Chapter 1 : Integration

Dr. A. N. Bhavale

Head, Department of Mathematics,
Modern College of Arts, Science and Commerce (Autonomous),
Shivajinagar, Pune-5.

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Integral (or Primitive) of a function :

If the differential coefficient of a function $f(x)$ is $F(x)$ then $f(x)$ is said to be an **Integral** or a **Primitive** of $F(x)$.

In symbols, we write it as follows.

If $\frac{df(x)}{dx} = F(x)$ then $\int F(x)dx = f(x)$.

The process of determining an integral of a function is called **Integration**, and the function to be integrated is called **Integrand**.

Examples :

1. $\int \cos x dx = \sin x$.
2. $\int \sin x dx = -\cos x$.
3. $\int 2x dx = x^2$.
4. $\int e^x dx = e^x$.
5. $\int \frac{1}{x} dx = \log |x|$.
6. $\int \sec^2 x dx = \tan x$.
7. $\int \csc^2 x dx = -\cot x$.
8. $\int \sec x \tan x dx = \sec x$.
9. $\int x^n dx = \frac{x^{n+1}}{n+1}$, where $n \neq -1$.
10. $\int a^x dx = \frac{a^x}{\log a}$, where $a > 0$ but $a \neq 1$.

Note that, if c is an arbitrary constant then $\frac{d}{dx}(f(x) + c) = F(x)$, and hence $\int F(x)dx = f(x) + c$, which is called **General Integral**.

Thus, it follows that integral of a function is not unique, and any two integrals of the same function differ by a constant.

11. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x = -\cos^{-1} x.$

12. $\int \frac{1}{1+x^2} dx = \tan^{-1} x = -\cot^{-1} x.$

13. $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x = -\csc^{-1} x.$

14. $\int \cosh x dx = \sinh x.$ 15. $\int \sinh x dx = \cosh x.$

Note that, $\cosh x = \frac{e^x+e^{-x}}{2}$ and $\sinh x = \frac{e^x-e^{-x}}{2}$.

Remark :

1. $\int af(x)dx = a\int f(x)dx$, where a is a constant.
2. $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx.$

If $f'(x) = F(x)$ then $\int_a^b F(x)dx = f(b) - f(a)$.
Geometrically, the definite integration represents the area under the curve $y = F(x)$, bounded by the lines $x = a, x = b$ and X-axis in the plane.

Remark :

1. $\int_a^b F(x)dx = -\int_b^a F(x)dx$.
2. $\int_a^b F(x)dx = \int_a^c F(x)dx + \int_c^b F(x)dx$.

1. Decomposition of the given integrand as a sum of integrands with known integrals.
2. Integration by substitution:
$\int f(x)dx = \int f(\phi(t))\phi'(t)dt.$
3. Integration by parts:
$\int f(x)g(x)dx =$
$f(x)\int g(x)dx - \int(\int g(x)dx)f'(x)dx.$
4. Integration by successive reduction.

Some important forms of integrals (using substitution) :

- $\int \frac{f'(x)}{f(x)} dx = \log f(x)$. It follows that
  1. $\int \tan x dx = \int \frac{\sec x \tan x}{\sec x} dx = \log \sec x$.
  2. $\int \cot x dx = \int \frac{\cos x}{\sin x} dx = \log \sin x$.
  3. $\int \sec x dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx$
     $= \log(\sec x + \tan x) = \log \tan(\frac{\pi}{4} + \frac{x}{2})$.
  4. $\int \csc x dx = \int \frac{\csc x(\csc x - \cot x)}{(\csc x - \cot x)} dx$
     $= \log(\csc x - \cot x) = \log \tan \frac{x}{2}$.
- $\int (f(x)^n) f'(x) dx = \frac{f(x)^{n+1}}{n+1}$, where $n \neq -1$.
- $\int f'(ax+b) dx = \frac{f(ax+b)}{a}$.

