Numerical Analysis

By Krishna Shinde
Department of Mathematics
Modern College of Arts, Science and Commerce(Autonomous)
Shivajinagar, Pune - 5

CHAPTER 1

Root Finding Methods

CONVERGENCE

The sequence $\{x_n\}$ is said to be converges to the value L if

$$\lim_{n\to\infty} x_n = L$$

or, equivalently,

$$\lim_{n\to\infty} |x_n - L| = 0$$

Rate of Convergence

Definition. Let $\{p_n\}$ be a sequence that converges to a number p. If there exists a sequence $\{\beta_n\}$ that converges to zero and a positive constant λ independent of n such that

$$|p_n - p| \leq \lambda|\beta_n|$$

for all sufficiently large values of n, then $\{p_n\}$ is said to be converges to p with rate of convergence $O(\beta_n)$.

The expression $O(\beta_n)$ is read as "big-O of β_n".

Comparing Rate of Convergence

Example. Consider the sequences $\left\{\dfrac{n+3}{n+7}\right\}$ and $\left\{\dfrac{2^n+3}{2^n+7}\right\}$

Numerical verification of rate of convergence

n	$\frac{n+3}{n+7}$	$\frac{2^n+3}{2^n+7}$
1	0.500000	0.555556
2	0.555556	0.636364
3	0.600000	0.733333
4	0.636364	0.826087
5	0.666667	0.897435
6	0.692308	0.943662
7	0.714286	0.970370
8	0.733333	0.984791
9	0.750000	0.992293

Since

$$\lim_{n\to\infty}\frac{n+3}{n+7}=1 \text{ and } \lim_{n\to\infty}\frac{2^n+3}{2^n+7}=1,$$

it follows that both the sequences converges to the limit 1. From the above table we see that the terms in the sequence $\left\{\frac{2^n+3}{2^n+7}\right\}$ converges to 1 more rapidly than the terms in the sequence $\left\{\frac{n+3}{n+7}\right\}$.

Algebraic verification of rate of convergence

Consider,

$$\left|\frac{n+3}{n+7}-1\right|=\left|\frac{n+3-n-7}{n+7}\right|=\left|\frac{-4}{n+7}\right|<4\cdot\frac{1}{n}.$$

Hence, we may choose $\lambda=4$ and $\beta_n=\frac{1}{n}$ in the definition of rate of convergence. Which gives the rate of convergence of $\left\{\frac{n+3}{n+7}\right\}$ is $O\left(\frac{1}{n}\right)$ or we can say the sequence $\left\{\frac{n+3}{n+7}\right\}$ converges to 1 with the rate of convergence $O\left(\frac{1}{n}\right)$. Similarly,

$$\left|\frac{2^n+3}{2^n+7}-1\right|=\left|\frac{2^n+3-2^n-7}{2^n+7}\right|=\left|\frac{-4}{2^n+7}\right|<4\cdot\frac{1}{2^n}.$$

Which follows that the sequence $\left\{\frac{2^n+3}{2^n+7}\right\}$ converges to 1 with the rate of convergence $O\left(\frac{1}{2^n}\right)$. These results confirm our numerical evidence because $\frac{1}{2^n}$ approaches to zero faster than $\frac{1}{n}$ as $n\to\infty$.

Taylor's Theorem

Suppose f is continuous on $[a,b]$, has n continuous derivatives on (a,b) and $f^{(n+1)}$ exists on $[a,b]$. Let $x_0\in[a,b]$. For every $x\in[a,b]$ there exists a number $\xi(x)$ between x and x_0 such that

$$f(x) \;=\; P_n(x)+R_n(x)$$

where, $P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$ and $R_n(x)=\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1}$.

Here, $P_n(x)$ is called the n^{th} degree Taylor's polynomial and $R_n(x)$ is called as remainder term associated with $P_n(x)$.

Definition. Let f be a function defined on the interval (a,b) that contains $x=0$, and suppose $\lim_{x\to0}f(x)=L$. If there exists a function g for which $\lim_{x\to0}g(x)=0$ and a positive constant K such that

$$|f(x)-L| \;\leq\; K|g(x)|$$

for all sufficiently small values of x, then $f(x)$ is said to be converges to L with rate of convergence $O(g(x))$.

