : l_1 \geq r_1 + 1, t \geq 2, l_1, r_1, l, r, t \in \mathbb{N}, l_1 + r_1 + l + r + t + 1 = n \geq 8\}$. Then it is clear that

$$|A_1| = \sum_t \sum_r \sum_l \sum_{r_1} |A_{r_1lrt}|. \tag{1}$$

$$\text{Now } |A_{r_1lrt}| = (n - r_1 - l - r - t - 1) \times (r_1). \tag{2}$$

This is nothing but the product of l_1, the number of possible positions for a_2 in the block B and r_1, the number of possible positions for a_3 in the block B. Also for fixed l, r and t, r_1 takes its maximum value when l_1 takes its minimum value $r_1 + 1$. Therefore $r_1 = n - l_1 - l - r - t - 1 \leq n - r_1 - 1 - l - r - t - 1 \Rightarrow 2r_1 \leq n - l - r - t - 2$ and hence we have

$$1 \leq r_1 \leq \left[\tfrac{n-l-r-t-2}{2}\right] \tag{3}$$

Again for fixed r and t, l takes maximum value when the other variables have minimum value. Therefore

$$1 \leq l \leq n - r - t - 4. \tag{4}$$

Now for fixed t, r takes maximum value when the other variables have minimum value. Therefore

$$1 \leq r \leq n - t - 5. \tag{5}$$

Finally t takes maximum value when other variables have minimum value. This gives us

$$2 \leq t \leq n - 6. \tag{6}$$

From $(1), (2), (3), (4), (5)$ and (6) we have $\forall\, n \geq 8$,

$$|A_1| = \sum_{t=2}^{(n-6)} \sum_{r=1}^{(n-t-5)} \sum_{l=1}^{(n-r-t-4)} \sum_{r_1=1}^{\left[\frac{(n-l-r-t-2)}{2}\right]} (e),$$

where $e = (n - r_1 - l - r - t - 1) \times (r_1)$.

Now in the following we find $|A_2|$.

For fixed value of l, r and t, let

$A_{lrt} = \{(1, l_1, r_1, l, r, t) : l_1 = r_1, t \geq 2, l \geq r + 1, l_1, r_1, l, r, t \in \mathbb{N}, l_1 + r_1 + l + r + t + 1 = n \geq 8\}$. Then it is clear that

$$|A_2| = \sum_t \sum_r \sum_l |A_{lrt}|. \tag{7}$$

Now $|A_{lrt}| = s \times s$, where $s = l_1 = r_1$. (8)

Note that l_1 is the number of possible positions for a_2 in the block B and r_1 is the number of possible positions for a_3 in the block B. Now $2s = l_1 + r_1 = n - l - r - t - 1$ implies that $s = \left[\frac{n-l-r-t-1}{2}\right]$. (9)

Also for fixed r and t, l takes maximum value when s take minimum value. Therefore

$$r + 1 \leq l = n - 2s - r - t - 1 \leq n - r - t - 3. \tag{10}$$

Again for fixed t, r takes maximum value when other variables have minimum value. Therefore

$1 \leq r \leq \left[\frac{n-t-4}{2}\right].$ (11)

Finally t takes maximum value when other variables have minimum value. This gives us

$2 \leq t \leq n-6.$ (12)

From $(7), (8), (9), (10), (11)$ and (12) we have $\forall\, n \geq 8$,

$$|A_2| = \sum_{t=2}^{(n-6)} \sum_{r=1}^{\left[\frac{n-t-4}{2}\right]} \sum_{l=r+1}^{(n-r-t-3)} \left(\left[\frac{n-l-r-t-1}{2}\right]\right)^2.$$

Now in the following we find $|A_3|$.

For fixed value of l and t, let

$A_{lt} = \{(1, l_1, r_1, l, r, t) : l_1 = r_1, l = r, t \geq 2, l_1, r_1, l, r, t \in \mathbb{N}, l_1 + r_1 + l + r + t + 1 = n \geq 7\}$. Then it is clear that

$|A_3| = \sum_t \sum_l |A_{lt}|.$ (13)

Now $|A_{lt}| = \frac{(u)(u+1)}{2}$, where $u = l_1 = r_1$. (14)

This is nothing but the total number of possible different positions for a_2 and a_3 in the block B, since $l = r$. Now $2u = l_1 + r_1 = n - l - r - t - 1 = n - 2l - t - 1$ lead us to conclude that $u = \left[\frac{n-2l-t-1}{2}\right]$. (15)

Also for fixed t, l takes maximum value when u take minimum value. Therefore $2l = l + r = n - l_1 - r_1 - t - 1 = n - 2u - t - 1 \leq n - t - 3$ implies that

$1 \leq l \leq \left[\frac{n-t-3}{2}\right].$ (16)

Finally t takes maximum value when the other variables have minimum value. This gives us

$2 \leq t = n - 2u - 2l - 1 \leq n - 5.$ (17)

From $(13), (14), (15), (16)$ and (17) we have $\forall\, n \geq 7$,

$$|A_3| = \sum_{t=2}^{(n-5)} \sum_{l=1}^{\left[\frac{n-t-3}{2}\right]} \frac{(u)(u+1)}{2}, \text{ where } u = \left[\tfrac{n-2l-t-1}{2}\right]. \qquad \square$$

Recall that, $\mathbb{B}_{11} = \{B \in \mathscr{B}''(n,3) \mid B_{11}$ (see Fig.10) is the basic block associated to $B\}$. Note that, the dual B^*_{11} of the basic block B_{11} is B_{10} (see Fig.10). In the following Corollary 5.2.17, we obtain the cardinality of the class $\mathbb{B}_{11}$.

