Linear Transformations

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Definition :

Let U and V be vector spaces over $\mathbb{R}$. A mapping $T : U \to V$ is called a **linear transformation** if it satisfies the following two conditions :

1. For all $u, v \in U$, $T(u+v) = T(u) + T(v)$.
2. For any $u \in U$ and for any $\alpha \in \mathbb{R}$, $T(\alpha u) = \alpha T(u)$.

In other words, T is a linear mapping if it preserves the basic operations of a vector space, that of vector addition and that of scalar multiplication.

Note that, $\mathbb{R}$ may be replaced by any field F.

For example, $T:\mathbb{R}^2 \to \mathbb{R}^3$ defined by
$T(x,y) = (x+y, 2x, x-y)$
is a linear transformation.

Theorem : Let U and V be vector spaces over $\mathbb{R}$. A mapping $T:U \to V$ is a linear transformation if it satisfies for all $u, v \in U$ and for any $\alpha, \beta \in \mathbb{R}$,
$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$.

Note that $T(\mathbf{0}) = \mathbf{0}'$, where $\mathbf{0}$ and $\mathbf{0}'$ are the zero vectors of U and V respectively.

Show that $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x + y, 2x, x - y)$ is a linear transformation.

Solution : Let $u = (a, b), v = (c, d) \in \mathbb{R}^2$. Then $T(u + v) = T(a + c, b + d) = ((a + c) + (b + d), 2(a+c), (a+c)-(b+d)) = (a+b, 2a, a-b)+(c+d, 2c, c - d) = T(a, b) + T(c, d) = T(u) + T(v)$.

Also, for $\alpha \in \mathbb{R}$ and $u = (a, b) \in \mathbb{R}^2$,

$T(\alpha u) = T(\alpha(a, b)) = T(\alpha a, \alpha b) = (\alpha a + \alpha b, 2(\alpha a), \alpha a - \alpha b) = (\alpha(a + b), \alpha(2a), \alpha(a - b))$
$= \alpha(a + b, 2a, a - b) = \alpha T(a, b) = \alpha T(u)$.

Exercise 1 :

Check whether the following mappings are linear transformations?

- $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x, y, 0)$.
- $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + 1, y)$.
- $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x, y) = xy$.
- $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y, 3x)$.
- $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x, yz)$.
- $T : U \to V$ defined by $T(u) = 0$, for all $u \in U$.
- $T : V \to V$ defined by $T(v) = v$, for all $v \in V$.

More examples of linear transformations :

- $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.
- $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T_A(X) = AX$, where $A = [a_{ij}]_{m \times n}$.
- $T : \mathcal{F}([0,1], \mathbb{R}) \to \mathcal{F}([0,1], \mathbb{R})$ defined by $T(f(x)) = f'(x)$, for all $f(x) \in \mathcal{F}([0,1], \mathbb{R})$.
- $T : C([0,1]) \to \mathbb{R}$ defined by $$T(f(x)) = \int_0^1 f(x)dx, \text{ for all } f(x) \in C([0,1]).$$
- $T : \mathcal{P}_n \to \mathcal{P}_{n+1}$ defined by $T(p(x)) = xp(x)$, for all $p(x) \in \mathcal{P}_n$.

Let $T : U \to V$ be a linear transformation. Then

- T is said to be **onto** or **surjective** if for all $v \in V$, there exists $u \in U$ such that $T(u) = v$. In other words, T is onto if $T(U) = V$.
- T is said to be **one-one** or **injective** if for any $u \neq v \in U$, $T(u) \neq T(v)$. Equivalently, T is one-one if $T(u) = T(v)$ implies $u = v$.
- T is said to be **bijective** if T is injective as well as surjective. If T is bijective then T^{-1} exists and it is also bijective.
- T is said to be an **isomorphism** if T is bijective. U and V are said to be **isomorphic** if T is an isomorphism.