- $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \frac{x}{a}$
- $\int \frac{1}{\sqrt{a^2+x^2}} dx = \sinh^{-1} \frac{x}{a}$
- $\int \frac{1}{\sqrt{x^2-a^2}} dx = \cosh^{-1} \frac{x}{a}$
- $\int \sqrt{a^2 - x^2} dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$
- $\int \sqrt{a^2 + x^2} dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2} \sinh^{-1} \frac{x}{a}$
- $\int \sqrt{x^2 - a^2} dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2} \cosh^{-1} \frac{x}{a}$

Note that $1 + \sinh^2 \theta = \cosh^2 \theta$. Therefore $\sinh^{-1} \frac{x}{a} = \log \frac{x+\sqrt{x^2+a^2}}{a}$, $\cosh^{-1} \frac{x}{a} = \log \frac{x+\sqrt{x^2-a^2}}{a}$.

Some important forms of integrals (using integration by parts) :

- $\int e^x(f(x) + f'(x))dx = e^x f(x)$.
- $\int e^{ax} \cos(bx + c)dx = \frac{e^{ax}}{a^2+b^2}(a\cos(bx + c) + b\sin(bx + c))$.
- $\int e^{ax} \sin(bx + c)dx = \frac{e^{ax}}{a^2+b^2}(a\sin(bx + c) - b\cos(bx + c))$.
- $\int x^n e^{ax} dx = x^n \frac{e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx$.
- $\int x^m \sin nx dx = -\frac{x^m \cos nx}{n} - \frac{m}{n} \int x^{m-1} \cos nx dx$.

The last two are reduction formulae.

Let $f(x) = a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m$ and $g(x) = b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n$ be two polynomials. Then the ratio $\frac{f(x)}{g(x)}$ is called a **rational function**, provided $g(x) \neq 0$.
By division algorithm, $f(x) = g(x)q(x) + r(x)$, where either $r(x) \equiv 0$ or $deg.r(x) < deg.g(x)$.
Thus, $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$. Therefore
$\int \frac{f(x)}{g(x)}dx = \int q(x)dx + \int \frac{r(x)}{g(x)}dx$.

As $q(x)$ is a quotient polynomial, its integration $\int q(x)dx$ can be obtained term by term.

Also $g(x)$ can be written as,

$$g(x) = c(c_1x + d_1)^{p_1}(c_2x + d_2)^{p_2} \cdots$$
$$\times(e_1x^2 + f_1x + g_1)^{q_1}(e_2x^2 + f_2x + g_2)^{q_2} \cdots,$$

where $p_i \geq 1$ and $q_j \geq 1$.

Thus, the factors of $g(x)$ are of the four types, linear non-repeated, linear repeated, quadratic non-repeated, and quadratic repeated.

Hence, $\frac{r(x)}{g(x)}$ can be written as sum of partial fractions of the forms $\frac{k}{(ax+b)^r}$ or $\frac{kx+l}{(ax^2+bx+c)^r}$, $r \geq 1$. Therefore $\int \frac{r(x)}{g(x)}dx$ can be obtained by integrating the partial fractions of these types. Consider the following four cases depending on the kind of factors of $g(x)$ through some illustrations.

Illustration 1 : Evaluate $\int \frac{x+1}{x^2+5x+6} dx$.
Solution : Note that $x^2 + 5x + 6 = (x+2)(x+3)$.
$\therefore \frac{x+1}{x^2+5x+6} = \frac{x+1}{(x+2)(x+3)} = \frac{A}{(x+2)} + \frac{B}{(x+3)}$,
where A, B are constants to be determined.
Now $x + 1 = A(x+3) + B(x+2)$.
$\therefore A + B = 1$ and $3A + 2B = 1$. $\therefore A = -1, B = 2$.
$\therefore \int \frac{x+1}{x^2+5x+6} dx = \int \frac{x+1}{(x+2)(x+3)} dx =$
$\int \left(\frac{-1}{(x+2)} + \frac{2}{(x+3)} \right) dx = \int \frac{(-1)}{(x+2)} dx + \int \frac{2}{(x+3)} dx =$
$-\log(x+2) + 2\log(x+3) + c = \log \frac{(x+3)^2}{(x+2)} + c$,
where c is a constant of integration.

