Differential Geometry

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CHAPTER 4

VECTOR FIELDS ON SURFACES; ORIENTATION

Definition. A vector field on an $n-$surface $S \subset \mathbb{R}^{n+1}$ is a function which assigns to each point p in S a vector $X(p) \in \mathbb{R}^{n+1}_p$ at p.
Definition. If $X(p)$ is tangent to S(i.e. $X(p) \in S_p$) for each $p \in S$, X is said to be a tangent vector field on S.
Definition. If $X(p)$ is orthogonal to S(i.e. $X(p) \in S_p^{\perp}$) for each $p \in S$, X is said to be a normal vector field on S.
Definition. A function $g : S \to \mathbb{R}^k$, where S is an $n-$surface in $\mathbb{R}^{n+1}$, is smooth if it is the restriction to S of a smooth function $\tilde{g} : V \to \mathbb{R}^k$ defined on some open set V in $\mathbb{R}^{k+1}$ containing S.
Definition. A vector field X on S is smooth if it is the restriction to S of a smooth vector field defined on some open set containing S. Thus S is smooth if and only if $X : S \to \mathbb{R}^{n+1}$ is smooth, where $X(p) = (p, X(p))$ for all $p \in S$.
Theorem. *Let S be an n-surface in $\mathbb{R}^{n+1}$, let X be a smooth tangent vector field on S, and let $p \in S$. Then there exists an open interval I containing 0 and a parametrized curve $\alpha : I \to S$ such that*
(i) $\alpha(0) = p$
(ii) $\dot{\alpha}(t) = X(\alpha(t))$ *for all* $t \in I$
(iii) *If $\beta : \tilde{I} \to S$ is any other parametrized curve in S satisfying (i) and (ii), then $\tilde{I} \subset I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$.*
PROOF. Since X is smooth vector field, there exists an open set V containing S and a smooth vector field $\tilde{X}$ on V such that $\tilde{X}(q) = X(q)$ for all $q \in S$. Let $f : U \to \mathbb{R}$ and $c \in \mathbb{R}$ be such that $S = f^{-1}(c)$ and $\nabla f(q) \neq 0$ for all $q \in S$. Let

$$W = \{q \in U \cap V : \nabla f(q) \neq 0\}.$$

Then W is an open set containing S, and both $\tilde{X}$ and f are defined on W. Let Y be the vector field on W, everywhere tangent to the level sets of f, defined by

$$Y(q) = \tilde{X}(q) - \frac{\tilde{X}(q) \cdot \nabla f(q)}{\|\nabla f(p)\|^2} \nabla f(q).$$

Note that $Y(q) = X(q)$ for all $q \in S$. Let $\alpha : I \to W$ be the maximal integral curve of Y through p. Then α maps I to S because

$$\begin{aligned}(f \circ \alpha)'(t) &= \nabla f(\alpha(t)) \cdot \dot{\alpha}(t) \\ &= \nabla f(\alpha(t)) \cdot Y(\alpha(t)) \\ &= 0\end{aligned}$$

and $(f \circ \alpha)(0) = f(\alpha(0)) = f(p) = c$, so $(f \circ \alpha)(t) = c$, for all $t \in I$. Therefore conditions (i) and (ii) holds and by theorem from Chapter 2, condition (iii) is satisfied because $\beta : \tilde{I} \to S$ satisfying (i) and (ii) is also an integral curve of the vector field Y on W. ■

Definition. A subset of $\mathbb{R}^{n+1}$ is said to be connected if for each pair (p, q) of points in S there is a parametrized map $\alpha : [a, b] \to S$ such that $\alpha(a) = p$ and $\alpha(b) = q$. Thus S is connected if each pair of points S can be joined by a continuous curve(not necessarily smooth which lies completely in S).

Example. Show that the unit n-sphere $x_1^2 + x_2^2 + ... + x_{n+1}^2 = 1$ is connected if $n \geq 1$.

Solution. Since the sphere $S = f^{-1}(1)$, where $f : U \to R^{n+1}$ is smooth function.

Let $n = 0$ then $f : \mathbb{R} \to \mathbb{R}$ and

$$\begin{aligned}S &= \{x_1 \in \mathbb{R} : x_1^2 = 1\} \\ &= \{x_1 \in \mathbb{R} : x_1 = \pm 1\}\end{aligned}$$

Let $n = 1$ then $f : \mathbb{R}^2 \to \mathbb{R}^2$ and

$$S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$$

Let $n = 2$ then $f : \mathbb{R}^3 \to \mathbb{R}^3$ and

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$$

For $n = 1, S$ is unit 1-sphere in $\mathbb{R}^2$ and for $n = 2, S$ is 2-sphere in $\mathbb{R}^3$ which are connected similarly we can see that for $n = k, S$ is $k-$sphere is connected surface in $\mathbb{R}^{k+1}$. But for $n = 0, S$ is set containing two points 1 and -1 so there is no path is which connects these two points hence not connected. Thus the unit $n-$sphere is connected for $n \geq 1$.

Theorem. *Let $S \subset \mathbb{R}^{n+1}$ be a connected $n-$surface in $\mathbb{R}^{n+1}$. Then there exist on S exactly two smooth unit normal vector field N_1 and N_2, and $N_2(p) = -N_1(p)$ for all $p \in S$.*

PROOF. Since S is smooth $n-$surface in $\mathbb{R}^{n+1} \implies S = f^{-1}(c)$, where $f : U \to \mathbb{R}$ and $\nabla f(p) \neq 0, \forall p \in S$.

Let N_1 be vector field defined by

$$N_1(p) = \frac{\nabla f(p)}{\|\nabla f(p)\|}, \quad \forall p \in S$$

Therefore, N_1 is unit normal vector field and $N_1(p) \in [\nabla f(p)]$. The another smooth unit normal vector field at the point p is

$$N_2(p) = -\frac{\nabla f(p)}{\|\nabla f(p)\|}, \quad \forall p \in S$$

Hence,

$$N_1(p) = N_2(p), \quad \forall p \in S$$

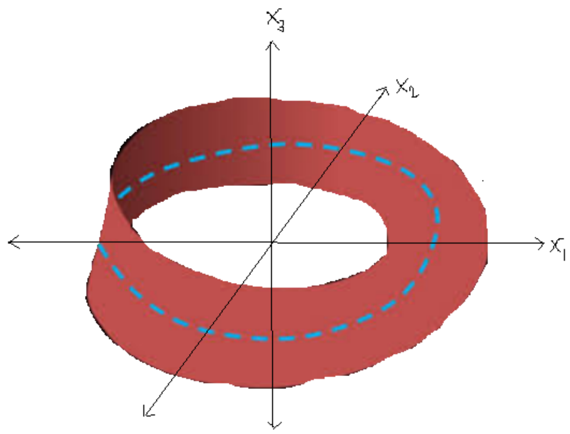
Therefore, N_1 and N_2 are both unit vector field on S.
Now show that these are the only two such vector fields. Suppose N_3 be another such vector field then for each $p \in S$.
Then, for each $p \in S$, $N_3(p)$ must be a multiple of $N_1(p)$ since both lies in 1-dimensional vector subspace $[\nabla f(p)]^{\perp} \subset \mathbb{R}^{n+1}_p$. Thus

$$N_3(p) = g(p)N_1(p), \quad \forall p \in S$$

where, $g : S \to \mathbb{R}$ is a smooth function on S.
Since $g(p) = N_3(p) \cdot N_1(p) = \pm 1$. $\because$ Both N_1 and N_2 are unit vectors.
Therefore, $N_3(p) = \pm N_1(p)$ which shows either $N_3 = N_1$ or $N_3 = N_2$. ■

Definition. A smooth unit normal vector field on an n-surface S in $\mathbb{R}^{n+1}$ is called an orientation on S. According to the theorem just proved , each connected n-surface in $\mathbb{R}^{n+1}$ has exactly two orientations. An n-surface together with a choice of orientation is called oriented surface.