- $T(\mathbf{0}) = \mathbf{0}' \implies T(-u) = -T(u)$ for all $u \in U$.
- $Ker(T) = \{u \in U | T(u) = \mathbf{0}'\}$ is called **kernel** of T. Note that $Ker(T)$ is a subspace of U.
- T is said to be **singular** if for some $u \neq \mathbf{0}$ in U, $T(u) = \mathbf{0}'$; Otherwise, T is called **non-singular.**
- T is one-one if $Ker(T) = \{\mathbf{0}\}$ (or T is non-singular).
- $T(U) = \{T(u) | u \in U\}$ is called **range** of T. Note that $T(U)$ is a subspace of V.
- Nullity$(T) = \dim(\text{Ker}(T))$ and Rank$(T) = \dim(T(U))$.
- Dimension Theorem : If U is of finite dimension then rank(T) + nullity$(T) = \dim(U)$.

Operations with linear mappings :

Let $T_1 : U \to V$ and $T_2 : U \to V$ be linear transformations. Define

1. The **sum** $T_1 + T_2 : U \to V$ as $(T_1 + T_2)(u) = T_1(u) + T_2(u)$, for all $u \in U$.
2. The **scalar multiplication** $\alpha T_1 : U \to V$ as $(\alpha T_1)(u) = \alpha T_1(u)$, for all $u \in U$.

Prove that both $T_1 + T_2$ and αT_1 are linear transformations.

Let $T_1 : U \to V$ and $T_2 : V \to W$ be two linear transformations.
Define a map $T_2 \circ T_1 : U \to W$ as
$(T_2 \circ T_1)(u) = T_2(T_1(u))$, for all $u \in U$.
The map $T_2 \circ T_1$ is called a **composition** of the linear transformations T_1 and T_2.

Prove that $T_2 \circ T_1$ is a linear transformation.

In general, $T_2 \circ T_1 \neq T_1 \circ T_2$.

The collection of all linear transformations from U to V, denoted by $Hom(U, V)$, is a vector space with the sum and the scalar multiplication of linear transformations defined as above.

For finite dimensional vector spaces U and V, $dim(Hom(U, V)) = dim(U) \times dim(V)$.

A linear transformation $T : V \to V$ is called a **linear operator**.
A linear operator T is said to be **invertible** if it has an inverse T^{-1}, that is, $T \circ T^{-1} = T^{-1} \circ T = I$.

Illustration :

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as $T(x, y, z) = (x + y, y + z, z + x)$. Show that T is one-one, onto, bijective, a linear transformation and isomorphism. Also find T^{-1}, Kernel of T and Range of T.

Solution : Clearly T is a linear transformation. Note that $\text{Ker}(T) = \{(0, 0, 0)\}$. Therefore T is one-one. Also for any $(x, y, z) \in \mathbb{R}^3$ (Codomain), there exists $(x', y', z') = (\frac{x+z-y}{2}, \frac{x+y-z}{2}, \frac{z+y-x}{2}) \in \mathbb{R}^3$ (Domain), such that $T(x', y', z') = (x' + y', y' + z', z' + x') = (x, y, z)$. This implies that T is onto. Thus, T is bijective. Hence T is an isomorphism. As T is onto, range of $T = T(\mathbb{R}^3) = \mathbb{R}^3$. Finally, $T^{-1}(x, y, z) = (\frac{x+z-y}{2}, \frac{x+y-z}{2}, \frac{z+y-x}{2})$.

Matrix of a linear transformation :

Let U and V be vector spaces of dimension n and m respectively. Let $T : U \to V$ be a linear transformation. Let $E = \{u_1, u_2, \ldots, u_n\}$ be a basis of U. Let $F = \{v_1, v_2, \ldots, v_m\}$ be a basis of V.
Now for each $i, 1 \leq i \leq n$, as $T(u_i) \in V$, we have
$T(u_i) = b_{i1}v_1 + b_{i2}v_2 + \cdots + b_{im}v_m$.
Therefore the coordinate vector of $T(u_i)$ is
$[T(u_i)]_F = (b_{i1}, b_{i2}, \ldots, b_{im})$.
If $B_{n\times m}$ is a matrix with i^{th} row $[b_{i1} \; b_{i2} \; \ldots \; b_{im}]$ then $A = B^t$, a transpose of B, is called the matrix representation of T relative to the bases E and F.