Illustration 2 : Evaluate $\int \frac{x^2+5x+41}{(x+3)(x-1)(2x-1)}dx$.

Solution : Let $\frac{x^2+5x+41}{(x+3)(x-1)(2x-1)} = \frac{A}{(x+3)} + \frac{B}{(x-1)} + \frac{C}{(2x-1)}$

where A, B, C are constants to be determined.

It follows that $A = 5/4, B = 47/4, C = -25.$

$\therefore \int \frac{x^2+5x+41}{(x+3)(x-1)(2x-1)}dx =$

$\int \left(\frac{(5/4)}{(x+3)} + \frac{(47/4)}{(x-1)} + \frac{(-25)}{(2x-1)} \right) dx$

$= (5/4)\log(x+3) + (47/4)\log(x-1) -$

$(25/2)\log(2x-1) + c,$

where c is a constant of integration.

Illustration 3 : Evaluate $\int \frac{x^2+x+1}{(x-1)^2(x-2)}dx$.
Solution : Consider
$\frac{x^2+x+1}{(x-1)^2(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-2)}$
where A, B, C are constants to be determined. Now
$x^2+x+1 = A(x-1)(x-2)+B(x-2)+C(x-1)^2$.
$\therefore A + C = 1$, $-3A + B - 2C = 1$ and
$2A - 2B + C = 1$. $\therefore A = -6, B = -3, C = 7$.
$\therefore \int \frac{x^2+x+1}{(x-1)^2(x-2)}dx = \int \left(\frac{(-6)}{(x-1)} + \frac{(-3)}{(x-1)^2} + \frac{7}{(x-2)} \right) dx =$
$-6\log(x-1) + 3(x-1)^{-1} + 7\log(x-2) + c$,
where c is a constant of integration.

Illustration 4 : Evaluate $\int \frac{x^2+1}{x^3+1} dx$.

Solution : Consider

$$\frac{x^2+1}{x^3+1} = \frac{x^2+1}{(x+1)(x^2-x+1)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2-x+1)}$$

where A, B, C are constants to be determined.

Now $x^2 + 1 = A(x^2 - x + 1) + (Bx + C)(x + 1)$.

$\therefore A + B = 1, -A + B + C = 0$ and $A + C = 1$.

$\therefore A = 2/3, B = 1/3, C = 1/3$.

$$\therefore \int \frac{x^2+1}{x^3+1} dx = \int \left(\frac{(2/3)}{(x+1)} + \frac{(1/3)x+(1/3)}{(x^2-x+1)} \right) dx$$

$$= (2/3)\log(x + 1) + (1/6)\log(x^2 - x + 1)$$

$+(1/\sqrt{3})\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + c$, where c is a constant.

Illustration 5 : Evaluate $\int \frac{2x-3}{(x^2+x+1)^2} dx$.

Solution : Expressing numerator in terms of the derivative of the denominator quadratic expression, we have $2x-3=(1)(2x+1)+(-4)$. Therefore

$$\int \frac{2x-3}{(x^2+x+1)^2} dx = \int \frac{(2x+1)}{(x^2+x+1)^2} dx + \int \frac{(-4)}{(x^2+x+1)^2} dx$$

$$= -\frac{1}{(x^2+x+1)} - 4 \int \frac{1}{(x^2+x+1)^2} dx + c, \text{ where}$$

$$\int \frac{1}{(x^2+x+1)^2} dx = \int \frac{1}{((x+(1/2))^2+(\sqrt{3}/2)^2)^2} dx =$$

$\frac{1}{3} \frac{2x+1}{x^2+x+1} + \frac{4}{3\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$, which follows from the

Reduction formula using integration by parts :

$$\int \frac{dy}{(y^2+k^2)^n} = \frac{y}{2(n-1)k^2(y^2+k^2)^{n-1}} + \frac{2n-3}{2(n-1)k^2} \int \frac{dy}{(y^2+k^2)^{n-1}}$$

Illustration 6 : Evaluate $\int \frac{2x^2-1}{(x^2-5)(x^2+4)} dx$.