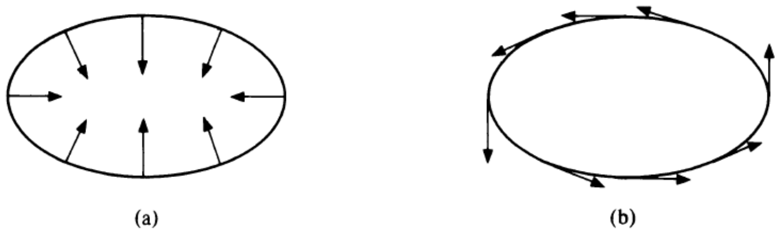
The Möbius band.
The Möbius band B, is surface in $\mathbb{R}^3$ obtained by taking a rectangular strip of paper, twisting one end 180°, and taping the ends together. That there is no smooth unit normal vector field on B can be seen by picking a unit normal vector at some point on the central circle and trying to extend it continuously to a unit normal vector field along this circle. After going around the circle once, the normal vector is pointing in the opposite directly! Since there is no smooth unit normal vector field on B, B cannot be expressed as a level set $f^{-1}(c)$ of some smooth function $f : U \to \mathbb{R}$ with $\nabla f(p) \neq 0$ for all $p \in S$, and hence B is not a 2-surface according to our definition. B is example of "unorientable 2-surface".



The Möbius band

Note. Here onward we will consider only orientable n-surfaces in $\mathbb{R}^{n+1}$.
Definition. A unit vector in $\mathbb{R}^{n+1}_p$ is called a direction at p.
Thus an orientation on an $n-$surface S in $\mathbb{R}^{n+1}$ is, a smooth choice of normal direction at each point of S.

On a plane curve, an orientation can be used to define tangent direction at each point of the curve. The positive tangent direction at the point p of the oriented plane curve C is the direction obtained by rotating the orientation normal direction at p through an angle $-\pi/2$, where the direction of positive rotation is counterclockwise



Orientation on a plane curve: (a) the chosen normal direction at each point determines (b) a choice of tangent direction at each point.

On a 2-surface in $\mathbb{R}^3$, an orientation can be used to define a direction of rotation in the tangent space at each point of the surface S is the linear transformation $\mathbb{R}_\theta : S_p \to S_p$ defined by $R_\theta(v) = (\cos\theta)v + (\sin\theta)N(p) \times v$ where $N(p)$ is the orientation normal direction at p. R_θ is usually described as "right-handed rotation about N(p) through the angle θ".

On a 3-surface in $\mathbb{R}^4$, an orientation can be used to define a sense of "handedness" in the tangent space at each point of the surface. Given an oriented 3-surface S and a point $p \in S$, an ordered orthonormal basis $\{e_1, e_2, e_3\}$ for the tangent space S_p to S at p is said to be right-handed if determinant

$$\det \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ N(p) \end{bmatrix}$$

is positive, where $N(p) = (p, N(p))$ is the orientation normal direction at p and $e_i = (p, e_i)$ for $i \in \{1, 2, 3\}$; the basis is left-handed if the determinant is negative.

On an n-surface in $\mathbb{R}^{n+1}$, an orientation can be used to partition the collection of all ordered basis for each tangent space into two subsets, those consistent with the orientation and those inconsistent with the orientation. An ordered basis $\{v_1, v_2, ..., v_n\}$ for the tangent space S_p at the point p of the oriented n-surface S is said to be consistent with the orientation N on S if the determinant

$$\det \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \\ N(p) \end{bmatrix}$$

is positive; the basis is inconsistent with N if the determinant is negative. Here, as usual $v_i = (p, v_i)$ and $N(p) = (p, N(p))$.

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CHAPTER 5

The Gauss Map

An oriented $n-$surface in $\mathbb{R}^{n+1}$ is more than just an $n-$surface. it is an n-surface S together with a smooth unit normal vector field N on S. The function $N : S \to \mathbb{R}^{n+1}$ associated with the vector field N by $N(p) = (p, N(p)), p \in S$, actually S maps to unit n-sphere $S^n \subset \mathbb{R}^{n+1}$, since $\|N(p)\| = 1, \quad \forall p \in S$.

Definition. Associated with each oriented n-surface S is a smooth map $N : S \to S^n$, called as Gauss map.

N may be thought as the map which assigns to every point $p \in S$ the point in $\mathbb{R}^{n+1}$ obtained by translating the unit normal vector $N(p)$ to the origin.

Definition. The image of the Gauss map,

$$N(s) \quad = \quad \{q \in S^n : q = N(p) \text{ for some } p \in S\}$$

is called the spherical image of the oriented n-surface S.

The spherical image of an oriented n-surface S records, the set of directions which occur as normal directions to S. Hence its size is a measure of how much surface curves around in $\mathbb{R}^{n+1}$. For an n-plane, which does not curve around at all, the spherical image is a single point. If an n-surface is compact then it must curve all the way around: the spherical image will be all of S^n.

Theorem. *Let S be a compact connected n-surface in $\mathbb{R}^{n+1}$ exhibited as a level set $f^{-1}(c)$ of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with $\nabla f(p) \neq 0$ for all $p \in S$. Then the Gauss map maps S into the unit sphere S^n.*

Example 1. Describe the spherical image when $n = 1$ and $n = 2$ of the given n-surface oriented by $\nabla f / \|\nabla f\|$, where f is a function defined by left hand side of the cone

$$-x_1^2 + x_2^2 + ... + x_{n+1}^2 \quad = \quad 0, x_1 > 0.$$

Solution. Here $f : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $f(x_1, x_2, ..., x_{n+1}) = -x_1^2 + x_2^2 + ... + x_{n+1}^2$.

$$\therefore S = f^{-1}(0) \quad = \quad \left\{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} : -x_1^2 + x_2^2 + ... + x_{n+1}^2 = 0\right\}$$

$$\nabla f(p) \quad = \quad (p, -2x_1, 2x_2, ..., 2x_{n+1})$$

$\Longrightarrow$

$$\|\nabla f(p)\| \quad = \quad \sqrt{4(x_1^2 + x_2^2 + ... + x_{n+1}^2)}$$

Consider,

$$\begin{aligned} N(p) &= \frac{\nabla f(p)}{\|\nabla f(p)\|} \\ &= \frac{(p, -2x_1, 2x_2, ..., 2x_{n+1})}{\sqrt{4(x_1^2 + x_2^2 + ... + x_{n+1}^2)}} \\ &= \frac{(p, -x_1, x_2, ..., x_{n+1})}{\sqrt{x_1^2 + x_2^2 + ... + x_{n+1}^2}} \end{aligned}$$

when $n = 1$,

$$f(x_1, x_2) = -x_1^2 + x_2^2$$

and

$$\begin{aligned} S = f^{-1}(0) &= \{(x_1, x_2) \in \mathbb{R}^2 : -x_1^2 + x_2^2 = 0\} \\ &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = \pm x_2\} \\ N(p) &= \frac{\nabla f(p)}{\|\nabla f(p)\|} \\ &= \frac{(p, -2x_1, 2x_2)}{\sqrt{4(x_1^2 + x_2^2)}} \\ &= \frac{(p, -x_1, x_2)}{\sqrt{x_1^2 + x_2^2}} \\ &= \left(-\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}}\right) \end{aligned}$$

If $x_1 = x_2$, then

$$\begin{aligned} N(p) &= \left(-\frac{x_1}{\sqrt{x_1^2 + x_1^2}}, \frac{x_1}{\sqrt{x_1^2 + x_1^2}}\right) \\ &= \left(-\frac{x_1}{\sqrt{2x_1^2}}, \frac{x_1}{\sqrt{2x_1^2}}\right) \\ &= \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \end{aligned}$$

If $x_1 = -x_2$, then

$$\begin{aligned} N(p) &= \left(-\frac{x_1}{\sqrt{x_1^2 + x_1^2}}, -\frac{x_1}{\sqrt{x_1^2 + x_1^2}}\right) \\ &= \left(-\frac{x_1}{\sqrt{2x_1^2}}, -\frac{x_1}{\sqrt{2x_1^2}}\right) \\ &= \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \end{aligned}$$

Therefore, the spherical image is $\left\{\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\}$.