Example 1. Find the rate of convergence of the functions $f(x) = \dfrac{\cos x - 1 + \frac{1}{2}x^2}{x^4}$.
Solution.
Consider the 5^{th} order Taylor's polynomial of $\cos x$,
That is,

$$\cos x = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{4!}x^4 - \tfrac{1}{6!}x^6 \cos \xi,$$

for some ξ between 0 and x, Hence

$$\frac{\cos x - 1 + \dfrac{1}{2}x^2}{x^4} = \frac{1}{24} - \frac{1}{720}x^2 \cos \xi$$

$$\implies \frac{\cos x - 1 + \dfrac{1}{2}x^2}{x^4} - \frac{1}{24} = -\frac{1}{720}x^2 \cos \xi$$

Taking modulus on both side we get,

$$\begin{aligned} \left| \frac{\cos x - 1 + \dfrac{1}{2}x^2}{x^4} - \frac{1}{24} \right| &= \left| -\frac{1}{720}x^2 \cos \xi \right| \\ &= \frac{1}{720}|x^2 \cos \xi| \\ &\leq \frac{1}{720}|x^2| \end{aligned}$$

Therefore, $\lim_{x \to 0} f(x) = \dfrac{1}{24}$ and rate of convergence is $O(x^2)$.
Example 2. Find the rate of convergence of function $f(x) = \dfrac{e^x - \cos x - x}{x^2}$.
Solution.
The Taylor's polynomial of e^x and $\cos x$ are,

$e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3!}e^{\xi}$

$\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}\cos \xi$, for some ξ between 0 and x. Hence,

$$\begin{aligned}
e^x - \cos x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}e^{\xi} - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\cos\xi\right) \\
&= x + 2\cdot\frac{x^2}{2} + \frac{x^3}{3!}e^{\xi} - \frac{x^4}{4!}\cos\xi \\
\implies e^x - \cos x - x &= x^2 + \frac{x^3}{3!}e^{\xi} - \frac{x^4}{4!}\cos\xi \\
\implies \frac{e^x - \cos x - x}{x^2} &= 1 + \frac{x}{3!}e^{\xi} - \frac{x^2}{4!}\cos\xi
\end{aligned}$$

Taking modulus on both side we get,

$$\begin{aligned}
\left|\frac{e^x - \cos x - x}{x^2} - 1\right| &= \left|\frac{x}{3!}e^{\xi} - \frac{x^2}{4!}\cos\xi\right| \\
&\leq \frac{1}{4!}\left|4x - x^2\right|
\end{aligned}$$

Therefore, $\lim_{x\to 0} \frac{e^x - \cos x - x}{x^2} = 1$ and rate of convergence is $O\left(4x - x^2\right)$.

Definition. A fixed point of the function g is any real number, p, for which $g(p) = p$.
Example. Find the fixed point of function defined by $g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$.
Solution. Suppose p is fixed point of $g(x)$.

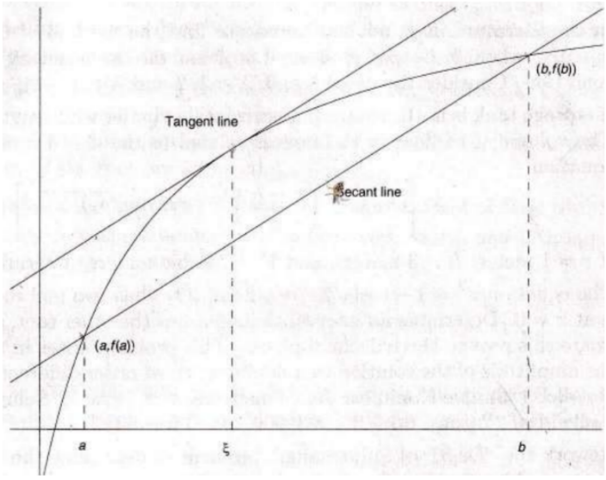
$$\begin{aligned}
\implies g(p) &= p \\
\implies \frac{1}{2}\left(p + \frac{a}{p}\right) &= p \\
\implies p + \frac{a}{p} &= 2p \\
\implies \frac{p^2 + a}{p} &= 2p \\
\implies p^2 + a &= 2p^2 \\
\implies p^2 &= a \\
\implies p &= \sqrt{a}
\end{aligned}$$

Therefore, $p = \sqrt{a}$ is fixed point of the function $g(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$.
Mean value theorem. If the function f is continuous on the closed interval $[a, b]$ and

differentiable on the open interval (a, b), then there exists a real number $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

PROOF. The quantity $\dfrac{f(b) - f(a)}{b - a}$ is slope of line passing through two points $(a, f(a))$ and $(b, f(b))$. The line passes through two point is called secant line.