Corollary 5.2.17. *For $n \geq 7$, $|\mathbb{B}_{11}| = |A_1| + |A_2| + |A_3|$, where*

$$|A_1| = \sum_{t=2}^{(n-6)} \sum_{r=1}^{(n-t-5)} \sum_{l=1}^{(n-r-t-4)} \sum_{r_1=1}^{\left[\frac{n-l-r-t-2}{2}\right]} (e),$$

where $e = (n - r_1 - l - r - t - 1)(r_1)$,

$$|A_2| = \sum_{t=2}^{(n-6)} \sum_{r=1}^{\left[\frac{n-t-4}{2}\right]} \sum_{l=r+1}^{(n-r-t-3)} \left[\frac{n-l-r-t-1}{2}\right]^2 \text{ and}$$

$$|A_3| = \sum_{t=2}^{(n-5)} \sum_{l=1}^{\left[\frac{n-t-3}{2}\right]} \frac{(u)(u+1)}{2}, \text{ where } u = \left[\tfrac{n-2l-t-1}{2}\right].$$

Proof. Clearly $|\mathbb{B}_{11}| = |\mathbb{B}_{10}|$, since $B \in \mathbb{B}_{11}$ if and only if the dual of B, $B^* \in \mathbb{B}_{10}$. Thus the proof follows from Proposition 5.2.16. $\square$

Using Proposition 5.2.9, Corollary 5.2.10, Proposition 5.2.12, Corollary 5.2.13, Corollary 5.2.14, Corollary 5.2.15, Proposition 5.2.16 and Corollary 5.2.17, we obtain the number of non-isomorphic blocks on n elements, having nullity three, and containing five reducible elements such that at least two of them are incomparable in the following Theorem 5.2.18. For the sake of brevity, we avoid the explicit formula here.

Theorem 5.2.18. *For $n \geq 7$,*

$$|\mathscr{B}''(n,3,5)| = \sum_{i=4}^{11} |\mathbb{B}_i|.$$

Proof. The proof follows from the fact that $\{\mathbb{B}_i : 4 \leq i \leq 11\}$ forms a partition of the class $\mathscr{B}''(n,3,5)$. □

5.2.4 Enumeration of blocks on six reducible elements

We now consider the problem of enumeration of blocks on six reducible elements; that is, to find $|\mathscr{B}''(n,3,6)|$. Recall that, $\{\mathbb{B}_i : 12 \leq i \leq 17\}$ forms a partition of the class $\mathscr{B}''(n,3,6)$. Therefore, it is required to find the cardinality of the class $\mathbb{B}_i$ for each i, $12 \leq i \leq 17$.

Recall that, $\mathbb{B}_{12} = \{B \in \mathscr{B}''(n,3) \mid B_{12}$ (see Fig.11) is the basic block associated to $B\}$. In the following Proposition 5.2.19, we obtain the cardinality of the class $\mathbb{B}_{12}$.

Proposition 5.2.19. *For $n \geq 10$,*

$$|\mathbb{B}_{12}| = \begin{cases} \displaystyle\sum_{n=i+j+2, i>j} (|\mathscr{L}(i,1)| \times |\mathscr{L}(j,1)|) & if\ \ n\ is\ odd; \\ \displaystyle\sum_{n=i+j+2, i>j} (|\mathscr{L}(i,1)| \times |\mathscr{L}(j,1)|) + E & if\ \ n\ is\ even, \end{cases}$$

where $E = \frac{|\mathscr{L}(\frac{n-2}{2},1)| \times (|\mathscr{L}(\frac{n-2}{2},1)|+1)}{2}$ *and*

$$|\mathscr{L}(i,1)| = \begin{cases} \frac{m(m-1)(4m+1)}{6} & if \quad i = 2m+1; \\ \frac{m(m-1)(4m-5)}{6} & if \quad i = 2m. \end{cases}$$

Proof. Let $B \in \mathbb{B}_{12}$. Then $B - \{0,1\}$ is the disjoint union of two sublattices, say Y_1 and Y_2 of B such that each of them is a 2-sum of two chains. By (ii) of Theorem 3.2.1, $Red(B) = Red(B_{12})$. Therefore, as $a, b, c, d \in Red(B_{12})$, let $Y_1 = C_1]_a^d C_2$ and let $Y_2 = C_3]_b^c C_4$ with $|Y_1| = i \geq 4$, $|Y_2| = j \geq 4$, where C_1, C_2, C_3 and C_4 are chains. Then without loss of generality, suppose $B = (\{0\} \oplus Y_1 \oplus \{1\})]_0^1 Y_2$ with $|Y_1| = i \geq |Y_2| = j$ and $|B| = n = i + j + 2 \geq 10$. It is clear that $Y_1 \in \mathscr{L}(i, 1)$ and $Y_2 \in \mathscr{L}(j, 1)$. Let $B' \in \mathbb{B}_{12}$ be such that $B' = (\{0\} \oplus Y_1' \oplus \{1\})]_0^1 Y_2'$. Then $B \cong B'$ if and only if $Y_1 \cong Y_1'$ and $Y_2 \cong Y_2'$. Therefore, if $i > j$ then there are $\sum\limits_{n=i+j+2} (|\mathscr{L}(i, 1)| \times |\mathscr{L}(j, 1)|)$ non-isomorphic blocks in $\mathbb{B}_{12}$. But if $i = j$ then n must be even and it seems that there are $|\mathscr{L}(i, 1)|^2$ blocks(all may not non-isomorphic). In fact, there are $\binom{|\mathscr{L}(i,1)|}{2}$ blocks which are counted twice, since $i = j$. Therefore in the case when $i = j$, there are $|\mathscr{L}(i, 1)|^2 - \binom{|\mathscr{L}(i,1)|}{2} = \frac{|\mathscr{L}(i,1)|(|\mathscr{L}(i,1)|+1)}{2}$ non-isomorphic blocks in $\mathbb{B}_{12}$. Thus the proof follows from Theorem 5.1.1. $\square$

For $n \geq 6$, let $\mathcal{B}_{1,4}^n$ be the class of all non-isomorphic blocks (of nullity two) of the type B, where $B = C_1]_{a_1}^{b_1} C_2]_{a_2}^{b_2} C_3$ and $0 = a_1 < a_2 < b_1 < b_2 = 1$.