Example 2. Find the spherical image of one sheet of a 2-sheeted hyperbola $x_1^2 - x_2^2 - ... - x_{n+1}^2 = 4, x_1 > 0$ oriented by $-\nabla f/\|\nabla f\|$.

Solution. Here

$$f(x_1, x_2, ..., x_{n+1}) = x_1^2 - x_2^2 - ... - x_{n+1}^2.$$

For $n = 1$,

$$\begin{aligned} f(x_1, x_2) &= x_1^2 - x_2^2 \\ \Longrightarrow f^{-1}(4) &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 - x_2^2 = 4\} \end{aligned}$$

which represent hyperbola and $x_1 > 0$, we take only one branch.
Here the orientation is given by $-\nabla f/\|\nabla f\|$.

$$\begin{aligned} \nabla f(x_1, x_2) &= (2x_1, -2x_2) \\ \Longrightarrow \|\nabla f(x_1, x_2)\| &= \sqrt{4x_1^2 + 4x_2^2} \end{aligned}$$

Therefore,

$$\begin{aligned} N(x_1, x_2) &= -\frac{\nabla f(x_1, x_2)}{\|\nabla f(x_1, x_2)\|} \\ &= -\frac{(2x_1, -2x_2)}{2\sqrt{x_1^2 + x_2^2}} \\ &= \frac{(-x_1, x_2)}{\sqrt{x_1^2 + x_2^2}} \\ &= \left(-\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}}\right) \\ &= \left(-\frac{x_1}{\sqrt{x_1^2 + x_1^2 - 4}}, \frac{\sqrt{x_1^2 - 4}}{\sqrt{x_1^2 + x_1^2 - 4}}\right) \\ &= \left(-\frac{x_1}{\sqrt{2}\sqrt{x_1^2 - 2}}, \frac{\sqrt{x_1^2 - 4}}{\sqrt{2}\sqrt{x_1^2 - 2}}\right) \end{aligned}$$

The spherical image is given by

$$\begin{aligned} N(x_1, x_2) &= \{q \in S^1 : q = N(p) \text{ for some } p \in S\} \\ N(x_1, x_2) &= \left\{q \in S^1 : q = \left(-\frac{x_1}{\sqrt{2}\sqrt{x_1^2 - 2}}, \frac{x_1^2 - 4}{\sqrt{2}\sqrt{x_1^2 - 2}}\right) \text{ for some } p \in S\right\} \end{aligned}$$

GEODESICS

Geodesics are curves in $n-$surface which play the same role as do straight line in $\mathbb{R}^n$. Before formulating a precise definition, we must introduce the process of differentiation of vector fields and functions defined along parametrized curves.

Definition. A vector field X along the parametrized curve $\alpha : I \to \mathbb{R}^{n+1}$ is a function which assigns to each $t \in I$ a vector $X(t)$ at $\alpha(t)$; i.e., $X(t) \in \mathbb{R}^{n+1}_{\alpha(t)}$ for all $t \in I$.

Definition. A function f along α is simply a function $f : I \to \mathbb{R}$.

Example. The velocity $\dot{\alpha}$ of the parametrized curve $\alpha : I \to \mathbb{R}^{n+1}$ is a vector field along

α; its length $\|\dot{\alpha}\| : I \to \mathbb{R}$, defined by $\|\dot{\alpha}\|(t) = \|\dot{\alpha}(t)\|$ for all $t \in I$, is a function along α. $\|\dot{\alpha}\|$ is called speed of α.

Vector fields and functions along parametrized curves frequently occur as restrictions. Thus X is a vector field on U, where U is an open subset of $\mathbb{R}^{n+1}$ containing image of α, then $X \circ \alpha$ is a vector field along α. Similarly $f \circ \alpha$ is a function along α whenever $f : U \to \mathbb{R}$, where $U \supset$ Image α.

Each vector field along α is of the form

$$X(t) = (\alpha(t), X_1(t), ..., X_{n+1}(t))$$

where each component X_i is a function along α. X is smooth if each $X_i : I \to \mathbb{R}$ is smooth. The derivative of a smooth vector field X along α is the vector field $\dot{X}$ along α defined by

$$\dot{X}(t) = \left(\alpha(t), \frac{dX_1}{dt}(t), ..., \frac{X_{n+1}}{dt}(t)\right).$$

$\dot{X}(t)$ measures the rate of change of the vector part $(X_1(t), X_2(t), ..., X_{n+1}(t))$ of $X(t)$ along α. Thus, for example, the acceleration $\ddot{\alpha}$ of a parametrized curve α is the vector field along α obtained by differentiating the velocity field $\dot{\alpha}$ $\left[\ddot{\alpha} = \dot{\left(\dot{\alpha}\right)}\right]$.
The differentiation of vector fields along parametrized curves has the following properties. For X and Y smooth vector field along the parametrized curve $\alpha : I \to \mathbb{R}^{n+1}$ and f a smooth function along α,
(i) $\dot{(X+Y)} = \dot{X} + \dot{Y}$
(ii) $\dot{(fX)} = f'X + f\dot{X}$
(iii) $(X \cdot Y)' = \dot{X} \cdot Y + X \cdot \dot{Y}$
where $X + Y, fX$ and $X \cdot Y$ are defined along α by

$$\begin{aligned} (X+Y)(t) &= X(t) + Y(t) \\ (fX)(t) &= f(t)X(t) \\ (X \cdot Y)(t) &= X(t) \cdot Y(t), \quad \forall t \in I. \end{aligned}$$

Definition. A geodesic in an $n-$surface $S \subset \mathbb{R}^{n+1}$ is a parametrized curve $\alpha : I \to S$ whose acceleration is everywhere orthogonal to S; that is $\ddot{\alpha} \in S^{\perp}_{\alpha(t)}$ for all $t \in I$.
Thus geodesic is a curve in S which always goes "straight ahead" in the surface. Its acceleration serves only to keep it in surface. It has no components of acceleration tangent to the surface.
Example 1. Prove that a geodesic has a constant speed.
PROOF. Suppose $\alpha : I \to S$ be a geodesic in an $n-$surface.
$\implies \dot{\alpha}(t) \in S_{\alpha(t)}$ and $\ddot{\alpha}(t) \in S^{\perp}_{\alpha(t)}$. Therefore, $\dot{\alpha}(t) \cdot \ddot{\alpha}(t) = 0$.

Now consider,

$$\begin{aligned}\frac{d}{dt}\|\dot{\alpha}(t)\|^2 &= \frac{d}{dt}\left[\dot{\alpha}(t)\cdot\dot{\alpha}(t)\right]\\ &= \dot{\alpha}(t)\cdot\ddot{\alpha}(t)+\ddot{\alpha}(t)\cdot\dot{\alpha}(t)\\ &= 2\dot{\alpha}(t)\cdot\ddot{\alpha}(t)\\ &= 0\end{aligned}$$

Therefore, $\|\dot{\alpha}(t)\|^2$ is constant.
Therefore, $\|\dot{\alpha}(t)\|$ is constant for all $t \in I$.
Therefore, speed of geodesic is constant.

Example 2. If an $n-$surface S contains a straight line segment $\alpha(t) = p + tv, \quad t \in I$, then prove that this segment is geodesic in S.

PROOF. Suppose $\alpha(t) = p + tv, \quad t \in I$ be a straight line segment in S.
$\Longrightarrow \dot{\alpha}(t) = v$ and $\ddot{\alpha}(t) = 0$.
Here $\dot{\alpha}(t)\cdot\ddot{\alpha}(t) = 0, \quad \forall t \in I$.
$\Longrightarrow \ddot{\alpha}(t) \perp \dot{\alpha}(t), \forall t \in I$.
$\Longrightarrow \ddot{\alpha}(t) \in S_{\alpha(t)}, \quad \forall t \in I$.
Therefore, straight line in S is geodesic in S.

