Simplex Method for LPP

Subject: Operations Research

For

T.Y.B.Sc. (Mathematics)

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Contents

1 Introduction

Simplex Algorithm was developed first by G. B. Dantzig in 1947. Any linear model for which the solution exists can be solved by this method.

This method is based on the fundamental theorem of linear programming:

Theorem 1.1 (Fundamental theorem of linear programming:) *The collection of all feasible solutions of linear programming problems constitutes a convex set whose extreme point correspond to the basic feasible solutions.*

(**Note:** A convex set in $n-$dimensional space is said to be *convex* if for any two points in that set, the line segment joining these points lies in the set.) i.e. start with some certain solution which is basic feasible solution and then we improve this initial solution in consecutive stages applying the steps of Simplex Algorithm until we arrive at the optimal solution which is acceptable by decision maker.

This method also helps the decision maker to identify the redundant constraints, an unbounded solution, alternate solution and an infeasible solution. The basis of the Simplex Method consists of two fundamental conditions:

a) ***The feasibility condition:*** It ensures that if the starting solution is basic feasible, only basic feasible solutions will be obtained during computation.

b) ***The optimality condition:*** It guarantees that only better solutions (as compared to the current solution) will be encountered.

The simplex algorithm is an iterative (step-by-step) procedure for solving linear programming problem (LPP). It follows the following steps:

i) Start with a trial basic feasible solution to constraint equations.

ii) Test whether it is optimal.

iii) Improve the first trial solution by a set of rules and repeating the process till an optimal solution is obtained.

It is very interesting to note that a feasible solution at any iteration is related to the feasible solution at the successive iteration. One of the non-basic variables (which is zero at initial stage) becomes basic variable (non-zero) at the following iteration and is called ***entering variable*** and to compensate one of the basic variables becomes non-basic variable at the same time which is called ***leaving variable***. The other non-basic variables remain zero and the other basic variables remain non-zero (though there value may change).
The entering variable can be selected so that it improves the value of the objective function so that the new solution is better than the previous one. This can be achieved by the *optimality condition* which selects that entering variable which produces the largest *per unit* gain in the objective function.
This procedure is repeated successively until no improvement in the value of the objective function is possible. The final solution is, then, called an optimal basic solution. This solution satisfies the objective function equation, the constraints as well as the non-negativity conditions.

2 Some Basic Concepts

Consider the general structure of Linear Programming Problem:

$$\text{optimize (Max. or Min.)} Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$
$$\text{subject to}$$
$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n(\leq, =, \geq)b_1\\
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n(\leq, =, \geq)b_1\\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad \vdots\\
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n(\leq, =, \geq)b_1\\
&x_1, x_2, \cdots, x_n \quad \geq \ 0
\end{aligned}$$

The general structure can be expressed in compact form as:

$$\text{optimize (Max. or Min.)} Z = \sum_{j=1}^{n} c_jx_j$$
$$\text{subject to}$$
$$\begin{aligned}
&\sum_{j=1}^{n} a_{ij}x_j = b_i; \quad i = 1, 2, \cdots, m\\
&x_j \quad \geq \ 0, \quad \forall \ j
\end{aligned}$$

Definition 2.1 (Feasible Solution:) *The set of values of decision variables $x_j, (j = 1, 2, \cdots, n)$ that satisfy all the constraints and non-negativity condition of an linear programming problem (LPP) simultaneously is said to constitute the feasible solution of that LPP.*

Definition 2.2 (Basic Solution:) *For a set of m simultaneous equations in n variables $(n > m)$, a solution obtained by setting $(n - m)$ variables equal to zero and solving for remaining m equations in m variables is called a basic solution.*

Definition 2.3 (Basic Feasible solution:) *A feasible solution of a LPP which is also the basic solution is called basic feasible solution. All basic variables have non-negative values.*

Basic feasible solutions are of two types:

Definition 2.4 ((a) Degenerate Solution:) *A basic feasible solution of a LPP is called degenerate solution in which one or more than one of the basic variable out of all m basic variables is equal to zero (≥ 0).*

Definition 2.5 ((b) Non-degenerate Solution:) *A basic feasible solution of a LPP is said to be non-degenerate solution if the value of all m basic variables are non-zero and positive (> 0).*

Definition 2.6 (Optimum basic feasible Solution:) *A basic feasible solution that optimizes (maximizes or minimizes) the objective function value of the given LPP is called an optimum basic feasible solution.*

Definition 2.7 (Unbounded Solution:) *If the value of the objective function can be increased or decreased indefinitely, then the solution is called unbounded solution.*

3 Conversion in Standard Form

The constraints of LPP may be of the type '$\leq$','$\geq$'. We have to convert them in the equality form before we use Simplex Algorithm by adding suitable variables as per the inequality.