Proposition 5.2.20. *[13]. For $n \geq 6$,*

$$|\mathcal{B}_{1,4}^n| = \sum_{r=1}^{(n-5)} \sum_{l=1}^{(n-r-4)} \sum_{s=1}^{(n-l-r-3)} (n - s - l - r - 2).$$

Recall that, $\mathbb{B}_{13} = \{B \in \mathscr{B}''(n, 3) \mid B_{13}$ (see Fig.11) is the basic block associated to $B\}$. In the following Proposition 5.2.21, we obtain the cardinality of the class $\mathbb{B}_{13}$.

Proposition 5.2.21. *For $n \geq 8$,*

$$|\mathbb{B}_{13}| = \sum_{j=1}^{(n-7)} \sum_{i=1}^{(n-j-6)} \sum_{r=1}^{(n-i-j-5)} \sum_{l=1}^{(n-i-j-r-4)} \sum_{s=1}^{(n-i-j-l-r-3)} (e),$$

where $e = (n-i-j-s-l-r-2)(r)$.

Proof. Let $B \in \mathbb{B}_{13}$. As nullity of B is 3, $B = C_1]_{a_1=0}^{b_1} C_2]_{a_2}^{b_2} C_3]_{a_3}^{b_3=1} C_4$, where C_1 is a maximal chain containing a_1, a_2, b_1, b_2, b_3 and C_2, C_3, C_4 are chains with $a_1 = 0 < a_2 = c < (b_1 = a \mid\mid a_3 = b) < b_2 = d < b_3 = 1$ and $a_3 \in C_3$. Note that, by (ii) of Theorem 3.2.1, $Red(B) = Red(B_{13})$ and $a, b, c, d \in Red(B_{13})$. Let $B' = (C_1 \cap [a_1, b_2])]_{a_1}^{b_1} C_2]_{a_2}^{b_2} C_3$, $C_1' = (C_1 \cap (b_2, b_3])$ and $C_2' = C_4$. Then $B = (B' \oplus C_1')]_{a_3}^{1} C_2'$. Let $|B'| = k \geq 6$, $|C_1'| = i \geq 1$ and $|C_2'| = j \geq 1$. Then $B' \in \mathcal{B}_{1,4}^k$ and $|B| = n = i + j + k \geq 8$. By Proposition 5.2.20,

$$|\mathcal{B}_{1,4}^k| = \sum_{r=1}^{(k-5)} \sum_{l=1}^{(k-r-4)} \sum_{s=1}^{(k-l-r-3)} (k-s-l-r-2),$$

where $s = |[a_1, a_2) \cap C_1|$, $l = |C_2|$, $r = |C_3|$ and for fixed s, l, r, $(k-s-l-r-2)$ is the number of possible positions of b_1 in the block B' and hence in the block $B \in \mathbb{B}_{13}$. Now for fixed i and j, $k = n-i-j \geq 6$. Therefore for fixed j, we have $1 \leq i = n-j-k \leq n-j-6$ and therefore $1 \leq j = n-i-k \leq n-1-6 = n-7$. Now a_3 takes $|C_3| = r$ number of positions in the block $B' \in \mathcal{B}_{1,4}^k$ and hence in the block $B \in \mathbb{B}_{13}$. Therefore we have for all $n \geq 8$,

$$|\mathbb{B}_{13}| = \sum_{j=1}^{(n-7)} \sum_{i=1}^{(n-j-6)} \sum_{r=1}^{(n-i-j-5)} \sum_{l=1}^{(n-i-j-r-4)} \sum_{s=1}^{(n-i-j-l-r-3)} (e),$$

where $e = (n-i-j-s-l-r-2) \times (r)$. □

For $n \geq 7$, let $\mathcal{B}_2^n$ be the class of all non-isomorphic blocks (of nullity two) of the type B, where $B = C_1]_{a_1}^{b_1} C_2]_{a_2}^{b_2} C_3$ and $0 = a_1 < a_2 < b_2 < b_1 = 1$.

Recall that, $\mathbb{B}_{14} = \{B \in \mathscr{B}''(n, 3) \mid B_{14}$ (see Fig.11) is the basic block associated to $B\}$. In the Proposition 5.2.23, we obtain the cardinality of the class $\mathbb{B}_{14}$ using the Proposition 5.2.22. For this purpose, we give the enumeration of the class $\mathcal{B}_2^n$ as it was not clearly mentioned in [13].

Proposition 5.2.22. *[13]. For $n \geq 7$,*

$$|\mathcal{B}_2^n| = \sum_{r=1}^{[\frac{n-5}{2}]} \sum_{l=1}^{n-2r-4} \sum_{s=1}^{n-l-2r-3} (n - s - l - 2r - 2).$$