Example 3. For $a, b, c, d \in \mathbb{R}$, show that the parametrized curve $\alpha(t) = (\cos(at + b), \sin(at + b), ct + d)$ is a geodesic in the cylinder $x_1^2 + x_2^2 = 1$ in $\mathbb{R}^3$.

PROOF. Suppose $f(x_1, x_2, x_3) = x_1^2 + x_2^2$ so that $S = f^{-1}(1)$ with $\nabla f(p) \neq 0, \quad \forall p \in S$.

$$\alpha(t) = (\cos(at+b), \sin(at+b), ct+d)$$

$$\Longrightarrow \dot{\alpha}(t) = (-a\sin(at+b), a\cos(at+b), c)$$

$$\begin{aligned}\Longrightarrow \ddot{\alpha}(t) &= (-a^2\cos(at+b), -a^2\sin(at+b), 0)\\ &= -a^2(\cos(at+b), \sin(at+b), 0). \qquad (1)\end{aligned}$$

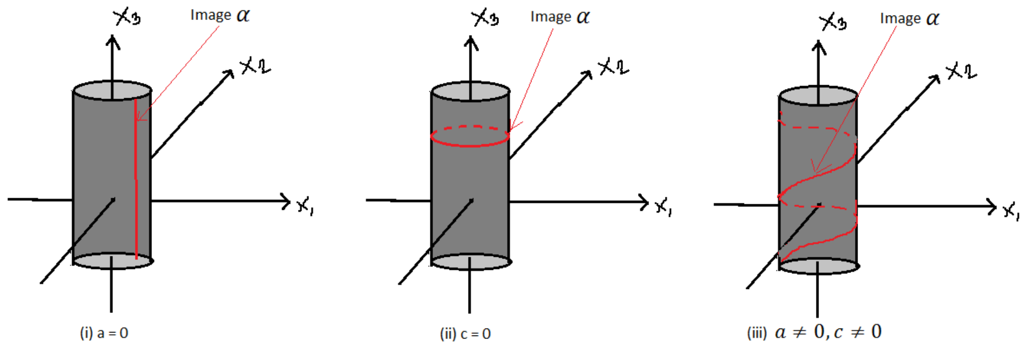
Now we shall find a unit normal vector field to the cylinder S.
$f(x_1, x_2, x_3) = x_1^2 + x_2^2 \Longrightarrow \nabla f(x_1, x_2, x_3) = (2x_1, 2x_2, 0)$.
Consider,

$$\begin{aligned}N(\alpha(t)) &= \frac{\nabla f(\alpha(t))}{\|\nabla f(\alpha(t))\|}\\ &= \frac{(2\cos(at+b), 2\sin(at+b), 0)}{2\sqrt{\cos^2(at+b)+\sin^2(at+b)}}\\ &= (\cos(at+b), \sin(at+b), 0). \qquad (2)\end{aligned}$$

From equation (1) and (2) we get,

$$\ddot{\alpha}(t) = \pm a^2 N(\alpha(t))$$

Therefore, $\alpha(t)$ is geodesic in the cylinder.



Geodesic $\alpha(t) = (\cos(at+b), \sin(at+b), ct+d)$ **in the cylinder** $x_1^2 + x_2^2 = 1$

Example 4. For each pair of orthogonal unit vectors $\{e_1, e_2\}$ in $\mathbb{R}^3$ and each $a \in \mathbb{R}$, the great circles $\alpha(t) = (\cos at)e_1 + (\sin at)e_2$ is a geodesic in the 2−sphere $x_1^2 + x_2^2 + x_3^2 = 1$ in $\mathbb{R}^3$.

Solution. Since $\alpha(t) = (\cos at)e_1 + (\sin at)e_2$.

$$\implies \dot{\alpha}(t) = -a(\sin at)e_1 + a(\cos at)e_2$$

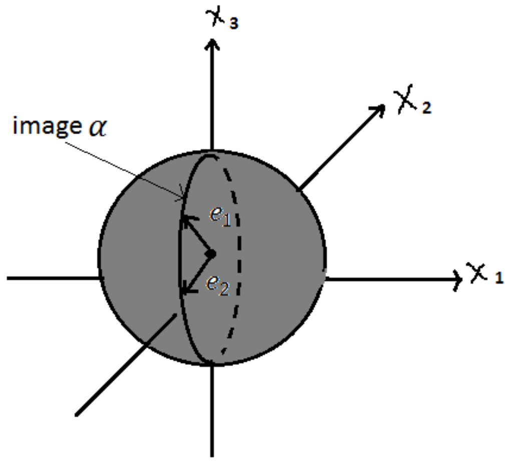
and

$$\ddot{\alpha}(t) = -a^2(\cos at)e_1 - a^2(\sin at)e_2$$

Now,

$$\begin{aligned} N(x_1, x_2, x_3) &= \frac{\nabla f(x_1, x_2, x_3)}{\|\nabla f(x_1, x_2, x_3)\|} \\ &= \frac{(2x_1, 2x_2, 2x_3)}{\|2\sqrt{x_1^2 + x_2^2 + x_3^2}\|} \\ &= (x_1, x_2, x_3) \\ \implies N(\alpha(t)) &= (\cos at, \sin at, 0) \\ &= (\cos at)e_1 + (\sin at)e_2 \end{aligned}$$

Therefore, $\ddot{\alpha}(t) = \pm a^2 N(\alpha(t))$.
$\implies \alpha$ is geodesic in the 2−sphere.



Great circles are geodesics in 2-sphere

Example 5. Let S denote the cylinder $x_1^2 + x_2^2 + x_3^2 = r^2$ in $\mathbb{R}^3$. Show that α is geodesic in S if and only if $\alpha(t) = (r\cos(at+b), r\sin(at+b), ct+d)$.

Solution. Since $\alpha(t) = (r\cos(at+b), r\sin(at+b), ct+d)$.

$$\Longrightarrow \dot{\alpha}(t) = (-ar\sin(at+b), ar\cos(at+b), c)$$

and

$$\begin{aligned}\Longrightarrow \ddot{\alpha}(t) &= (-a^2 r\cos(at+b), -a^r\sin(at+b), c)\\ &= -a^2 r(\cos(at+b), \sin(at+b), c)\end{aligned}$$

Also, we have $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 \Longrightarrow S = f^{-1}(1)$.

Consider,

$$\begin{aligned}N(x_1, x_2, x_3) &= \frac{\nabla f(x_1, x_2, x_3)}{\|\nabla f(x_1, x_2, x_3)\|}\\ &= \frac{(2x_1, 2x_2, 2x_3)}{\sqrt{4x_1^2 + 4x_2^2 + 4x_3^2}}\\ &= \frac{(x_1, x_2, x_3)}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\\ &= (x_1, x_2, x_3)\\ \Longrightarrow N(\alpha(t)) &= r(\cos(at+b), \sin(at+b), ct+d)\end{aligned}$$

$$\begin{aligned}\Longrightarrow \ddot{\alpha}(t) &= \pm a^2 N(\alpha(t))\\ \Longrightarrow \ddot{\alpha}(t) &\in S^{\perp}_{\alpha(t)} \qquad (1)\end{aligned}$$

Suppose $\alpha(t) = (r\cos(at+b), r\sin(at+b), ct+d)$ is geodesic in S.

$\Longleftrightarrow \ddot{\alpha}(t) \in S^{\perp}_{\alpha(t)}$.