Solution : In the integrand, replacing x^2 by t for the moment, we have

$$\frac{2x^2-1}{(x^2-5)(x^2+4)} = \frac{2t-1}{(t-5)(t+4)} = \frac{1}{t-5} + \frac{1}{t+4} = \frac{1}{x^2-5} + \frac{1}{x^2+4}$$

$$\therefore \int \frac{2x^2-1}{(x^2-5)(x^2+4)} dx = \int \frac{1}{x^2-5} dx + \int \frac{1}{x^2+4} dx$$

$$= \frac{1}{2\sqrt{5}} \log\left(\frac{x-\sqrt{5}}{x+\sqrt{5}}\right) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c,$$

where c is a constant of integration.

Show that

$$\int \frac{x^2}{(x^2+1)(3x^2+1)} dx = \frac{1}{2}\tan^{-1} x - \frac{1}{2\sqrt{3}}\tan^{-1}(\sqrt{3}x) + c.$$

Illustration 7 : Evaluate $\int \frac{\cos x}{(1+\sin x)(2+\sin x)} dx$.
Solution : Put $\sin x = t$. $\therefore \cos x dx = dt$. Hence
$\int \frac{\cos x}{(1+\sin x)(2+\sin x)} dx =$
$\int \frac{1}{(1+t)(2+t)} dt = \log\left(\frac{1+t}{2+t}\right) + c = \log\left(\frac{1+\sin x}{2+\sin x}\right) + c$.

Evaluate the following.

1. $\int \frac{1}{\sin x+\sin 2x} dx$. (Hint : Put $\cos x = t$)
2. $\int \frac{1}{e^x-1} dx$. (Hint : Put $e^x = t$)
3. $\int \frac{\log x}{x(1+\log x)(2+\log x)} dx$. (Hint : Put $\log x = t$)
4. $\int \frac{1}{x(x^2+1)^3} dx$. (Hint : Divide and multiply by x, and put $x^2 + 1 = t$)

Illustration 8 : Evaluate $\int \frac{x^2+1}{x^4+1} dx$.

Solution : Consider $\int \frac{x^2+1}{x^4+1} dx = \int \frac{1+(1/x^2)}{x^2+(1/x^2)} dx =$ $\int \frac{1+(1/x^2)}{(x-(1/x))^2+2} dx = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + c$, where c is a constant of integration. [Hint : Put $x-(1/x)=t$]

Evaluate the following.

1. $\int \frac{x^2-1}{x^4+1} dx$. [Hint : Put $x+(1/x)=t$]
2. $\int \frac{1}{x^4+1} dx$. [Hint : $\frac{1}{x^4+1} = \frac{1}{2}\frac{(x^2+1)-(x^2-1)}{x^4+1}$]
3. $\int \frac{x^2+1}{x^4-x^2+1} dx$. [Hint : Put $x-(1/x)=t$]
4. $\int \frac{x^2-1}{x^4-x^2+1} dx$. [Hint : Put $x+(1/x)=t$]

Type 1 : Evaluation of $\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$.

Hint : $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)}$.

Illustration 1 : Evaluate $\int \frac{3x+2}{\sqrt{2x^2+2x+1}}dx$.

Solution : Clearly $3x + 2 = (3/4)(4x + 2) + (1/2)$.

$\therefore \int \frac{3x+2}{\sqrt{2x^2+2x+1}}dx =$

$\frac{3}{4}\int \frac{4x+2}{\sqrt{2x^2+2x+1}}dx + \frac{1}{2}\int \frac{1}{\sqrt{2x^2+2x+1}}dx =$

$\frac{3}{2}\sqrt{2x^2 + 2x + 1} + \frac{1}{2\sqrt{2}} \log\left((x + \frac{1}{2}) + \sqrt{x^2 + x + \frac{1}{2}}\right)$

$+ c$, where c is a constant of integration.