Proof. Let $A = \{(s, m, t, l, r) : s \geq 1, t \geq 2, m, l, r \in \mathbb{N}, s+m+t+l+r = n\}$ and $\phi : A \to \mathcal{B}_2^n$ be a map defined as $\phi(s, m, t, l, r) = (C_1]_{a_1}^{b_1} C_2)]_{a_2}^{b_2} C_3$, where $C_1 \equiv x_1 < x_2 < \ldots < x_{s+m+t}$, $C_2 \equiv y_1 < y_2 < \ldots < y_l$ and $C_3 \equiv z_1 < z_2 < \ldots < z_r$ are disjoint chains with $a_1 = x_1, a_2 = x_{s+1}, b_2 = x_{s+m+1}$ and $b_1 = x_{s+m+t}$. For $B \in \mathcal{B}_2^n$ with adjunct representation $(C_1]_{a_1=0}^{b_1} C_2)]_{a_2}^{b_2} C_3$, let $s = |[a_1, a_2) \cap C_1|, m = |[a_2, b_2) \cap C_1|, t = |[b_2, b_1] \cap C_1|, l = |C_2|$ and $r = |C_3|$. Then $(s, m, t, l, r) \in A$ and $\phi(s, m, t, l, r) = B$. This shows that ϕ is onto. It is evident that $\phi(s, m, t, l, r) \in \mathcal{B}_2^n$ if and only if $t \geq 2$ and $s \geq 1$. Also $\phi(s_1, m_1, t_1, l_1, r_1) \cong \phi(s_2, m_2, t_2, l_2, r_2)$ if and only if either $s_1 = s_2, m_1 = m_2, t_1 = t_2, l_1 = l_2$ and $r_1 = r_2$ or $s_1 = s_2, t_1 = t_2, l_1 = l_2$, $m_2 = r_1+1$ and $r_2 = m_1-1$. Hence by imposing the additional condition $m \geq r+1$ on elements of A we observe that the class $\mathcal{B}_2^n$ and the set $\{(s, m, t, l, r)/s, m, t, l, r \in \mathbb{N},\ s+m+t+l+r = n,\ t \geq 2,\ s \geq 1,\ m \geq r+1\}$ become numerically equivalent. Therefore $|\mathcal{B}_2^n| = |\{(s, m, t, l, r)/s, m, t, l, r \in \mathbb{N},\ s+m+t+l+r = n,\ t \geq 2,\ s \geq 1,$

$m \geq r+1\}|$.

For fixed value of r, l and s, let

$A_{rls} = \{(s,m,t,l,r) | s+m+t+l+r = n, t \geq 2, m \geq r+1\}$. Then it is clear that

$$|\mathcal{B}_2^n| = \sum_r \sum_l \sum_s |A_{rls}|. \tag{A}$$

For any ordered 5-tuple in A_{rls}, $r+1 \leq m = n-s-t-l-r \leq n-l-r-s-2$, and $m+t = n-s-l-r$ (fixed). Hence $|A_{rls}| = n-s-l-r-(r+1)-1$. That is,

$$|A_{rls}| = n-s-l-2r-2. \tag{B}$$

This is nothing but the number of possible positions for b_2 in the block B. Also for fixed r and l, s takes maximum value when m and t take minimum values $r+1$ and 2 respectively. Therefore $1 \leq s \leq n-l-2r-3$. (C)

Again for fixed r, l takes maximum value when the other variables have minimum value, *i.e.*, $s=1, m=r+1$, and $t=2$. Therefore $1 \leq l \leq n-2r-4$. (D)

Finally r takes maximum value when other variables have minimum values, *i.e.*, $s=1, m=r+1, t=2$ and $l=1$. This gives us $r \leq n-r-5$. *i.e.*, $2r \leq n-5$ leads to

$$1 \leq r \leq \left[\frac{n-5}{2}\right]. \tag{E}$$

From $(A), (B), (C), (D)$ and (E) we have

$$|\mathcal{B}_2^n| = \sum_{r=1}^{\left[\frac{n-5}{2}\right]} \sum_{l=1}^{n-2r-4} \sum_{s=1}^{n-l-2r-3} (n-s-l-2r-2).$$

□

Using Proposition 5.2.22, we prove the following Proposition 5.2.23.

Proposition 5.2.23. *For $n \geq 9$,*

$$|\mathbb{B}_{14}| = \sum_{j=1}^{(n-8)} \sum_{i=1}^{(n-j-7)} \sum_{r=1}^{[\frac{n-i-j-5}{2}]} \sum_{l=1}^{n-i-j-2r-4} \sum_{s=1}^{n-i-j-l-2r-3} (e),$$

where $e = (n-i-j-s-l-2r-2)(l)$.

Proof. Let $B \in \mathbb{B}_{14}$. As nullity of B is 3, $B = C_1]_{a_1=0}^{b_1} C_2]_{a_2}^{b_2} C_3]_{a_3}^{b_3=1} C_4$, where C_1 is a maximal chain containing a_1, a_2, b_1, b_2, b_3 and C_2, C_3, C_4 are chains with $a_1 = 0 < (a_2 = b < b_2 = d) \mid\mid a_3 = a) < b_1 = c < b_3 = 1$ and $a_3 \in C_2$. Note that, by (ii) of Theorem 3.2.1, $Red(B) = Red(B_{14})$ and $a, b, c, d \in Red(B_{14})$. Let $B' = (C_1 \cap [a_1, b_1])]_{a_1}^{b_1} C_2]_{a_2}^{b_2} C_3$, $C_1' = (C_1 \cap (b_1, b_3])$ and $C_2' = C_4$. Then $B = (B' \oplus C_1')]_{a_3}^{b_3=1} C_2'$. Let $|B'| = k \geq 7$, $|C_1'| = i \geq 1$ and $|C_2'| = j \geq 1$. Then $B' \in \mathcal{B}_2^k$ (see [13]) and $|B| = n = i + j + k \geq 9$. By above Proposition 5.2.22,

$$|\mathcal{B}_2^k| = \sum_{r=1}^{[\frac{k-5}{2}]} \sum_{l=1}^{k-2r-4} \sum_{s=1}^{k-l-2r-3} (k-s-l-2r-2).$$

where $s = |[a_1, a_2) \cap C_1|$, $l = |C_2|$, $r = |C_3|$ and for fixed s, l, r, $(k-s-l-2r-2)$ is the number of possible positions of b_2 in the block B' and hence in the block $B \in \mathbb{B}_{14}$. Now for fixed i and j, $k = n-i-j \geq 7$. Therefore for fixed j, we have $1 \leq i = n-j-k \leq n-j-7$ and therefore $1 \leq j = n-i-k \leq n-1-7 = n-8$. Now a_3 takes $|C_2| = l$ number of positions in the block $B' \in \mathcal{B}_2^k$ and hence in the block $B \in \mathbb{B}_{14}$. Therefore we have for all $n \geq 9$,

$$|\mathbb{B}_{14}| = \sum_{j=1}^{(n-8)} \sum_{i=1}^{(n-j-7)} \sum_{r=1}^{[\frac{n-i-j-5}{2}]} \sum_{l=1}^{n-i-j-2r-4} \sum_{s=1}^{n-i-j-l-2r-3} (e),$$

where $e = (n - i - j - s - l - 2r - 2) \times (l)$. $\square$

Recall that, $\mathbb{B}_{15} = \{B \in \mathscr{B}''(n, 3) \mid B_{15}$ (see Fig.11) is the basic block associated to $B\}$. Note that, the dual B^*_{15} of the basic block B_{15} is B_{14} (see Fig.11). In the Corollary 5.2.24, we obtain the cardinality of the class $\mathbb{B}_{15}$.