Definition 3.1 (Slack Variable:) *A slack variable represents an unused resource. It may be in the form of in the form of time on machine, labour hours, money, warehouse space etc. The slack variable is added to the left hand side of the constraint if the constraint is of the type '$\leq$' to convert the constraint into an equality.*

Definition 3.2 (Surplus variable:) *A surplus variable represents the amount by which solution values exceed a resource. The surplus variable is subtracted from the left hand side of the constraint if the constraint is of the type '$\geq$' to convert the constraint into an equation.*
*Surplus variable is also called '**negative slack variable**'.*

The standard form of LPP should have the following characteristics:

i) All the constraints should be expressed as equations by adding slack or surplus (and/or artificial[1]) variables.

ii) The right hand side of each constraint should be made non-negative. If not, then this should be done by multiplying the resulting constraint by -1.

iii) The slack or surplus (and/or artificial) should be added in the objective function with appropriate coefficients.
Slack and surplus variable carry zero coefficient in the objective function.

The standard form of the LPP (all constraints '$\leq$' type) can be expressed as:

$$\text{optimize (Max. or Min.)} Z = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} 0 s_i \quad \text{(Objective function)}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i; \quad i = 1, 2, \cdots, m \quad \text{(constraints)}$$

$$x_j,\ s_i \quad \geq 0, \quad \forall\ i,\ j \quad \text{(Non-negativity conditions)}$$

4 Simplex Algorithm for Maximization Case

We consider the LPP of maximization case with all '$\leq$' constraints.

Step 1: Write LPP in standard form as per given in section (3).

Step 2: Set up the initial basic feasible solution (IBFS) by setting all non-basic variables equal to zero. Assume that the profit earned is zero which will possible if all decision variables are set equal to zero. These variables are called non-basic variables.

[1] Artificial variables will be explained in Big -M Method.

Step 3: Set up the initial table as follows:

	c_j	c_1	c_2		c_n	
c_B	x_B/x_j	x_1	x_2	$\cdots$	x_n	B
c_{B1}	s_1	a_{11}	a_{12}	$\cdots$	a_{1n}	$x_{B1} = b_1$
c_{B2}	s_2	a_{21}	a_{22}	$\cdots$	a_{2n}	$x_{B2} = b_2$
c_{B3}	s_3	a_{31}	a_{32}	$\cdots$	a_{3n}	$x_{B3} = b_3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
c_{Bm}	s_m	a_{m1}	a_{m2}		a_{mn}	$x_{Bm} = b_m$
$Z = \sum c_{Bi}x_{Bi}$	$z_j = \sum c_{Bi}x_j$	0	0	$\cdots$	0	-
	$z_j - c_j$	$z_1 - c_1$	$z_2 - c_2$	$\cdots$	$z_n - c_n$	-

where

$c_B =$ Coefficients of basic variables in the objective function

$x_B =$ Basic variables in standard form of LPP

$a_{ij} =$ Substitution rates or exchange coefficients

It represents the rate at which i^{th} resource is consumed by each unit of j^{th} activity

$z_j =$ Amount by which the value of objective function Z would be decreased (or increased) if one unit of the given variable is added to the new solution.

$z_j - c_j =$ net effect $= z_j$(outgoing total profit/cost) $- c_j$(incoming unit profit/cost)

$$z_j = \sum_{j=1}^{n} [(\text{coefficient of basic variables column}) \times (\text{coefficient in the column of } x_j)]$$

(Each of values in the $z_j - c_j$ row represents the net amount of increase (or decrease) in the objective function that would occur when one unit of the variable represented by the column head is introduced into the solution.) The variables corresponding to the columns of the identity matrix are called *basic variables* and the remaining ones are called *non-basic variables*. (Slack variables in constraints being coefficient 1 are always considered as the basic variable at the initial stage.)