$\Longleftrightarrow \ddot{\alpha}(t) \cdot \dot{\alpha}(t) = 0, \quad \forall \dot{\alpha}(t) \in S_{\alpha(t)}$ this will follows from equation (1).
Therefore, α is geodesic in S if and only if $\alpha(t) = (r\cos(at+b), r\sin(at+b), ct+d)$.
Theorem. *Let S an $n-$surface in $\mathbb{R}^{n+1}$ let $p \in S$, and let $v \in S_p$. Then there exists an open interval I containing 0 and a geodesic $\alpha : I \to S$ such that*
(i) *$\alpha(0) = p$ and $\dot{\alpha}(0) = v$.*
(ii) *If $\beta : \tilde{I} \to S$ is any other geodesic in S with $\beta(0) = p$ and $\dot{\beta}(0) = v$, then $\tilde{I} \subset I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$.*
PROOF. Suppose S is n-surface in $\mathbb{R}^{n+1} \Longrightarrow S = f^{-1}(c)$ for some $c \in \mathbb{R}$ and some smooth function $f : U \to \mathbb{R}$(U open in $\mathbb{R}^{n+1}$) with $\nabla f(p) \neq 0$ for all $p \in S$. Since $\nabla f(p) \neq 0$ for all p in some open set containing S, we may assume(by shrinking U if necessary) that $\nabla f(p) \neq 0$ for all $p \in U$. Set $N = \nabla f / \|\nabla f\|$.

By definition, a parametrized curve $\alpha : I \to S$ is geodesic of S if and only if its acceleration is everywhere perpendicular to S; that is, if and only if $\ddot{\alpha}(t)$ is multiple of $N(\alpha(t))$ for all $t \in I$:

$$\ddot{\alpha} = g(t)N(\alpha(t))$$

for all $t \in I$, where $g : I \to \mathbb{R}$. Taking the dot product of both side of this equation with $N(\alpha(t))$ we find

$$\begin{aligned} \ddot{\alpha} \cdot N(\alpha(t)) &= g(t)N(\alpha(t)) \cdot N(\alpha(t)) \qquad (1) \\ &= g(t), \quad \forall t \in I \\ \Longrightarrow g &= \ddot{\alpha} \cdot N \circ \alpha \\ &= (\dot{\alpha} \cdot N \circ \alpha)' - \dot{\alpha} \cdot N\dot{\circ}\alpha \\ &= -\dot{\alpha} \cdot N\dot{\circ}\alpha \end{aligned}$$

Because $\dot{\alpha} \perp N \circ \alpha = 0$. Substituting this value of g in equation (1) we get,

$$\begin{aligned} \ddot{\alpha}(t) &= -\left(\dot{\alpha} \cdot N\dot{\circ}\alpha\right)(N \circ \alpha) \\ \Longrightarrow \ddot{\alpha}(t) + \left(\dot{\alpha} \cdot N\dot{\circ}\alpha\right)(N \circ \alpha) &= 0 \end{aligned}$$

Thus $\alpha : I \to S$ is geodesics if it satisfies the differential equation

$$\ddot{\alpha}(t) + \left(\dot{\alpha} \cdot N\dot{\circ}\alpha\right)(N \circ \alpha) = 0$$

which is a second order differential equation in α. If we express $\alpha(t) = (x_1, x_2, ..., x_{n+1})$ then equating the i^{th} component on both side we get,

$$\frac{d^2 x_i}{dt^2} + \sum_{j,k=1}^{n+1} N_i(x_1, x_2, ..., x_{n+1}) \frac{\partial N_j}{\partial x_k}(x_1, x_2, ..., x_{n+1}) \frac{dx_j}{dt}\frac{dx_k}{dt} = 0$$

where the N_j ($j \in \{1, 2, ..., n+1\}$) are the components of N. By existence theorem for the solution of such equations there is an interval I_1 about 0 and a solution $\beta_1 : I_1 \to U$ of this differential equation satisfying the initial conditions $\beta(0) = p$ and $\dot{\beta}(0) = v$

(that is, satisfying $x_i(0) = p_i$ and $(dx_i/dt)(0) = v_i$ for $i \in \{1, 2, ..., n+1\}$, where $p = (p_1, p_2, ..., p_{n+1})$ and $v = (v_1, v_2, ..., v_{n+1})$). Moreover, this solution is unique in the sense that if $\beta_2 : I_2 \to U$ is another solution, with $\beta_2(0) = p$ and $\dot{\beta}_2(0) = v$, then $\beta_1(t) = \beta_2(t)$ for all $t \in I_1 \cap I_2$. It follows that there exists a maximal open interval I and a unique solution $\alpha : I \to U$ of given system of equations satisfying $\alpha(0) = p$ and $\dot{\alpha}(0) = v$. Furthermore if $\beta : \tilde{I} \to I$ is any another solution with $\beta(0) = p$ and $\dot{\beta}(0) = v$ then $\tilde{I} \subset I$ and β is the restriction of α to $\tilde{I}$.

Now to complete the proof it only remains to show that the solution α is curve in S. For, if so, it must be geodesic because it satisfies the geodesic equation, and rest of the theorem follows from the uniqueness statement above. To see that α is in fact a curve in S, note first that every solution $\alpha : I \to U$ of second order differential equation given above, $\dot{\alpha} \cdot N \circ \alpha = 0$.

$$\Longrightarrow (\dot{\alpha} \cdot N \circ \alpha)' = \ddot{\alpha} \cdot N \circ \alpha + \dot{\alpha} \cdot N \dot{\circ} \alpha = 0$$

Therefore, $\dot{\alpha} \cdot N \circ \alpha$ is constant along α, and

$$(\dot{\alpha} \cdot N \circ \alpha)(0) = \dot{\alpha}(0) \cdot N(\alpha(0)) = v \cdot N(p) = 0$$

Because $v \in S_p$ and $N(p) \perp S_p$.

$$(f \circ \alpha)'(t) = \nabla f(\alpha(t)) \cdot \dot{\alpha}(t) = \|\nabla f(\alpha(t))\| N(\alpha(t)) \cdot \dot{\alpha}(t) = 0$$

for all $t \in I$ so $f \circ \alpha$ is constant along α, and $f(\alpha(0)) = f(p) = c$ so $f(\alpha(t)) = c$ for all $t \in I$.
That is, Image $\alpha \subset f^{-1}(c) = S$. ■

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CHAPTER 6
PARALLEL TRANSPORT

Definition. A vector field X along a parametrized curve $\alpha : I \to S$ in an $n-$surface S is tangent to S along α if $X(t) \in S_{\alpha(t)}$ for all $t \in I$. The derivative $\dot{X}$ of such a vector field is, however, generally not tangent to S. We can, obtain a vector field tangent to S by projecting $\dot{X}(t)$ on to $S_{\alpha(t)}$ for each $t \in I$. This process of differentiating and then projecting onto the tangent space to S defines an operation with the same properties as differentiation, except that now differentiation or vector fields tangent to S yields vector fields tangent to S. This process is called covariant differentiation.

Let S be an $n-$surface in $\mathbb{R}^{n+1}$, let $\alpha : I \to S$ be a parametrized curve in S, and let X be a smooth vector field tangent to S along α. The covariant derivative of X is the vector field X' tangent to S along α defined by

$$X'(t) \quad = \quad \dot{X}(t) - \Big[\dot{X} \cdot N(\alpha(t))\Big] N(\alpha(t)),$$

where N is an orientation on S. Note that $X'(t)$ is independent of the choice of N since replacing N by $-N$ has no effect on the above formula.

Exercise. Verify that the following properties of covariant differentiation for smooth vector fields X and Y tangent to S along α,

(i) $(X + Y)' = X' + Y'$

(ii) $(fX)' = f'X + fX'$

(iii) $(X \cdot Y)' = X' \cdot Y + X \cdot Y'$

Covariant derivative X' measures the rate of change of X along α as seen from the surface S. Note that a parametrized curve $\alpha : I \to S$ is a geodesic in S if and only if its covariant acceleration $(\dot{\alpha}')$ is zero along α.

The covariant derivative leads naturally to a concept of parallelism on an $n-$surface. In $\mathbb{R}^{n+1}$, $v = (p, v) \in \mathbb{R}^{n+1}_p$ and $w = (q, w) \in \mathbb{R}^{n+1}_q$ are said to be Euclidean parallel if $v = w$. A vector field X along a parametrized curve $\alpha : I \to \mathbb{R}^{n+1}$ is Euclidean parallel if $X(t_1) = X(t_2)$ for all $t_1, t_2 \in I$, where $X(t) = (\alpha(t), X(t))$ for $t \in I$. Thus X is Euclidean parallel along α if and only if $\dot{X} = 0$.