Also, $f'(\xi)$ gives slope of tangent line to the graph f at $x = \xi$. Since f is continuously differentiable function hence we will get at least one point on the graph of function at which the tangent line which is parallel to secant line, which is our required point $x = \xi$.
Therefore, $f'(\xi) = \dfrac{f(b) - f(a)}{b - a}$. ■

Note: We will use a slightly different formulation of LMVT. Taking the equation

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

and multiplying both side by $b - a$ we get,

$$f(b) - f(a) = f'(\xi)(b - a).$$

Theorem 1. *Let g be continuous on the closed interval $[a, b]$ with $g : [a, b] \to [a, b]$. Then g has a fixed point $p \in [a, b]$. Furthermore, if g is differentiable on the open interval (a, b) and there exists a positive constant $k < 1$ such that $|g'(x)| \leq k < 1$ for all $x \in (a, b)$, then the fixed point in $[a, b]$ is unique.*

PROOF. (1) Existence
Assume that $g : [a, b] \to [a, b]$ is continuous function. Define a auxiliary function $h(x) = g(x) - x$. Since h is difference of two continuous functions on $[a, b]$ hence it is continuous on $[a, b]$. Also, the roots of $h(x)$ are precisely fixed points of g.
Since, $\min\limits_{x\in[a,b]} g(x) \geq a$ and $\max\limits_{x\in[a,b]} g(x) \leq b$, it follows that

$$h(a) = g(a) - a \geq 0 \text{ and } h(b) = g(b) - b \leq 0.$$

If either $h(a) = 0 \Longrightarrow g(a) = a$ or $h(b) = 0 \Longrightarrow g(b) = b$, then we are through. If neither $h(a) = 0$ nor $h(b) = 0$, then $h(b) < 0 < h(a)$. Therefore, by intermediate value theorem there exists $p \in [a, b]$ such that $h(p) = 0$, which implies $g(p) = p$.
(2) Uniqueness
Suppose p and q are two fixed points of function on the interval $[a, b]$, with $p \neq q$. Therefore, by definition of fixed point, $g(p) = p$ and $g(q) = q$. Then

$$\begin{aligned} |p - q| &= |g(p) - g(q)| \\ &= |g'(\xi)(p - q)| \qquad \text{by Mean Value Theorem} \\ &= |g'(\xi)||p - q| \\ &\leq k|p - q| \\ &< |p - q|, \end{aligned}$$

which is a contradiction. Hence $p = q$ and fixed point is unique. ■

Order of Convergence

Definition. Let $\{p_n\}$ be a sequence that converges to a number p. Let $e_n = p_n - p$ for all $n \geq 0$. If there exist positive constant λ and α such that

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^\alpha} = \lambda,$$

then p_n is said to be converges to p with order of convergence α and asymptotic error constant λ. ■

Fixed point iteration scheme. To approximate the fixed point, p, of a function g, generates the sequence $\{p_n\}$ by the rule $p_n = g(p_{n-1})$ for all $n \geq 1$, given a starting point p_0.

Example. Find the fixed point of function $g(x) = e^{-x}$ using fixed point iteration scheme by using starting point $p_0 = 0$.

Solution. Generate the sequence $\{p_n\}$ using fixed point iteration scheme $g(p_n) = p_{n+1}$.

$$\begin{aligned} p_1 &= g(p_0) = 1.0000000000 \\ p_2 &= g(p_1) = 0.3678794412 \\ p_3 &= g(p_2) = 0.6922006276 \\ p_4 &= g(p_3) = 0.5004735006 \\ p_5 &= g(p_4) = 0.6062435351 \\ p_6 &= g(p_5) = 0.5453957860 \\ p_7 &= g(p_6) = 0.5796123355 \\ p_8 &= g(p_7) = 0.5601154614 \\ p_9 &= g(p_8) = 0.5711431151 \\ p_{10} &= g(p_9) = 0.5648793474 \end{aligned}$$

The sequence generated by this scheme converges to fixed point very slowly. It takes more than 20 iterations for p_n to agree with the exact fixed point to at least 5 significant decimal digits.