Step 4: ***Optimality Test***: Calculate the $z_j - c_j$ value for all non-basic variables. Examine the values of $z_j - c_j$

(a) If all $z_j - c_j \geq 0$, then the basic feasible solution is optimal.

(b) If at least one of $z_j - c_j < 0$, then it indicates that improvement in the objective function Z is possible.

Step 5: Select a variable that has the smallest value among all negative $z_j - c_j$. Then the corresponding column is called ***key column*** and the corresponding variable x_j is called ***entering variable***.

Step 6: Find the ratio: $\left(\dfrac{\text{column of constants } B}{\text{corresponding coefficient of entering variable } x_j}\right) = \dfrac{x_{Br}}{a_{rj}}$, where $a_{rj} > 0$.
Select the ***minimum ratio*** among all ratios, then the corresponding row is called ***key column*** and the corresponding basic variable is the ***leaving***

variable.
It may be noted that division by a negative or by a zero element in a key column is not permitted.
The element lies at the intersection of the key row and key column of the simplex table is called ***key or pivot. element***.

Step 7: Apply Gauss Jordan Elimination Method. i.e Reduce the pivot to 1 and reduce the entries below and above pivot to zero by performing the suitable row operations on coefficient matrix and column of constants.

Step 8: Go to **step 4** and repeat the procedure till we get all $z_j - c_j \geq 0$.The solution in the final simplex table will be optimal solution.

5 Simplex Algorithm for Minimization Case

We consider LPP with minimization of objective function which contains '$\geq$' and '$=$' type constraints. We know that by adding negative surplus variable we can convert '$\geq$' type constraint into an equation(Refer definition (3.2)). Then the standard form of LPP can be given as:

$$\text{Min} \quad Z = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} 0 s_i \quad \text{(Objective function)}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j - s_i = b_i; \quad i = 1, 2, \cdots, m \quad \text{(constraints)}$$

$$x_j,\ s_i \quad \geq 0, \quad \forall\ i,\ j \quad \text{(Non-negativity conditions)}$$

Letting $x_j = 0, \quad j = 1, 2, \cdots, n$, we get IBFS as $-s_i = b_i$ or $s_i = -b_i$.
This solution can not be considered as the IBFS as it violets the non-negativity condition of surplus variables as $s_i \geq 0$.
Similarly if some k^{th} constraint is '$=$' type i.e. of the type $\sum_{j=1}^{n} a_{kj} x_j = b_k$, then substituting $x_j = 0, \quad j = 1, 2, \cdots, n$, we get $0 = b_k$ which is absurd when $b_k \neq 0$.
It is not also acceptable when $b_k = 0$.
In such condition, we have to add artificial variable in left hand side of that corresponding constraints.
Then the resulting system of equations becomes:

$$\text{Max} \quad Z = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} 0 s_i - \sum_{i=1}^{\leq m} M A_i \qquad \text{(if objective function is of maximization)}$$

OR

$$\text{Min} \quad Z = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} 0 s_i + \sum_{i=1}^{m} M A_i \qquad \text{(if objective function is of minimization)}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j - s_i + A_i = b_i; \quad i = 1, 2, \cdots, m \quad \text{(constraints)}$$

$$x_j,\ s_i, A_i \quad \geq 0, \quad \forall\ i,\ j \quad \text{(Non-negativity conditions)}$$

Definition 5.1 (Artificial Variable:) *An artificial variable is the* ***fictitious variable*** *and* ***has no physical meaning*** *and are only used as a tool for generating an initial basic feasible solution to an LPP. They assume the role of slack variables in the first iteration. Before the final simplex solution is reached, all artificial variables must be dropped out from the solution mix. This is done by assigning appropriate coefficients to those variables in the objective function.*
These variables are added to the constraints of the type '$\geq$' and '$=$'.