Given an $n-$surface S in $\mathbb{R}^{n+1}$ and a parametrized curve $\alpha : I \to S$, a smooth vector field X tangent to S along α is said to be Levi-Civita parallel, or simply parallel, if $X' = 0$. Therefore, X is parallel along α if X is a constant vector field along α. Levi-Civita parallelism has following properties:

(i) If X is parallel along α, then X has constant length.

PROOF. Suppose X is parallel along α.

$$\begin{aligned} \frac{d}{dt}\|X\|^2 &= \frac{d}{dt}(X \cdot X) \\ 0 &= X'X + XX' \\ 0 &= 2X'X \end{aligned}$$

Therefore, length of X is constant along α.
(ii) If X and Y are two parallel vector field along α, then $X \cdot Y$ is constant.
PROOF. Since X and Y are parallel vector fields along $\alpha \Longrightarrow X' = 0, Y' = 0$.
Consider,

$$\begin{aligned}(X \cdot Y)' &= X \cdot Y' + X' \cdot Y \\ &= 0\end{aligned}$$

Therefore, $X \cdot Y$ is constant along α.
(iii) If X and Y are parallel along α, then the angle between $\cos^{-1}(X \cdot Y/\|X\|\|Y\|)$ between X and Y is constant along α.
PROOF. Since $X \cdot Y$, $\|X\|$ and $\|Y\|$ are each constant along α.
Therefore, $\cos^{-1}(X \cdot Y/\|X\|\|Y\|)$ is constant along α.
(iv) If X and Y are parallel along α then $X + Y$ and cX, for all $c \in \mathbb{R}$ are parallel along α.
(v) The velocity vector field along a parametrized curve α in S is parallel if and only if α is a geodesic.
Theorem. *Let S be an $n-$surface in $\mathbb{R}^{n+1}$, let $\alpha : I \to S$ be a parametrized curve in S, let $t_0 \in I$, and let $v \in S_{\alpha(t_0)}$. Then there exists a unique vector field V, tangent to S along α, which is parallel and has $V(t_0) = v$.*
PROOF. We require a vector field V tangent to S along α satisfying $V' = 0$. But

$$\begin{aligned}V' &= \dot{V} - (\dot{V} \cdot N \circ \alpha)N \circ \alpha \\ &= \dot{V} - \left[(V \cdot N \circ \alpha)' - V \cdot N\dot{\circ}\alpha\right] N \circ \alpha \\ &= \dot{V} + (V \cdot N\dot{\circ}\alpha)N \circ \alpha\end{aligned}$$

$V' = 0$ if and only if V satisfied the differential equation

$$\dot{V} + (V \cdot N\dot{\circ}\alpha)N \circ \alpha = 0$$

This is first order differential equation in V. If we write $V(t) = (\alpha(t), V_1(t), V_2(t), ..., V_{n+1}(t))$, the vector differential equation becomes the system of first order differential equations

$$\frac{dV_i}{dt} + \sum_{j=1}^{n+1}(N_j \circ \alpha)(N_j \circ \alpha)'V_j = 0$$

where $N_j (j \in \{1, 2, ..., n+1\})$ are the components of N. By existence and uniqueness theorem for solutions of first order differential equations, there exist a unique vector field V along α satisfying the above differential equation together with initial conditions $V(t_0) = v$ (that is, satisfying $V_i(t_0) = v_i$ for $i \in \{1, 2, ..., n+1\}$, where $v = (\alpha(t_0), v_1, v_2, ..., v_{n+1})$). The existence and uniqueness theorem does not guarantee that V is tangent to S along α.

To show V is tangent to S, from the above differential equation we have,

$$\begin{aligned}(V \cdot N \circ \alpha)' &= \dot{V} \cdot N \circ \alpha + V \cdot N \dot{\circ} \alpha \\ &= [-(V \cdot N \dot{\circ} \alpha) N \circ \alpha] \cdot N \circ \alpha + V \cdot N \dot{\circ} \alpha \\ &= -V \cdot N \dot{\circ} \alpha + V \cdot N \dot{\circ} \alpha \\ &= 0\end{aligned}$$

so $V \cdot N \circ \alpha$ is constant along α and, since $(V \cdot N \circ \alpha)(t_0) = v \cdot N(\alpha(t_0)) = 0$, this constant must be 0. Finally, this vector field V, tangent to S along α, is parallel because it satisfies above differential equation. ■

Corollary. *Let S be a 2-surface in $\mathbb{R}^3$ and let $\alpha : I \to S$ be a geodesic in S with $\dot{\alpha} \neq 0$. Then a vector field X tangent to S along α is parallel along α if and only if both $\|X\|$ and the angle between X along $\dot{\alpha}$ are constant along α.*

PROOF. Suppose a vector field X tangent to S is parallel along α then by property (i) $\|X\|$ is constant along α. Also we know that the geodesic has constant speed along α. That is $\|\dot{\alpha}\|$ is constant along α. Since the angle between X and $\dot{\alpha}$ is given by $\theta = \cos^{-1}\left(\frac{X \cdot \dot{\alpha}}{\|X\|\|\dot{\alpha}\|}\right)$ and $\|X\|, \|\dot{\alpha}\|$ and $X \cdot \dot{\alpha}$ are constant. Which implies θ is constant along α.

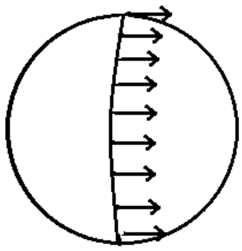
Conversely, Suppose $\|X\|$ and the angle θ between X and $\dot{\alpha}$ is constant along α. Let $t_0 \in I$ and let $v \in S_{\alpha(t_0)}$ be a unit vector orthogonal to $\dot{\alpha}(t_0)$. Let V be unique parallel vector field along α such that $V(t_0) = v$. Then $\|V\| = 1$ and $V \cdot \dot{\alpha} = 0$ along α so $\left\{\dot{\alpha(t)}, V(t)\right\}$ is and orthogonal basis for $S_{\alpha(t)}$, for each $t \in I$. In particular, there exist an smooth functions $f, g : I \to \mathbb{R}$ such that $X = f\dot{\alpha} + gV$. Since

$$\begin{aligned}\cos\theta &= \frac{X \cdot \dot{\alpha}}{\|X\|\|\dot{\alpha}\|} \\ &= \frac{f\dot{\alpha} \cdot \dot{\alpha} + gV \cdot \dot{\alpha}}{\|X\|\|\dot{\alpha}\|} \\ &= \frac{f\|\dot{\alpha}\|^2}{\|X\|\|\dot{\alpha}\|} \\ &= \frac{f\|\dot{\alpha}\|}{\|X\|}\end{aligned}$$

and

$$\|X\|^2 = f^2\|\dot{\alpha}\|^2 + g^2$$

the constancy of $\theta, \|X\|$ and $\|\dot{\alpha}\|$ along α implies that f and g are constant along α. Hence X is parallel along α, by property (iv) above. ■



Levi-Civita parallel vector fields along geodesic in the 2-sphere

Parallelism can be use to transport tangent vectors from one point of an n-surface to another. Given two points p and q in an n-surface S, a parametrized curve in S from p to q is a smooth map $\alpha : [a,b] \to S$, from a closed interval $[a,b]$ into S, with $\alpha(a) = p$ and $\alpha(b) = q$. By smoothness of a map α defined on a closed interval we mean that α is restriction to $[a,b]$ into S. Each parametrized curve $\alpha : [a,b] \to S$ from q to q determines a map $P_\alpha : S_p \to S_q$ by

$$P_\alpha(v) = V(b)$$

where, for $v \in S_p$, V is the unique parallel vector field along α with $V(a) = v$. $P_\alpha(v)$ is called the parallel transport of v along α to q.

