Partial derivatives

Partial derivative of a function w.r.t.x

A partial derivative of function $f(x,y)$ w.r.t. x at point (x_0,y_0) is

$\frac{\partial f}{\partial x}|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$, if limit exists

It is denoted by $f_x(x_0,y_0)$

Partial derivative of a function w.r.t.y

A partial derivative of function $f(x,y)$ w.r.t. y at point (x_0,y_0) is

$\frac{\partial f}{\partial y}|_{(x_0,y_0)} = \lim_{k\to 0} \frac{f(x_0,y_0+k)-f(x_0,y_0)}{k}$, if limit exists

It is denoted by $f_y(x_0,y_0)$

Examples:

1. Find partial derivatives of the following functions.

$a)f(x,y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$ at point $(2,-3)$

$b)f(x,y) = sin^2(x-3y)$

Solution: (a) Let $f(x,y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$ at point $(2,-3)$

$\frac{\partial f}{\partial x} = 5y - 14x + 3$

$\frac{\partial f}{\partial x}|_{(2,-3)} = 5(-3) - 14(2) + 3 = -40$

$\frac{\partial f}{\partial y} = 5x - 2y - 6$

$\frac{\partial f}{\partial y}|_{(2,-3)} = 5(2) - 2(-3) - 6 = 10$

(b) Let $f(x,y) = sin^2(x-3y)$

$\frac{\partial f}{\partial x} = 2sin(x-3y)\frac{\partial[sin(x-3y)]}{\partial x}$

$\frac{\partial f}{\partial x} = 2sin(x-3y)cos(x-3y)\frac{\partial(x-3y)}{\partial x}$

$\frac{\partial f}{\partial x} = 2sin(x-3y)cos(x-3y)(1)$

$\frac{\partial f}{\partial x} = sin[2(x-3y)]$

$\frac{\partial f}{\partial y} = 2sin(x-3y)\frac{\partial[sin(x-3y)]}{\partial y}$

$\frac{\partial f}{\partial x} = 2sin(x-3y)cos(x-3y)\frac{\partial(x-3y)}{\partial y}$

$\frac{\partial f}{\partial x} = 2sin(x-3y)cos(x-3y)(-3)$

$\frac{\partial f}{\partial x} = -3sin[2(x-3y)]$

2.Find f_x, f_y, f_z of the following functions

$(a)f(x,y,z) = x - \sqrt{y^2+z^2}$

$(b)f(x,y,z) = sin^{-1}(xyz)$

Solution: (a) Let $f(x,y,z) = x - \sqrt{y^2+z^2}$

$f_x = 1$

$f_y = \frac{\partial(x-\sqrt{y^2+z^2})}{\partial y}$

$f_y = -\frac{2y}{2\sqrt{y^2+z^2}} = -\frac{y}{\sqrt{y^2+z^2}}$

$f_z = \frac{\partial(x-\sqrt{y^2+z^2})}{\partial z}$

$f_z = -\frac{1}{2\sqrt{y^2+z^2}}\frac{\partial(y^2+z^2)}{\partial z} = -\frac{2z}{2\sqrt{y^2+z^2}} = -\frac{z}{\sqrt{y^2+z^2}}$

(b) Let $f(x,y,z) = sin^{-1}(xyz)$

$f_x = \frac{1}{\sqrt{1-x^2y^2z^2}} \frac{\partial(xyz)}{\partial x} = \frac{yz}{\sqrt{1-x^2y^2z^2}}$
$f_y = \frac{xz}{\sqrt{1-x^2y^2z^2}}$
$f_z = \frac{xy}{\sqrt{1-x^2y^2z^2}}$

3. By using limit definition of partial derivatives, Compute the partial derivatives of $f(x, y) = 4 + 2x - 3y - xy^2$ at $(2, -1)$
Solution: Let $f(x, y) = 4 + 2x - 3y - xy^2$
Since $\frac{\partial f}{\partial x}|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$
$\frac{\partial f}{\partial x}|_{(-2,1)} = \lim_{h\to 0} \frac{f(-2 + h, 1) - f(-2, 1)}{h}$
$= \lim_{h\to 0} \frac{[4 + 2(-2 + h) - 3 - (-2 + h)] - [4 - 4 - 3 + 2]}{h}$
$= \lim_{h\to 0} \frac{[4 - 4 + 2h - 3 + 2 - h] - [-1]}{h}$
$= \lim_{h\to 0} \frac{h}{h} = 1$
Similarly $\frac{\partial f}{\partial y}|_{(x_0,y_0)} = \lim_{k\to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}$
$\frac{\partial f}{\partial x}|_{(-2,1)} = \lim_{k\to 0} \frac{f(-2, 1 + k) - f(-2, 1)}{k}$
$\frac{\partial f}{\partial y}|_{(-2,1)} \lim_{k\to 0} \frac{[4 + 2(-2) - 3(1 + k) - (-2)(1 + k)^2] - [] - 1}{k}$
$\lim_{k\to 0} \frac{2k^2 + k}{k} = \lim_{k\to 0} 2k + 1 = 1$