Coefficient of artificial variable in objective function of maximization case is $-M$ and coefficient of artificial variable in objective function of minimisation case is $+M$.
The summary of the extra variables to be added in the given LPP in order to covert it into a standard form is given in the following table:

Type of constraint	Extra variable needed	Coefficient in objective function		Coefficient in constraints	Nature of variable in IBFS
		Max Z	Min Z		
$\leq$	Add slack variable $(s_1, s_2, \cdots)$	0	0	1	basic
$\geq$	Subtract surplus variable $(s_1, s_2, \cdots)$	0	0	-1	non-basic
	Add artificial variable $(A_1, A_2, \cdots)$	-M	+M	1	basic
$=$	Add only artificial variable $(A_1, A_2, \cdots)$	-M	+M	1	basic

There are two methods for eliminating artificial variables from the solution

- Big-M Method
- Two-phase Method

6 Big-M Method

Consider the LPP of minimization case which may contains all types of constraints. All the steps are same as the simplex algorithm for maximisation case except for the selection of entering variable and optimality condition.
For Minimization case:

1. If all $z_j - c_j \leq 0$, then the basic feasible solution is optimal.
2. If at least one of $z_j - c_j > 0$, then it indicates that improvement in the objective function Z is possible.
3. Select a variable that has the largest positive $z_j - c_j$. The corresponding variable x_j is called ***entering variable***.
4. Rest fo the operations are same as per simplex algorithm for maximization case.

7 Resolution of complications

Unrestricted variable: In LPP, it is assumed that all variables $x_j, j = 1, 2, \cdots$ should be non-negative values. However, in practical situation, one or more variables may be zero or positive, negative. Such variables are called *unrestricted variables.* All variables must be non-negative as per the requirement of simplex method. If any LPP consists of unrestricted variables, then first we have to convert LPP into an equivalent problem having only non-negative variables.
Procedure to resolve: Express each of unrestricted variables as the difference of two non-negative variables and re-write the LPP in the form of x_r^+ and x_r^-.
Let variable x_r be unrestricted in sign. Then define two new variables say x_r^+ and x_r^- such that $x_r = x_r^+ - x_r^-, \quad x_r^+, x_r^- \geq 0$
Note: Variables x_r^+, x_r^- can not appear simultaneously in the basis. Thus any of the following three cases may arise at the optimal solution:
(i) if $x_r^- = 0 \Rightarrow x_r = x_r^+$ i.e. positive.
(ii) if $x_r^+ = 0 \Rightarrow x_r = -x_r^-$ i.e. negative.
(iii)if $x_r^- = 0 = x_r^+ \Rightarrow x_r = 0$ i.e. zero.

Tie for Entering Basic Variable

1. If there is tie between two decision variables, then the selection can be made arbitrary.
2. If there is a tie between a decision variable and a slack (or surplus) variable, then select the decision variable to enter into basis first
3. If there is a tie between two slack (or surplus) variable, then selection can be made arbitrary.

Tie for leaving Basic Variable

1. If there is tie between two slack (or surplus) variables, then the selection can be made arbitrary.
2. If there is tie between two slack and decision variables, then the slack variable leave the basis first.
3. If there is a tie between slack and surplus variables, then the selection can be made arbitrary.
4. If there is a tie between slack (or surplus) and artificial variables, then artificial variable leave the basis first.
5. If there is tie between two artificial variables, then the selection can be made arbitrary.

8 Types of Linear Programming Solutions

Alternative Optimal Solution: Alternative optimal solution arise when $z_j - c_j = 0$ for at least one non-basic variable in the final optimal table of linear programming problem.
How to obtain alternate solution? - To obtain alternate solution enter the non-basic variable for which $z_j - c_j = 0$ into basis.
Unbounded Solution: Unbounded solution occurs when value of decision variables in the solution of linear programming problem becomes infinitely large without violating any given constraints.

Definition 8.1 *Unbounded solution of LPP occurs when minimum ratio cannot be obtained due to all the coefficients in entering variable are less than or equals to zero.*

Infeasible solution or no solution:

Definition 8.2 *Infeasible solution of LPP occurs when there exist at least one artificial variable in the column of basis variable with positive value in the final optimal table.*

When an infeasible solution exists, the linear programming model should be reformulated. This may occur because of the fact that the model is either improperly formulated or because two or more of the constraints are incompatible.