Example. For $\theta \in \mathbb{R}$, let $\alpha_\theta : [0,\pi] \to S^2$ be the parametrized curve in the unit $2-$sphere S^2, from the north pole $p = (0,0,1)$ to the south pole $q = (0,0,-1)$, defined by

$$\alpha_\theta(t) = (\cos\theta \sin t, \sin\theta \sin t, \cos t)\,.$$

Thus, for each θ, α_θ is half of a great circle on S^2. Let $v = (p,1,0,0) \in S^2_p$. Since α_θ is geodesics in S^2, a vector field tangent to S^2 along α_θ will be parallel if and only if it has constant length and keeps constant angle with $\dot{\alpha}_\theta$. The one with initial value v is

$$V_\theta(t) = (\cos\theta)\dot{\alpha}_\theta(t) - (\sin\theta)N(\alpha_\theta(t)) \times \dot{\alpha}_\theta(t),$$

where N is the outward orientation on S^2. Hence

$$\begin{aligned} P_{\alpha_\theta}(v) &= V_\theta(\pi) \\ &= (\cos\theta)(q, -\cos\theta, -\sin\theta, 0) - (\sin\theta)(q, -\sin\theta, \cos\theta, 0) \\ &= -(q, \cos 2\theta, \sin 2\theta, 0) \end{aligned}$$

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CHAPTER 7

THE WEINGARTEN MAP

Definition. Given a smooth function f defined on an open set U in $\mathbb{R}^{n+1}$ and a vector $v \in \mathbb{R}_p^{n+1}, p \in U$, the derivative of f with respect to v is the real number

$$\nabla_v f = (f \circ \alpha)'(t_0)$$

where $\alpha : I \to U$ is any parametrized curve in U with $\dot{\alpha}(t_0) = v$. The value of $\nabla_v f$ does not depend on the choice of α.

$$\nabla_v f = (f \circ \alpha)'(t_0) = \nabla f(\alpha(t_0)) \cdot \dot{\alpha}(t_0) = \nabla f(p) \cdot v$$

This formula also shows that the function which sends v into $\nabla_v f$ is a linear map from $\mathbb{R}_p^{n+1}$ to $\mathbb{R}$,

$$\nabla_{v+w} f = \nabla_v f + \nabla_w f$$

and

$$\nabla_{cv} f = c \nabla_v f$$

for all $v, w \in \mathbb{R}_p^{n+1}$ and $c \in \mathbb{R}$.

Note that $\nabla_v f$ depends on the magnitude of v as well as direction of v. When $\|v\| = 1$, the derivative $\nabla_v f$ is called the directional derivative of f at p in the direction v.

Given an n-surface S in $\mathbb{R}^{n+1}$ and a smooth function $f : S \to \mathbb{R}$, it's derivative with respect to a v tangent to S is defined similarly, by

$$\nabla_v f = (f \circ \alpha)'(t_0)$$

where $\alpha : I \to S$ is any parametrized curve in S with $\dot{\alpha}(t_0) = v$. Note that the value of $\nabla_v f$ is independent of the curve α in S passing through p with velocity v, since

$$\nabla_v f = (\tilde{f} \circ \alpha)'(t_0) = \nabla \tilde{f}(\alpha(t_0)) \cdot \dot{\alpha}(t_0) = \nabla \tilde{f}(p) \cdot v$$

where $\tilde{f} : U \to \mathbb{R}$ is any smooth function, defined on an open set U containing S, whose restriction to S is f. It also follows from this last formula that the function which sends v in to $\nabla_v f$ is a linear map from S_p to $\mathbb{R}$.

The derivative of a smooth vector field X on an open set U in $\mathbb{R}^{n+1}$ with respect to a vector $v \in \mathbb{R}_p^{n+1}, p \in U$ is defined by

$$\nabla_v X = (X \dot{\circ} \alpha)(t_0)$$

where $\alpha : I \to U$ is any parametrized curve in U such that $\dot{\alpha}(t_0) = v$. For X a smooth vector field on an n-surface S in $\mathbb{R}^{n+1}$ and v a vector tangent to S at $p \in S$, the derivative $\nabla_v X$ is defined by the same formula, where now α is required to be a parametrized curve in S with $\dot{\alpha}(t_0) = v$. Note that, in both situations, $\nabla_v X \in \mathbb{R}^{n+1}$ and that

$$\begin{aligned}\nabla_v X &= (\alpha(t_0), (X_1 \circ \alpha)'(t_0), ..., (X_{n+1} \circ \alpha)'(t_0)) \\ &= (p, \nabla_v X_1, ..., \nabla_v X_{n+1})\end{aligned}$$

where X_i are the components of X. The differentiation of vector field satisfies following properties:
(i) $\nabla_v(X+Y) = \nabla_v X + \nabla_v Y$
(ii) $\nabla_v(fX) = (\nabla_v f)X(t) + f(p)(\nabla_v X)$
(iii) $\nabla_v(X \cdot Y) = (\nabla_v X) \cdot Y(p) + X(p) \cdot (\nabla_v Y)$
for all smooth vector fields X and Y on U(or on S) and all smooth functions $f : U \to \mathbb{R}$. The sum $X+Y$ of two vector fields X and Y is the vector field defined by $(X+Y)(q) = X(q)+Y(q)$, the product of a function f and a vector field X is the vector field defined by $(Xf)(q) = f(q)X(q)$ and the dot product of vector fields X and Y is the function defined by $(X \cdot Y)(q) = X(q) \cdot Y(q)$, for all $q \in U$. Moreover, for each vector field X, the function which sends v in to $\nabla_v X$ is a linear map, from $\mathbb{R}^{n+1}_p$ into $\mathbb{R}^{n+1}_p$ if X is vector field on an open set U, and from S_p into $\mathbb{R}^{n+1}_p$ if X is a vector field on an $n-$surface S.

The derivative $\nabla_v X$ of a tangent vector field X on an $n-$surface S with respect to a vector v tangent to S at $p \in S$ will not in general be tangent to S. Consider the tangential component of $D_v X$ of $\nabla_v X$:

$$D_v X = \nabla_v X - (\nabla_v X \cdot N(p))N(p),$$

where N is an orientation on S. $D_v X$ is called the covariant derivative of the tangent vector field X with respect to $v \in S_p$. Covariant derivative has the same properties as ordinary differentiation. For each smooth tangent vector field X on S, the function which sends v into $D_v X$ is a linear map from S_p into S_p.

Suppose N is a normal direction on an n-surface in $\mathbb{R}^{n+1}$. For $p \in S$ and $v \in S_p$, the derivative $\nabla_v N$ is tangent to S since.

$$\begin{aligned} 0 &= \nabla_v(1) \\ &= \nabla_v(N \cdot N) \\ &= (\nabla_v N) \cdot N(p) + N(p) \cdot (\nabla_v N) \\ &= 2(\nabla_v N) \cdot N(p) \end{aligned}$$

The linear map $L_p : S_p \to S_p$ defined by

$$L_p(v) = -\nabla_v N$$

is called Weingarten map of S at p. The geometric meaning of L_p seen from the formula

$$\nabla_v N = (N\dot{\circ}\alpha)(t_0)$$

where, $\alpha : I \to S$ is any parametrized curve in S with $\dot{\alpha}(t_0) = v$. $L_p(v)$ measures the rate of change of N as one passes through p along any such curve α. Since the tangent space $S_{\alpha(t)}$ to S at $\alpha(t)$ is just $[\nabla N(\alpha(t))]^{\perp}$, the tangent space turns as the normal N turns and so $L_p(v)$ can be interpreted as a measure of the turning of the tangent space as one passes through p along α. Thus L_p contains information about the shape of S. For this reason L_p is sometimes called the shape operator S at p.

$L_p(v)$ can be obtained from the formula

$$\begin{aligned} L_p(v) &= -\nabla_v N \\ &= -(p, \nabla_v N_1, \nabla_v N_2, ..., \nabla_v N_{n+1}) \\ &= -(p, \nabla N_1 \cdot v, \nabla N_2 \cdot v, ..., \nabla N_{n+1} \cdot v) \\ &= -(p, \nabla \tilde{N}_1 \cdot v, \nabla \tilde{N}_2 \cdot v, ..., \nabla \tilde{N}_{n+1} \cdot v), \end{aligned}$$

where $\tilde{N}$ is any smooth vector field defined on an open set U containing S with $\tilde{N}(q) = N(q)$ for all $q \in S$.