Corollary 5.2.24. *For $n \geq 9$,*

$$|\mathbb{B}_{15}| = \sum_{j=1}^{(n-8)} \sum_{i=1}^{(n-j-7)} \sum_{r=1}^{[\frac{n-i-j-5}{2}]} \sum_{l=1}^{n-i-j-2r-4} \sum_{s=1}^{n-i-j-l-2r-3} (e),$$

where $e = (n - i - j - s - l - 2r - 2)(l)$.

Proof. Clearly $|\mathbb{B}_{15}| = |\mathbb{B}_{14}|$, since $B \in \mathbb{B}_{15}$ if and only if the dual of B, $B^* \in \mathbb{B}_{14}$. Thus the proof follows by Proposition 5.2.23. $\square$

Recall that, $\mathbb{B}_{16} = \{B \in \mathscr{B}''(n, 3) \mid B_{16}$ (see Fig.11) is the basic block associated to $B\}$. In the Proposition 5.2.25, we obtain the cardinality of the class $\mathbb{B}_{16}$.

Proposition 5.2.25. *For $n \geq 8$,*

$$|\mathbb{B}_{16}| = \sum_{j=1}^{(n-7)} \sum_{i=1}^{(n-j-6)} \sum_{r=1}^{(n-i-j-5)} \sum_{l=1}^{(n-i-j-r-4)} \sum_{s=1}^{(n-i-j-l-r-3)} (e),$$

where $e = (n - i - j - s - l - r - 2)(l)$.

Proof. Let $B \in \mathbb{B}_{16}$. As nullity of B is 3,
$B = C_1]^{b_1}_{a_1=0} C_2]^{b_2}_{a_2} C_3]^{b_3=1}_{a_3} C_4$, where C_1 is a maximal chain containing a_1, a_2, b_1, b_2, b_3 and C_2, C_3, C_4 are chains with $a_1 = 0 < (a_2 = b \mid\mid a_3 = a) < b_1 = c < b_2 = d < b_3 = 1$ and $a_3 \in C_2$. Note that, by (ii)

of Theorem 3.2.1, $Red(B) = Red(B_{16})$ and $a, b, c, d \in Red(B_{16})$. Let $B' = (C_1 \cap [a_1, b_2])]_{a_1}^{b_1} C_2]_{a_2}^{b_2} C_3$, $C_1' = (C_1 \cap (b_2, b_3])$ and $C_2' = C_4$. Then $B = (B' \oplus C_1')]_{a_3}^{b_3=1} C_2'$. Let $|B'| = k \geq 6$, $|C_1'| = i \geq 1$ and $|C_2'| = j \geq 1$. Then $B' \in \mathcal{B}_{1,4}^k$ and $|B| = n = i + j + k \geq 8$. By Proposition 5.2.20,

$$|\mathcal{B}_{1,4}^k| = \sum_{r=1}^{(k-5)} \sum_{l=1}^{(k-r-4)} \sum_{s=1}^{(k-l-r-3)} (k - s - l - r - 2),$$

where $s = |[a_1, a_2) \cap C_1|$, $l = |C_2|$, $r = |C_3|$ and for fixed s, l, r, $(k - s - l - r - 2)$ is the number of possible positions of b_1 in the block $B' \in \mathcal{B}_{1,4}^k$ and hence in the block $B \in \mathbb{B}_{16}$.

Now for fixed i and j, $k = n - i - j \geq 6$. Therefore for fixed j, we have $1 \leq i = n - j - k \leq n - j - 6$ and therefore $1 \leq j = n - i - k \leq n - 1 - 6 = n - 7$. Now a_3 takes $|C_2| = l$ number of positions in the block $B' \in \mathcal{B}_{1,4}^k$ and hence in the block $B \in \mathbb{B}_{16}$. Therefore we have for all $n \geq 8$,

$$|\mathbb{B}_{16}| = \sum_{j=1}^{(n-7)} \sum_{i=1}^{(n-j-6)} \sum_{r=1}^{(n-i-j-5)} \sum_{l=1}^{(n-i-j-r-4)} \sum_{s=1}^{(n-i-j-l-r-3)} (e),$$

where $e = (n - i - j - s - l - r - 2) \times (l)$. □

Recall that, $\mathbb{B}_{17} = \{B \in \mathscr{B}''(n, 3) \mid B_{17}$ (see Fig.11) is the basic block associated to $B\}$. Note that, the dual B_{17}^* of the basic block B_{17} is B_{16} (see Fig.11). In the Corollary 5.2.26, we obtain the cardinality of the class $\mathbb{B}_{17}$.

Corollary 5.2.26. *For $n \geq 8$,*

$$|\mathbb{B}_{17}| = \sum_{j=1}^{(n-7)} \sum_{i=1}^{(n-j-6)} \sum_{r=1}^{(n-i-j-5)} \sum_{l=1}^{(n-i-j-r-4)} \sum_{s=1}^{(n-i-j-l-r-3)} (e),$$

where $e = (n-i-j-s-l-r-2)(l)$.