9 Exercise

Solve the following LPP by simplex/Big-M method:

1. $MaxZ = 5x_1 + 3x_2$
   subject to
   $x_1 + x_2 \leq 2$
   $5x_1 + 2x_2 \leq 10$
   $3x_1 + 8x_2 \leq 12$
   $x_1, x_2 \geq 0$

2. $MaxZ = 5x_1 + 4x_2$
   subject to
   $6x_1 + 4x_2 \leq 24$
   $x_1 + 2x_2 \leq 6$
   $-x_1 + x_2 \leq 1$
   $x_2 \leq 2$
   $x_1, x_2 \geq 0$

3. $MaxZ = x_1 + x_2 + x_3$
   subject to
   $3x_1 + 2x_2 + x_3 \leq 7$
   $2x_1 + x_2 + x_3 \leq 2$
   $x_1, x_2, x_3 \geq 0$

4. $MaxZ = 2x_1 + x_2 - 3x_3 + 5x_4$
   subject to
   $x_1 + 2x_2 + 2x_3 + 4x_4 \leq 40$
   $2x_1 - x_2 + x_3 + 2x_4 \leq 8$
   $4x_1 - 2x_2 + x_3 - x_4 \leq 10$
   $x_1, x_2, x_3, x_4 \geq 0$

5. $MaxZ = x_1 - x_2 + 3x_3$
   subject to
   $x_1 + x_2 + x_3 \leq 10$
   $2x_1 - x_3 \leq 2$
   $2x_1 - 2x_2 + 3x_3 \leq 0$
   $x_1, x_2, x_3 \geq 0$

6. $MinZ = 2x_1 + 8x_2$
   subject to
   $2x_1 + 3x_2 \geq 3$
   $x_1 + 4x_2 \geq 2$
   $x_1 + 2x_2 \geq 3$
   $x_1, x_2 \geq 0$

7. $MinZ = 4x_1 + x_2$
   subject to
   $3x_1 + x_2 = 3$
   $4x_1 + 3x_2 \geq 6$
   $x_1 + 2x_2 \leq 4$
   $x_1, x_2 \geq 0$

8. $MinZ = 2x_1 + 3x_2 - 5x_3$
   subject to
   $x_1 + x_2 + x_3 = 7$
   $2x_1 - 5x_2 + x_3 \geq 10$
   $x_1, x_2, x_3 \geq 0$

9. $MaxZ = 2x_1 + 5x_2$
   subject to
   $3x_1 + 2x_2 \geq 6$
   $2x_1 + x_2 \leq 2$
   $x_1, x_2 \geq 0$

10. $MaxZ = 3x_1 + 9x_2$
    subject to
    $x_1 + 4x_2 \leq 8$
    $x_1 + 2x_2 \leq 4$
    $x_1, x_2 \geq 0$

11. $MaxZ = 3x_1 + 2x_2$
    subject to
    $4x_1 - x_2 \leq 4$
    $4x_1 + 3x_2 \leq 6$
    $4x_1 + x_2 \leq 4$
    $x_1, x_2 \geq 0$

12. $MaxZ = 2x_1 - x_2 + 3x_3$
    subject to
    $x_1 - x_2 + 5x_3 \leq 5$
    $2x_1 - x_2 + 3x_3 \leq 20$
    $x_1, x_2, x_3 \geq 0$

13. $MaxZ = 2x_1 + 3x_2 + 4x_3$
    subject to
    $3x_1 + x_2 + 4x_3 \leq 600$
    $2x_1 + 4x_2 + 2x_3 \geq 480$
    $2x_1 + 3x_2 + 3x_3 = 540$
    $x_1, x_2, x_3 \geq 0$

14. $MaxZ = 20x_1 + 5x_2 + x_3$
    subject to
    $3x_1 + 5x_2 - 5x_3 \leq 504$
    $x_1 \leq 10$
    $x_1 + 3x_2 - 4x_3 \leq 20$
    $x_1, x_2, x_3 \geq 0$

15. $MaxZ = 3x_1 + 2x_2 + 3x_3$
    subject to
    $2x_1 + x_2 + x_3 \leq 4$
    $3x_1 + 4x_2 + 2x_3 \geq 16$
    $x_1, x_2, x_3 \geq 0$

16. $MaxZ = 3x_1 + 5x + 4x_3$
    subject to
    $2x_1 + 3x_2 \leq 8$
    $2x_2 + 5x_3 \leq 10$
    $3x_1 + 2x_2 + 4x_3 \leq 15$
    $x_1, x_2, x_3 \geq 0$

17. $MaxZ = 3x_1 + 2x_2 + 2x_3$
    subject to
    $5x_1 + 7x_2 + 4x_3 \leq 7$
    $-4x_1 + 7x_2 + 5x_3 \geq -2$
    $x_1, x_2, x_3 \geq 0$