4. Find $\frac{\partial x}{\partial z}$ at $(1, -1, -3)$, if the equation $xz + ylnx - x^2 + 4 = 0$ defines x as a function of y, z ad partial derivative exists.
Solution: Let $xz + ylnx - x^2 + 4 = 0$
differentiating partially w.r.t. z
$x + z\frac{\partial x}{\partial z} + \frac{y}{x}\frac{\partial x}{\partial z} - 2x\frac{\partial x}{\partial z} = 0$
$\therefore x + (z + \frac{y}{x} - 2x)\frac{\partial x}{\partial z} = 0$
$\therefore \frac{\partial x}{\partial z} = \frac{-z}{zx+y-2x^2}$
At a point $(1, -1, -3)$
$\frac{\partial x}{\partial z} = \frac{1}{6}$

5. If resistors of R_1, R_2, R_3 ohms are connected i parallel to make an R ohm resistors, the value of R as $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$. Find $\frac{\partial R}{\partial R_2}$, when $R_1 = 30, R_2 = 45, R_3 = 90ohms.$
Solution: Since $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$
$\frac{\partial}{\partial R_2}(\frac{1}{R}) = \frac{\partial}{\partial R_2}(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3})$
$-\frac{1}{R^2}\frac{\partial}{\partial R_2} = -\frac{1}{R_2^2}\ \frac{\partial}{\partial R_2} = (\frac{R}{R_2})^2$
If $R_1 = 30, R_2 = 45, R_3 = 90ohms$ $\frac{\partial R}{\partial R_2} = 1/9$

Second order partial derivatives
If we partially differentiate $f(x, y)$ twice , we get second order partial derivatives.

It is denoted by
$\frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial x^2} = (f_x)_x = f_{xx} = f_{x^2}$
$\frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial x \partial y} = (f_y)_x = f_{yx}$
$\frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y = f_{xy}$
$\frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial y^2} = (f_y)_y = f_{yy} = f_{y^2}$
The second order partial derivative at point (x_0, y_0) are defined as

$$f_{xx}(x_0, y_0) = \lim_{h \to 0} \frac{f_x(x_0 + h, y_0) - f_x(x_0, y_0)}{h}$$
$$f_{xy}(x_0, y_0) = \lim_{k \to 0} \frac{f_x(x_0, y_0 + k) - f_x(x_0, y_0)}{k}$$
$$f_{yx}(x_0, y_0) = \lim_{h \to 0} \frac{f_y(x_0 + h, y_0) - f_y(x_0, y_0)}{h}$$
$$f_{yy}(x_0, y_0) = \lim_{k \to 0} \frac{f_y(x_0, y_0 + k) - f_y(x_0, y_0)}{k}$$

Example
1. Find all second order partial derivatives of function $f(x, y) = tan^{-1}(\frac{y}{x})$
Solution: Let $f(x, y) = tan^{-1}(\frac{y}{x})$
$$\frac{\partial f}{\partial x} = \frac{1}{1 + (\frac{y}{x})^2} \frac{\partial}{\partial x}(\frac{y}{x})$$
$$= \frac{x^2}{x^2 + y^2}(\frac{-y}{x^2})$$
$$= \frac{-y}{x^2 + y^2}$$
$$\frac{\partial f}{\partial y} = \frac{1}{1 + (\frac{y}{x})^2} \frac{\partial}{\partial y}(\frac{y}{x})$$
$$= \frac{x^2}{x^2 + y^2}(\frac{1}{x^2})$$
$$= \frac{x}{x^2 + y^2}$$
$$\frac{\partial f}{\partial x} = \frac{1}{1 + (\frac{y}{x})^2} \frac{\partial}{\partial x}(\frac{y}{x})$$
$$= \frac{x^2}{x^2 + y^2}(\frac{-y}{x^2})$$
$$= \frac{-y}{x^2 + y^2}$$
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial}{\partial x}(\frac{-y}{x^2 + y^2}) = \frac{2xy}{(x^2 + y^2)^2}$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial x}(\frac{x}{x^2 + y^2}) = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$
Similarly $\frac{\partial^2 f}{\partial y \partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial x}(\frac{x}{x^2 + y^2}) = \frac{-2xy}{(x^2 + y^2)^2}$$

2.Verify that $W_{xy} = W_{yx}$ for $W = e^x + xlny + ylnx$
Solution: let $W = e^x + xlny + ylnx$
$$\frac{\partial W}{\partial x} = W_x = e^x + lny + \frac{y}{x}$$
$$\frac{\partial W}{\partial x} = W_y = \frac{x}{y} + lnx$$
$$\frac{\partial^2 W}{\partial x \partial y} = W_{yx} = \frac{\partial}{\partial x}(\frac{x}{y} + lnx) = \frac{1}{y} + \frac{1}{x}$$
$$\frac{\partial^2 W}{\partial y \partial x} = W_{xy} = \frac{\partial}{\partial y}(e^x + \frac{y}{x} + lny) = \frac{1}{y} + \frac{1}{x}$$
$W_{xy} = W_{yx}$