Type 2 : Evaluation of $\int (Ax+B)\sqrt{ax^2+bx+c}\,dx$.

Illustration 1 : Evaluate $\int (2x-5)\sqrt{2+3x-x^2}dx$.

Solution : Clearly $2x-5=(-1)(3-2x)+(-2)$.

$$\therefore \int (2x-5)\sqrt{2+3x-x^2}dx =$$

$$-\int (3-2x)\sqrt{2+3x-x^2}dx - 2\int \sqrt{2+3x-x^2}dx$$

$$= -\frac{(2+3x-x^2)^{3/2}}{3/2} - 2\int \sqrt{2+\frac{9}{4}-\frac{9}{4}+3x-x^2}dx$$

$$= -\frac{2}{3}(2+3x-x^2)^{3/2} - 2\int \sqrt{\frac{17}{4}-(x-\frac{3}{2})^2}dx$$

$$= -\frac{2}{3}(2+3x-x^2)^{3/2}$$

$$-2\left[\frac{(x-\frac{3}{2})\sqrt{2+3x-x^2}}{2} + \frac{(17/4)}{2}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\sqrt{17}/2}\right)\right] + c.$$

Type 3 : Evaluation of $\int \frac{1}{(Ax+B)\sqrt{ax^2+bx+c}}dx$.

Hint : Put $Ax+B=\frac{1}{t}$.
$\therefore x=\frac{1}{A}(\frac{1}{t}-B)$ and $dx=-\frac{1}{At^2}dt$.

Illustration 1 : Evaluate $I=\int \frac{1}{(x+1)\sqrt{2x^2+3x+4}}dx$.

Solution : Put $x+1=\frac{1}{t}$ $\quad \therefore t=\frac{1}{x+1}$

$\therefore x=\frac{1}{t}-1$ and $dx=-\frac{1}{t^2}dt$.

$\therefore 2x^2+3x+4=2(\frac{1}{t}-1)^2+3(\frac{1}{t}-1)+4$

$=\frac{2(1-t)^2+3t(1-t)+4t^2}{t^2}=\frac{3t^2-t+2}{t^2}$

$\therefore I=-\int \frac{dt}{\sqrt{3t^2-t+2}}=-\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{(t-\frac{1}{6})^2+(\frac{\sqrt{23}}{6})^2}}$

$\therefore I=-\frac{1}{\sqrt{3}}\sinh^{-1}\left(\frac{t-\frac{1}{6}}{\frac{\sqrt{23}}{6}}\right)=-\frac{1}{\sqrt{3}}\sinh^{-1}\left(\frac{5-x}{\sqrt{23}(x+1)}\right)$.

Type 4 : Evaluation of $\int (ax+b)^{1/n} dx$.

Hint : Put $ax+b=t^n$.

$\therefore x=\frac{t^n-b}{a}$ and $dx=\frac{nt^{n-1}}{a}dt$.

Illustration 1 : Evaluate $I=\int \frac{x}{(2x+3)^{1/3}}dx$.

Solution : Put $2x+3=t^3$. $\quad \therefore dx=\frac{3t^2}{2}dt$.

$\therefore I=\int \frac{(t^3-3)}{2t}\frac{3t^2}{2}dt=\frac{3}{4}\int (t^4-3t)dt=\frac{3}{4}(\frac{t^5}{5}-\frac{3t^2}{2})$.

Thus $I=\frac{3t^5}{20}-\frac{9t^2}{8}=\frac{3t^5}{20}-\frac{9t^2}{8}=t^2(\frac{3t^3}{20}-\frac{9}{8})$.

Hence $I=(2x+3)^{2/3}\left(\frac{3(2x+3)}{20}-\frac{9}{8}\right)+c$.

That is, $I=(2x+3)^{2/3}\left(\frac{3x}{10}-\frac{27}{40}\right)+c$,

where c is a constant of integration.

Thank you