18. $MinZ = 3x_1 - x_2$
    subject to
    $2x_1 + x_2 \geq 2$
    $x_1 + 3x_2 \leq 3$
    $x_2 \geq 4$
    $x_1, x_2 \geq 0$

19. $MaxZ = 6x_1 + 3x_2$
subject to
$2x_1 + x_2 \leq 8$
$3x_1 + 3x_2 \leq 18$
$x_2 \leq 3$
$x_1, x_2 \geq 0$

20. $MaxZ = 3x_1 + 6x_2$
subject to
$3x_1 + 4x_2 \geq 12$
$-2x_1 + x_2 \leq 4$
$x_1, x_2 \geq 0$

21. $MaxZ = 2x_1 + 3x_2$
subject to
$x_1 - x_2 \geq 4$
$x_1 + x_2 \leq 6$
$x_1 \leq 2$
$x_1, x_2 \geq 0$

22. $MaxZ = x_1 - 2x_2 + 2x_3$
subject to
$3x_1 - 2x_2 + 3x_3 \leq 7$
$-2x_1 + 4x_2 \leq 12$
$x_2 + 8x_3 \leq 10$
$x_1, x_2, x_3 \geq 0$

23. $MaxZ = x_1 + x_2 + x_3$
subject to
$4x_1 + 5x_2 + 3x_3 \leq 15$
$10x_1 + 7x_2 + x_3 \leq 12$
$x_1, x_2, x_3 \geq 0$

24. $MaxZ = x_1 + x_2 + x_3$
subject to
$3x_1 + 2x_2 + x_3 \leq 3$
$2x_1 + x_2 + 2x_3 \leq 2$
$x_1, x_2, x_3 \geq 0$

25. $MaxZ = x_1 - x_2 + 3x_3$
subject to
$x_1 + x_2 + x_3 \leq 10$
$2x_1 - x_3 \leq 2$
$2x_1 - 2x_2 + 3x_3 \leq 0$
$x_1, x_2, x_3 \geq 0$

26. $MaxZ = 20x_1 + 10x_2 + x_3$
subject to
$3x_1 - 3x_2 + 5x_3 \leq 50$
$x_1 + x_3 \leq 10$
$x_1 - x_2 + 4x_3 \leq 20$
$x_1, x_2, x_3 \geq 0$

10 Two Phase method:

This method has two phases. The solution of the LPP is completed in two phases. The purpose of this method is to remove artificial variables and minimize calculations.

First write the given LPP in the standard form.

Phase I:

In this phase, the sum of artificial variables is minimized subject to the given constraints in order to get the basic feasible solution of LPP.

1. Consider the objective function always as
Min r = sum of all artificial variables introduced in standard form of given LPP regardless whether the LPP is of maximization or minimization.
2. Obtain r-equation and construct the initial table for phase I.
3. Reduce the coefficients of artificial variables to zero by applying suitable row operations in order to get the new r-equation. Solve it .
4. (a) If Min r = 0 and the solution is optimal,then proceed for Phase II.
   (b) If Min r = 0 and the solution is optimal but at least one artificial variable is present in the column of basic variables, LPP has no feasible solution.

Phase II:

In this phase, we have to start with the original objective function and with the basic feasible solution obtained at the end of Phase I.

1. Construct the table containing original objective function(first row) and the constraints from the basic feasible solution obtained at the end of Phase I. **Delete the columns of artificial variables.**
2. Apply the usual simplex algorithm to the modified simplex table in order to get the optimal solution to the original LPP.

11 Duality in LPP

The concept of duality is very useful in Mathematics, Physics, Statistics, Engineering and managerial decision making. For example, in a two-person game theory, one competitor's problem is the dual of the opponent's problem. The main focus of a dual problem is to find for each resource its best marginal value.

Formulation of Dual LPP:

The given original LPP is called primal LPP. A formulation can be summarized as follows:

1. In maximization primal LPP, all '$\geq$' type constraints are converted to '$\leq$' type. Put '=' type constraints as it is.
2. In minimization primal LPP, all '$\leq$' type constraints are converted to '$\geq$' type. Put '=' type constraints as it is.
3. The dual variable corresponding to '=' type constraint in primal LPP will be unrestricted variable in dual LPP and vice-versa.