Theorem:The Mixed derivative theorem (Clairaut's) theorem
Statement: If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, f_{yx} are defined throughout an open region containing a point (a, b) and all are continuous at (a, b) then $f_{xy}(a, b) = f_{yx}(a, b)$
Proof: Let f_x, f_y, f_{xy}, f_{yx} are defined throughout an open region containing a point (a, b) and all are continuous at (a, b)
Claim: $f_{xy}(a, b) = f_{yx}(a, b)$
Since $f, f_x, f_y, f_{xy}, f_{yx}$ are defined in the interior of rectangle R in the xy plane containing point (a, b)
Let h and k be the numbers such that the point $(a+h, b+k)$ lies also in the interior of R
Consider $\triangle = F(a + h) - F(a)$...(1)
Where $F(x) = f(x, b + k) - f(x, b)$...(2)
Apply the mean value theorem to F on $(a, a + h)$, which is continuous because it is differentiable. $\therefore$ equation (1) becomes
$\triangle = hF'(c_1), c_1 \in (a, a + h)$......(3)
From equation (2) $F'(x) = f_x(x, b + k) - f_x(x, b)$
Equation (3) becomes $\triangle = h[f_x(c_1, b + k) - f_x(c_1, b)]$.....(4)
Apply mean value theorem to function $g(y) = f_x(c_1, y)$
$\therefore g(b + k) - g(b) = kg'(d_1), d_1 \in (b, b + k)$
$\therefore f_x(c_1, b + k) - f_x(c_1, b) = kf_{xy}(c_1, d_1)$
equation (4) becomes $\triangle = hkf_{xy}(c_1, d_1)$ for some point $(c_1, d_1) \in R'$.....(5)
now by using equation (2) equation (1) becomes
$\triangle = f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b)$
$\triangle = [f(a + h, b + k) - f(a, b + k)] - [f(a + h, b) - f(a, b)]$
Let $\triangle = \phi(b + k) - \phi(b)$.....(6)
where $\phi(y) = f(a + h, y) - f(a, y)$.....(7)
Apply mean value theorem to equation (6) we get
$\triangle = k\phi'(d_2), d_2 \in (b, b + k)$.....(8)
from equation (7)
$\phi'(y) = f_y(a + h, y) - f_y(a, y)$....(9)
equation (8) becomes
$\triangle = k[f_y(a + h, d_2) - f_y(a, d_2)]$
Apply mean value theorem to $f_y(x.d_2)$ we get

$f_y(a+h,d_2) - f_y(a,d_2) = hf_{yx}(c_2,d_2), c_2 \in (a,a+h)$
$\therefore \triangle = khf_{yx}(c_2,d_2).....(10)$
from equation (5) and (10)
$f_{xy}(c_1,d_1) = f_{yx}(c_2,d_2)$
where c_1, d_1 both lies in R'
Since f_{xy} and f_{yx} are both continuous at point (a,b)
$\therefore f_{xy}(c_1,d_1) = f_{xy}(a,b) + \epsilon_1$
and $\therefore f_{yx}(c_2,d_2) = f_{yx}(a,b) + \epsilon_2$
Since $(\epsilon_1,\epsilon_2) \to (0,0)$ as $(h,k) \to (0,0)$
$\therefore as(h,k) \to (0,0)$
$f_{xy}(a,b) = f_{yx}(a,b)$

Higher order Partial derivative
Higher order partial derivatives are $f_{xyxx}, f_{xxxx}, f_{yyyyx}$
For example:Find f_{yxyz} if $f(x,y,z) = 1 - 2xy^2z + x^2y$
Solution: Let $f(x,y,z) = 1 - 2xy^2z + x^2y$
First we differentiate $f(x,y,z)$ with respect to y then x then y and then z
$\therefore f_y = -4xyz + x^2$
$f_{yx} = -4yz + 2x$
$f_{yxy} = -4z$
$f_{yxyz} = -4$

Example: Show that $f(x,y,z) = 2z^3 - 3(x^2+y^2)z$ satisfies a Laplace equation.
Solution: Let $f(x,y,z) = 2z^3 - 3(x^2+y^2)z$

$$\frac{\partial f}{\partial x} = -6xz$$
$$\frac{\partial^2 f}{\partial x^2} = -6z$$
$$\frac{\partial f}{\partial y} = -6yz$$
$$\frac{\partial^2 f}{\partial y^2} = -6z$$
$$\frac{\partial f}{\partial z} = 6z^2 - 3(x^2+y^2)$$
$$\frac{\partial^2 f}{\partial z^2} = 12z$$
$$\therefore \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -6z - 6z + 12z = 0$$