Illustration 1 :

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation given by $T(x, y, z) = (-x - y + z, x - 4y + z, 2x - 5y)$. Determine the matrix of T with respect to the basis $E = \{u_1 = (1, 0, 2), u_2 = (2, 1, 0), u_3 = (1, 0, 1)\}$.

Solution : Note that $T(u_1) = (1, 3, 2)$, $T(u_2) = (-3, -2, -3)$ and $T(u_3) = (0, 2, 2)$. Also, the coordinate vectors $[T(u_1)]_E = (7, 3, -12)$, $[T(u_2)]_E = (-4, -2, 5)$ and $[T(u_3)]_E = (6, 2, -10)$. Therefore $B = \begin{bmatrix} 7 & 3 & -12 \\ -4 & -2 & 5 \\ 6 & 2 & -10 \end{bmatrix}$. Hence matrix of T is $A = \begin{bmatrix} 7 & -4 & 6 \\ 3 & -2 & 2 \\ -12 & 5 & -10 \end{bmatrix}$.

Let $A = \begin{bmatrix} 3 & 2 & -2 \\ -1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix}$ be the matrix of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to a basis $B = \{u_1 = (5, 1, 3), u_2 = (3, 2, 2), u_3 = (1, 2, 1)\}$. Determine the linear transformation T.

Solution : Note that, the coordinate vectors $[T(u_1)]_B = (3, -1, 2)$, $[T(u_2)]_B = (2, 0, 1)$ and $[T(u_3)]_B = (-2, 1, -1)$.

$\therefore T(u_1) = (3)u_1 + (-1)u_2 + (2)u_3 = (14, 5, 9)$,
$T(u_2) = (2)u_1 + (0)u_2 + (1)u_3 = (11, 4, 7)$ and
$T(u_3) = (-2)u_1 + (1)u_2 + (-1)u_3 = (-8, -2, -5)$.

Let $u = (x, y, z) \in \mathbb{R}^3$.
Suppose $u = k_1 u_1 + k_2 u_2 + k_3 u_3$.
Then we get
$5k_1 + 3k_2 + k_3 = x, k_1 + 2k_2 + 2k_3 = y, 3k_1 + 2k_2 + k_3 = z$.
$\therefore k_1 = -2x - y + 4z, k_2 = 5x + 2y - 9z, k_3 = -4x - y + 7z$.

But $T(u) = k_1 T(u_1) + k_2 T(u_2) + k_3 T(u_3)$. Therefore

$T(x, y, z) = (59x + 16y - 99z, 18x + 5y - 30z, 37x + 10y - 62z)$.

Theorem : Let V be a vector space with $dim(V) = n$. Let $E = \{u_1, u_2, \ldots, u_n\}$ and $F = \{v_1, v_2, \ldots, v_m\}$ be two bases of V. Then there exists a non-singular matrix $P = [p_{ij}]$ of size n such that $v_i = p_{1i}u_1 + p_{2i}u_2 + \cdots + p_{ni}u_n, \forall i, 1 \leq i \leq n$.

Note that $P_i = [p_{1i} \ p_{2i} \ \cdots \ p_{ni}]^t$, the i^{th} column of P, is the coordinate vector of v_i with respect to the basis E for each i, $1 \leq i \leq n$.
The matrix P in the above theorem is the coordinate transformation matrix, called a **transition matrix** from F to E.

Also, if X and Y are the coordinate vectors of $u \in V$ with respect to the bases E and F respectively then $Y = P^{-1}X$.

Theorem : Let V be a vector space with $dim(V) = n$ and $T : V \to V$ be a linear operator. Let $E = \{u_1, u_2, \ldots, u_n\}$ and $F = \{v_1, v_2, \ldots, v_m\}$ be two bases of V, and let P be the transition matrix from F to E. Then $P^{-1}AP$ is the matrix of T w.r.t. the basis F whenever A is the matrix of T w.r.t. the basis E.

Note that $P^{-1}AP$ and A are similar matrices.

Illustration : Let $E = \{(1,0),(0,1)\}$ and $F = \{(1,-1),(2,1)\}$ be two bases of $\mathbb{R}^2$. Verify the above theorem for $T(x,y) = (x+y, x-2y)$.

Thank you