Proof. Clearly $|\mathbb{B}_{17}| = |\mathbb{B}_{16}|$, since $B \in \mathbb{B}_{17}$ if and only if the dual of B, $B^* \in \mathbb{B}_{16}$. Thus the proof follows by Proposition 5.2.25. $\square$

Using Proposition 5.2.19, Proposition 5.2.21, Proposition 5.2.23, Corollary 5.2.24, Proposition 5.2.25 and Corollary 5.2.26, we obtain the number of non-isomorphic blocks on n elements, having nullity three, and containing six reducible elements such that at least two of them are incomparable in the following Theorem 5.2.27. For the sake of brevity, we avoid the explicit formula here also.

Theorem 5.2.27. *For* $n \geq 8$,

$$|\mathscr{B}''(n,3,6)| = \sum_{i=12}^{17} |\mathbb{B}_i|.$$

Proof. The proof follows from the fact that $\{\mathbb{B}_i : 12 \leq i \leq 17\}$ forms a partition of the class $\mathscr{B}''(n,3,6)$. $\square$

We now obtain the number ($|\mathscr{B}''(n,3)|$) of non-isomorphic blocks (that is, lattices in which 0 and 1 are reducible elements) on n elements, having nullity three, and in which at least two of the reducible elements are incomparable.

As $\mathscr{B}''(n,3) = \mathscr{B}''(n,3,4) \dot{\cup} \mathscr{B}''(n,3,5) \dot{\cup} \mathscr{B}''(n,3,6)$, we have using Theorem 5.2.8, Theorem 5.2.18 and Theorem 5.2.27, the following Theorem 5.2.28.

Theorem 5.2.28. *For all* $n \geq 7$,

$$|\mathscr{B}''(n,3)| = |\mathscr{B}''(n,3,4)| + |\mathscr{B}''(n,3,5)| + |\mathscr{B}''(n,3,6)|.$$

Definition 5.2.4. Let $\mathscr{L}''(n,k)$ be the subclass of $\mathscr{L}(n,k)$ such that at least two of the reducible elements in each lattice in it are incomparable.

It is clear that, $\mathscr{L}(n,k) = \mathscr{L}'(n,k) \,\dot{\cup}\, \mathscr{L}''(n,k)$. Using Theorem 5.2.28, we prove that, the number ($|\mathscr{L}''(n,3)|$) of non-isomorphic lattices on n elements, having nullity three, and in which at least two of the reducible elements are incomparable is given by the following Theorem 5.2.29. Again for the sake of brevity, we avoid the explicit formula here.

Theorem 5.2.29. *For all $n \geq 7$,*

$$|\mathscr{L}''(n,3)| = \sum_{i=0}^{n-7}(i+1) \times |\mathscr{B}''(n-i,3)|.$$

Proof. It is clear that a lattice $L \in \mathscr{L}''(n,3)$ if and only if $L = C \oplus B \oplus C'$, where $B \in \mathscr{B}''(n-i,3)$ and C, C' are chains with $|C| + |C'| = i$. For fixed $i \geq 0$, the i elements can be allocated to the chains C and C' in $i+1$ ways. Let $j = n-i$. Now for any $B \in \mathscr{B}''(j,3)$, $j \geq 7$. Therefore $i = n-j \leq n-7$. Thus $0 \leq i \leq n-7$. Hence the proof is complete. □

5.3 Enumeration of lattices of nullity three

In this section, we obtain the number ($|\mathscr{L}(n,3)|$) of all non-isomorphic lattices on n elements, having nullity three. As $\mathscr{L}(n,3) = \mathscr{L}'(n,3) \dot{\cup} \mathscr{L}''(n,3)$, we first obtain the cardinality of the class $\mathscr{L}'(n,3)$.
By Proposition 4.3.1, we have the following (Note that $k = 3$).

Corollary 5.3.1. *For any $2 \leq s \leq 6$, for any $n \geq s+3$ and for $Bb \in B'(3,s)$, the number N^s_{Bb} of non-isomorphic blocks in $\mathscr{B}'(n,3,s)$*

which are associated by the basic block Bb is given by

$$N^s_{Bb} = \sum_{n-s=u_1+u_2+\cdots+u_{m+p+l}} \left(\prod_{i=1}^{m} P^{m_i+1}_{u_i}\right) \times \left(\prod_{i=m+1}^{m+p} P^{p_i}_{u_i}\right),$$

where $u_i \geq m_i + 1$*, for* $1 \leq i \leq m$*,* $u_i \geq p_i$*, for* $m+1 \leq i \leq m+p$ *and* $u_i \geq 0$*, for* $m+p+1 \leq i \leq m+p+l$*.*

Note that for $2 \leq s \leq 6$, the values of $B'(3,s)$ are given in the Table 9. By Proposition 4.3.2, we have the following.

Corollary 5.3.2. *For any* $2 \leq s \leq 6$ *and for any* $n \geq s+3$*,*

$$|\mathscr{B}'(n,3,s)| = \sum_{Bb \in B'(3,s)} N^s_{Bb}.$$

By Proposition 4.3.3, we have the following.

Corollary 5.3.3. *For any* $n \geq 6$*,*

$$|\mathscr{B}'(n,3)| = \sum_{s=2}^{6} |\mathscr{B}'(n,3,s)|.$$

Therefore by Theorem 4.3.4, we have the following.