The relation between primal and dual is given as follows:

Primal LPP	**Dual LPP**
Objective function is of maximization	Objective function becomes of minimization
Every primal variable	Corresponds to the constraint
Every primal constraint	Corresponds to dual variable
Primal variable x_j unrestricted in sign	Corresponding j^{th} dual constraint is '=' type
i^{th} primal constraint '='	Corresponding dual variable y_i is unrestricted in sign
If primal constraints of '$\leq$' type	Corresponding dual constraint is of '$\geq$' type

11.1 Standard results on duality

1. The dual of the dual LPP is again the primal problem.
2. If either the primal or dual problem has an unbounded objective function value, the other problem has no feasible solution.
3. If either the primal or dual problem has a finite optimal solution, the other one also possesses the same and optimal value of the objective functions of the two problems are equal. i.e. Max Z_x = Min Z_y. This analytical result is known as *fundamental primal-dual relationship*.
4. ***Complementary slackness*** property of primal-dual relationship states that for a positive basic variable in the primal, the corresponding dual variable will be equal to zero. Alternatively, for a non-basic variable in the primal (which is zero), the corresponding dual variable will be basic and positive.

11.2 Comparison of solutions

1. The slack variables in the primal, correspond to the dual basic variables in the optimal optimal solution and vice-versa.
2. The absolute value of $z_j - c_j$ row under the columns of the slack/surplus variables directly gives the optimal values of the dual/primal basic variables.

11.3 Advantages of Duality

1. It is advantageous to solve the dual of a primal that has a less number of constraints because the number of constraints usually equals th number of iterations required to solve the problem.

2. This avoids the necessity for adding surplus or artificial variables and solves the problem quickly. In economics, duality is useful in the formulation in the input and output systems. It is useful in physics, engineering, mathematics etc.

3. The dual variables provide an important economic interpretation of the final solution of LPP.

4. It is quite useful when investigating changes in the parameters of LPP. (the technique is known as the *sensitivity analysis*)

5. Duality is used to solve LPP by simplex method in which the initial solution is infeasible. (the technique is known as dual simplex)

12 Exercise

Ques. 1: Write the dual of the following primal LPP:

1. $MaxZ = 2x_1 + 5x_2 + 6x_3$
   subject to
   $5x_1 + 6x_2 - x_3 \leq 3$
   $-2x_1 + x_2 + 4x_3 \leq 4$
   $x_1 - 5x_2 + 3x_3 \leq 1$
   $x_1, x_2, x_3 \geq 0$

2. $MinZ = 7x_1 + 3x_2 + 8x_3$
   subject to
   $8x_1 + 2x_2 + x_3 \geq 3$
   $3x_1 + 6x_2 + 4x_3 \geq 4$
   $4x_1 + x_2 + 5x_3 \geq 1$

3. $MaxZ = 2x_1 + 3x_2 + 4x_3$
   subject to
   $2x_1 + 3x_2 + 5x_3 \geq 2$
   $3x_1 + x_2 + 7x_3 = 4$
   $x_1 + 4x_2 + 6x_3 \geq 5$
   $x_1, x_2 \geq 0, x_3$ unrestricted

4. $MaxZ = 3x_1 + 6x_2$
   subject to
   $3x_1 + 4x_2 \geq 12$
   $-2x_1 + x_2 \leq 4$
   $x_1, x_2 \geq 0$

Ques. 2: Write the dual of the following primal LPP and obtain the solution of primal LPP:

1. $MaxZ = 2x_1 + x_2$
   subject to
   $x_1 + x_2 \geq 2$
   $x_1 + 3x_2 \leq 3$
   $x_1, x_2 \geq 0$

2. $MinZ = -2x_1 + 3x_2 + 4x_3$
   subject to
   $-2x_1 + x_2 \geq 3$
   $-x_1 + 3x_2 + x_3 \geq -1$
   $x_1, x_2, x_3 \geq 0$

3. $MinZ = x_1 + x_2$
   subject to
   $x_1 + 2x_2 \geq 2$
   $x_1 + 7x_2 \geq 7$
   $x_1, x_2 \geq 0$

4. $MaxZ = x_1 + 5x_2$
   subject to
   $x_1 + 2x_2 \leq 3$
   $-2x_1 + x_2 \leq -4$
   $x_1, x_2 \geq 0$

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