Example 1. Let S be the n-sphere $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = r^2$ of radius $r > 0$, oriented by the inward unit normal vector field N.

$$\begin{aligned} N(q) &= (q, -q/\|q\|) \\ &= (q, -q/r) \end{aligned}$$

for $q \in S$. Setting $\tilde{N}(q) = (q, -q/r)$ for $q \in \mathbb{R}^{n+1}$.

$$\begin{aligned} \tilde{N}(x_1, x_2, ..., x_{n+1}) &= \left(x_1, x_2, ..., x_{n+1}, -\frac{x_1}{r}, -\frac{x_2}{r}, ..., -\frac{x_{n+1}}{r}\right) \\ &= (q, -q/r) \end{aligned}$$

we have, for $p \in S$ and $v \in S_p$,

$$\begin{aligned} L_p(v) &= -\nabla_v N \\ &= -\left(p, \nabla_v N_1, \nabla_v N_2, ..., \nabla_v N_{n+1}\right) \\ &= -\left(p, \nabla_v \left(-\frac{x_1}{r}\right), \nabla_v \left(-\frac{x_2}{r}\right), ..., \nabla_v \left(-\frac{x_{n+1}}{r}\right)\right) \\ &= \frac{1}{r}\left(p, \nabla_v x_1, \nabla_v x_2, ..., \nabla_v x_{n+1}\right) \end{aligned}$$

But for each $i \in \{1, 2, ..., n+1\}$,

$$\begin{aligned} \nabla_v x_i &= \nabla x_i \cdot v \\ &= (p, 0, 0, .., 1, ..., 0) \cdot (p, v_1, v_2, ..., v_{n+1}) \\ &= v_i \end{aligned}$$

Therefore,

$$\begin{aligned} L_p(v) &= \frac{1}{r}\left(p, v_1, v_2, ..., v_{n+1}\right) \\ &= \frac{1}{r} v \end{aligned}$$

Therefore, the Weingarten map of the $n-$sphere of radius r is simply multiplication by $1/r$. If S is oriented by outward normal $-N$, the Weingartern map will be multiplication by $-1/r$.

Theorem 1. *Let S be an n-surface in $\mathbb{R}^{n+1}$, oriented by the unit normal vector field N. Let $p \in S$ and $v \in S_p$. Then for every parametrized curve $\alpha : I \to S$, with $\dot{\alpha}(t_0) = v$ for some $t_0 \in I$,*

$$\ddot{\alpha}(t_0) \cdot N(p) \quad = \quad L_p(v) \cdot v$$

PROOF. Since α is a parametrized curve in $S \Longrightarrow \dot{\alpha}(t) \in S_{\alpha(t)} = [N(\alpha(t))]^{\perp}, \quad \forall t \in I.$ That is, $\dot{\alpha} \cdot N(\alpha(t)) = 0$ along α. Hence

$$\begin{aligned}
0 &= [\dot{\alpha} \cdot (N \circ \alpha)]'(t_0) \\
&= \ddot{\alpha}(t_0) \cdot (N \circ \alpha)(t_0) + (\alpha)(t_0) \cdot \dot{(N \cdot \alpha)}(t_0) \\
&= \ddot{\alpha}(t_0) \cdot (N \circ \alpha)(t_0) + v \cdot \nabla_v N \\
&= \ddot{\alpha}(t_0) \cdot (N \circ \alpha)(t_0) - v \cdot L_p(v)
\end{aligned}$$

Therefore, $\ddot{\alpha}(t_0) \cdot (N \circ \alpha)(t_0) = L_p(v) \cdot v$. ■

Theorem 2. *The Weingarten map L_p is self-adjoint; that is,*

$$L_p(v) \cdot w = v \cdot L_p(w)$$

for all $v, w \in S_p$.

PROOF. Let $f : U \to \mathbb{R}(U$ open in $\mathbb{R}^{n+1})$ be such that $S = f^{-1}(c)$ for some $c \in \mathbb{R}$ such that $N(p) = \nabla f(p)/\|\nabla f(p)\|$ for all $p \in S$. Then

$$\begin{aligned}
L_p(v) \cdot w &= (-\nabla_v N) \cdot w \\
&= -\nabla_v \left(\frac{\nabla f}{\|\nabla f\|} \right) \cdot w \\
&= -\left[\nabla_v \left(\frac{1}{\|\nabla f\|} \right) \nabla f(p) + \left(\frac{1}{\|\nabla f\|} \right) \nabla_v (\nabla f) \right] \cdot w \\
&= -\nabla_v \left(\frac{1}{\|\nabla f\|} \right) \nabla f(p) \cdot w - \left(\frac{1}{\|\nabla f\|} \right) \nabla_v (\nabla f) \cdot w
\end{aligned}$$

Since $\nabla f(p) \cdot w = 0$, the first term drops out. Thus

$$\begin{aligned}
L_p(v) \cdot w &= -\frac{1}{\|\nabla f(p)\|} \nabla_v(\nabla f) \cdot w \\
&= -\frac{1}{\|\nabla f(p)\|} \nabla_v \left(p, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_{n+1}}\right) \cdot w \\
&= -\frac{1}{\|\nabla f(p)\|} \left(p, \nabla_v \frac{\partial f}{\partial x_1}, \nabla_v \frac{\partial f}{\partial x_2}, ..., \nabla_v \frac{\partial f}{\partial x_{n+1}}\right) \cdot w \\
&= -\frac{1}{\|\nabla f(p)\|} \left(p, \nabla \left(\frac{\partial f}{\partial x_1}\right)(p) \cdot v, \nabla \left(\frac{\partial f}{\partial x_2}\right)(p) \cdot v, ..., \nabla \left(\frac{\partial f}{\partial x_{n+1}}\right)(p) \cdot v\right) \cdot w \\
&= -\frac{1}{\|\nabla f(p)\|} \left(p, \sum_{i=1}^{n+1} \frac{\partial^2 f}{\partial x_i \partial x_1}(p) \cdot v_i, \sum_{i=1}^{n+1} \frac{\partial^2 f}{\partial x_i \partial x_2}(p) \cdot v_i, ..., \sum_{i=1}^{n+1} \frac{\partial^2 f}{\partial x_i \partial x_{n+1}}(p) \cdot v_i\right) \cdot w \\
&= -\frac{1}{\|\nabla f(p)\|} \sum_{i,j=1}^{n+1} \frac{\partial^2 f}{\partial x_i \partial x_j}(p) v_i w_j,
\end{aligned}$$

where $v = (v_1, v_2, ..., v_{n+1})$ and $w = (w_1, w_2, ..., w_{n+1})$. The same computation, with v and w interchanged, shows that

$$L_p(w) \cdot v = -\frac{1}{\|\nabla f(p)\|} \sum_{i,j=1}^{n+1} \frac{\partial^2 f}{\partial x_i \partial x_j}(p) w_i v_j,$$

Since $\dfrac{\partial^2 f}{\partial x_i \partial x_j} = \dfrac{\partial^2 f}{\partial x_j \partial x_i}$ for all $1 \leq i, j \leq n+1$.
Therefore,

$$\begin{aligned}
L_p(v) \cdot w &= -\frac{1}{\|\nabla f(p)\|} \sum_{i,j=1}^{n+1} \frac{\partial^2 f}{\partial x_i \partial x_j}(p) v_i w_j \\
&= -\frac{1}{\|\nabla f(p)\|} \sum_{i,j=1}^{n+1} \frac{\partial^2 f}{\partial x_j \partial x_i}(p) v_i w_j \\
&= -\frac{1}{\|\nabla f(p)\|} \sum_{i,j=1}^{n+1} \frac{\partial^2 f}{\partial x_j \partial x_i}(p) w_j v_i \\
&= L_p(w) \cdot v
\end{aligned}$$

which shows Weingarten map is self-adjoint. ■

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