Corollary 5.3.4. *For any* $n \geq 6$*,*

$$|\mathscr{L}'(n,3)| = \sum_{i=0}^{n-6} (i+1)|\mathscr{B}'(n-i,3)|.$$

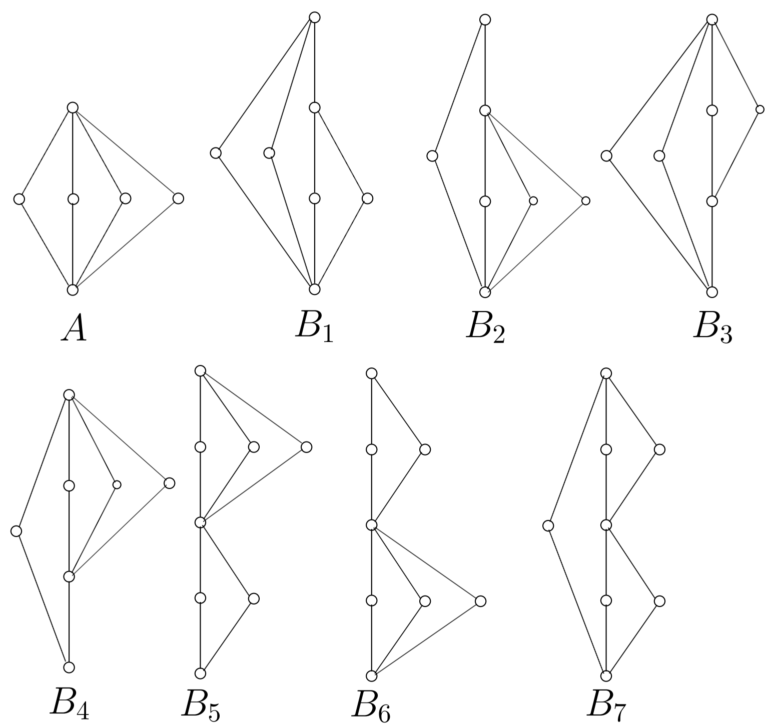
As $\mathscr{L}(n,3) = \mathscr{L}'(n,3) \dot\cup \mathscr{L}''(n,3)$, using Corollary 5.3.4 and Theorem 5.2.29, we obtain the following.

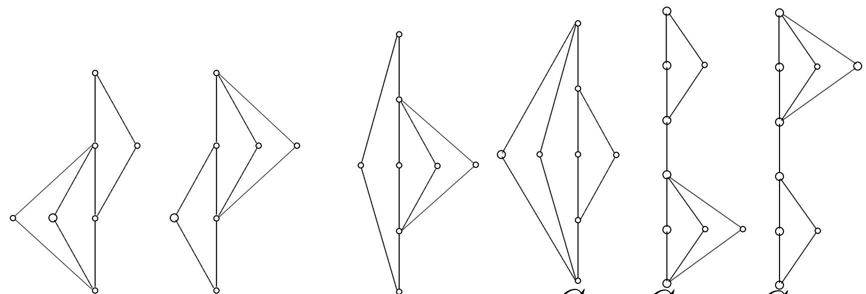
Theorem 5.3.5. *For all* $n \geq 6$*,*

$$|\mathscr{L}(n,3)| = |\mathscr{L}'(n,3)| + |\mathscr{L}''(n,3)|.$$

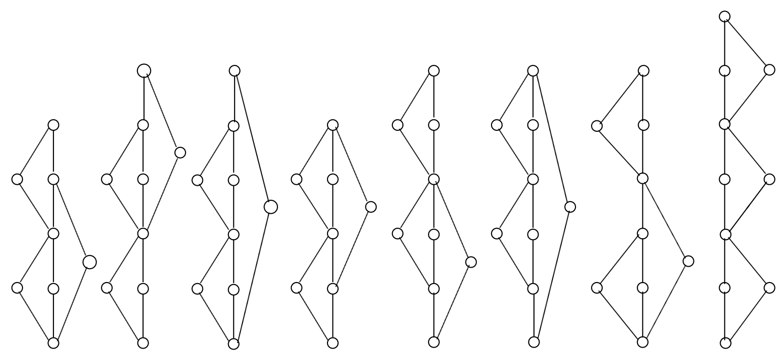
Appendix

Basic blocks and fundamental basic blocks of nullity three.

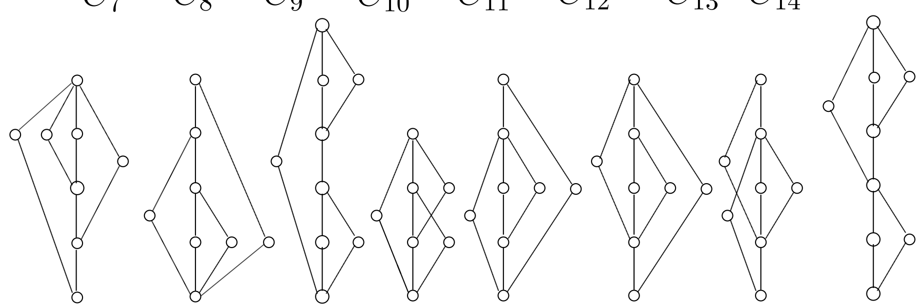




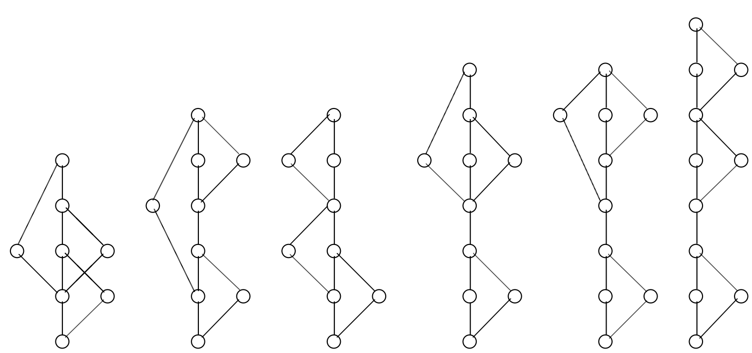
C_1 C_2 C_3 C_4 C_5 C_6



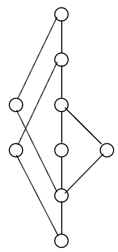
C_7 C_8 C_9 C_{10} C_{11} C_{12} C_{13} C_{14}



C_{15} C_{16} C_{17} C_{18} C_{19} C_{20} C_{21} C_{22}



D_1 D_2 D_3 D_4 D_5 D_6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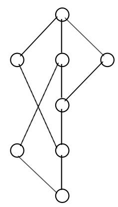
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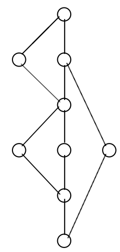
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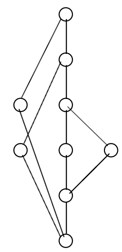
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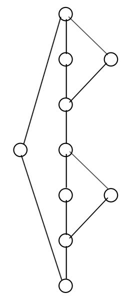
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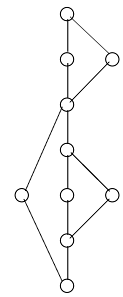
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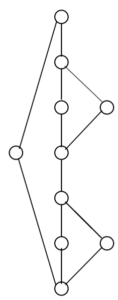
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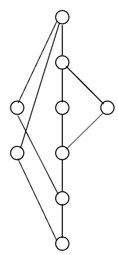
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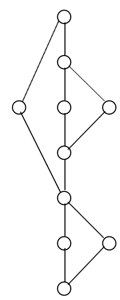
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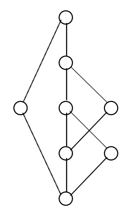
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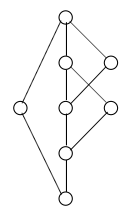
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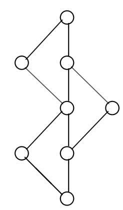
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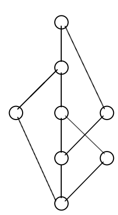
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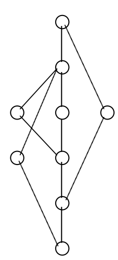
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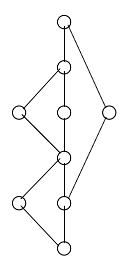
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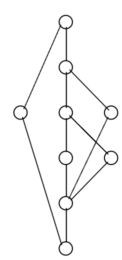
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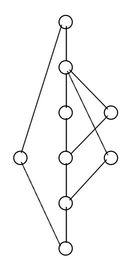
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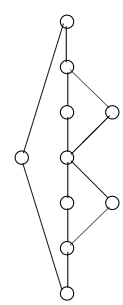
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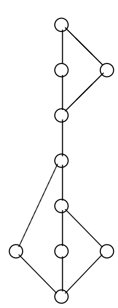
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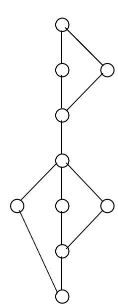
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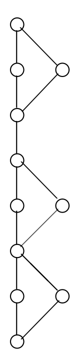
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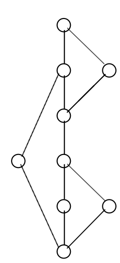
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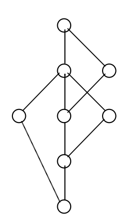
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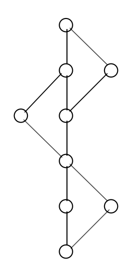
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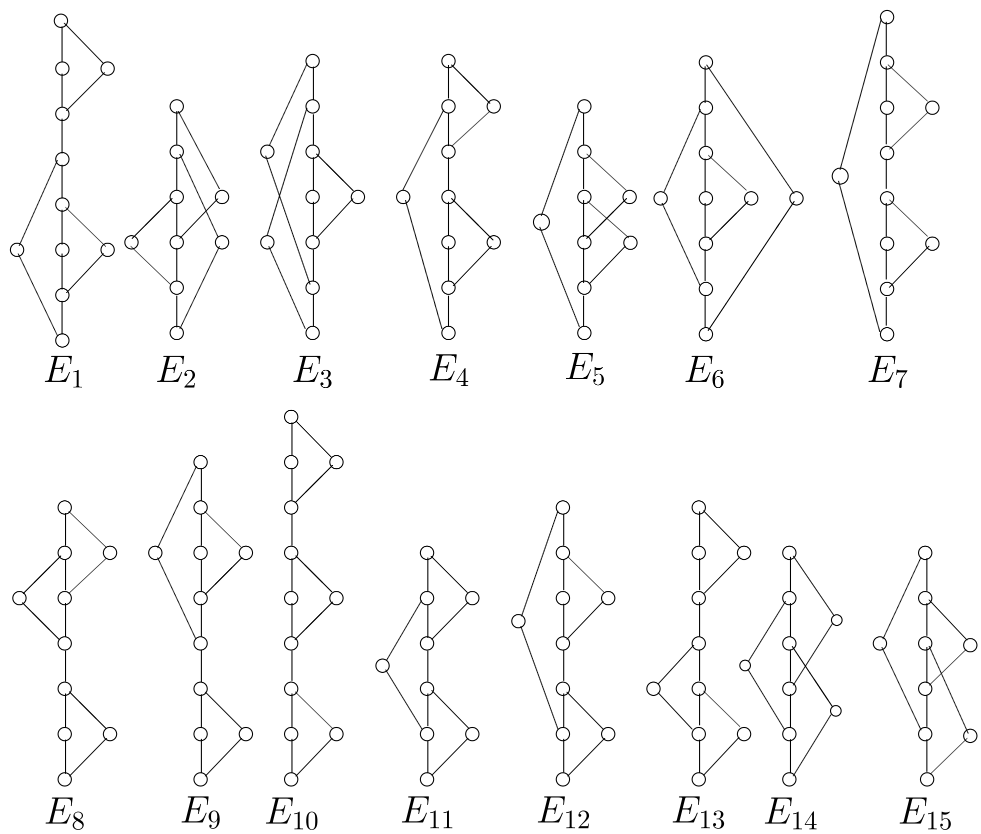

Thus there are in all 75 non-isomorphic basic blocks of nullity three, in which all the reducible elements are comparable. There are in all 17 non-isomorphic basic blocks of nullity three, in which at least two of the reducible elements are incomparable (see the figures Fig.9, Fig.10 and Fig.11). Therefore there are in all 92 non-isomorphic basic blocks of nullity three.

It can also be observed that there are in all 79 non-isomorphic fundamental basic blocks of nullity three.

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