Example. Find the order of convergence of following recursive formula.

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

Solution. We have given,

$$\begin{aligned} x_{n+1} &= \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \\ \Longrightarrow g(x_n) &= \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \end{aligned}$$

Also, we know that $p = \sqrt{a}$ is fixed point of $g(x_n)$.
Now to find the order of convergence consider,

$$\begin{aligned} x_{n+1} - \sqrt{a} &= \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) - \sqrt{a} \\ &= \frac{x_n^2 + a}{2x_n} - \sqrt{a} \\ &= \frac{x_n^2 + a - 2x_n\sqrt{a}}{2x_n} \\ &= \frac{(x_n - \sqrt{a})^2}{2x_n} \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \frac{|x_{n+1} - \sqrt{a}|}{|x_n - \sqrt{a}|} = \lim_{n\to\infty} \frac{1}{2x_n} = \frac{1}{2\sqrt{a}}.$$

Hence, the sequence generated by this scheme has order of convergence 2 and asymptotic error constant $1/(2\sqrt{a})$.
Theorem 2. *Let g be continuous on the closed interval $[a, b]$ with $g : [a, b] \to [a, b]$. Furthermore, suppose g is differentiable on the open interval (a, b) and there exists a positive constant $k < 1$ such that $|g'(x)| \leq k < 1$ for all $x \in (a, b)$. Then*
(1) *The sequence $\{p_n\}$ generated by $p_n = g(p_{n-1})$ converges to p for any $p_0 \in [a, b]$.*
(2) $|p_n - p| \leq k^n \max(p_0 - a, b - p_0)$; *and,*
(3) $|p_n - p| \leq \dfrac{k^n}{1-k}|p_1 - p_0|$.
PROOF. (1) Since $g : [a, b] \to [a, b]$ is continuous function defined on $[a, b]$ and, hence it has fixed point $p \in [a, b]$.
Now we have to show the sequence $\{p_n\}$ generated by $p_n = g(p_{n-1})$ converges to p.

Consider,

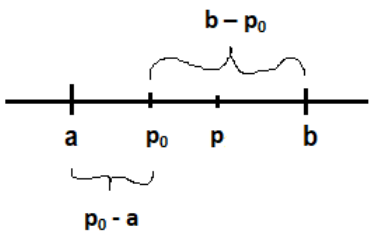
$$\begin{aligned}
|p_n - p| &= |g(p_{n-1}) - g(p)| && \text{By definition of } p_n \text{ and } p \\
&= |g'(\xi)(p_{n-1} - p)| && \text{By Mean Value Theorem} \\
&\leq k|p_{n-1} - p| && \text{By hypothesis on } g' \\
&\leq k|g(p_{n-2}) - g(p)| && \text{By definition of } p_n \text{ and } p \\
&\leq k|g'(\xi)(p_{n-2} - p)| && \text{By Mean Value Theorem} \\
&\leq k \cdot k|p_{n-2} - p| \\
&\leq k^2|p_{n-2} - p| \\
&. \\
&. \\
&. \\
&\leq k^n|p_0 - p|.
\end{aligned}$$

Therefore, $|p_n - p| \leq k^n|p_0 - p|$

Taking limit as $n \to \infty$ on both side we get,

$$\lim_{n\to\infty} |p_n - p| \leq \lim_{n\to\infty} k^n|p_0 - p| = |p_0 - p| \lim_{n\to\infty} k^n = 0 \qquad \because k < 1 \implies \lim_{n\to\infty} k^n = 0$$

(2) By part (1) we have,
$|p_n - p| \leq k^n|p_0 - p|$



From the figure we see that $|p_0 - p| \leq \max(p_0 - a, b - p_0)$.
$\implies |p_n - p| \leq k^n \max(p_0 - a, b - p_0)$.

(3) Consider,

$$
\begin{aligned}
|p_{n+1} - p_n| &= |g(p_n) - g(p_{n-1})| && \text{By definition of } p_n \\
&= |g'(\xi)(p_n - p_{n-1})| && \text{By Mean Value Theorem} \\
&\leq k|p_n - p_{n-1}| && \text{By hypothesis on } g' \\
&\leq k|g(p_{n-1}) - g(p_{n-2})| && \text{By definition of } p_n \\
&\leq k|g'(\xi)(p_{n-1} - p_{n-2})| && \text{By Mean Value Theorem} \\
&\leq k \cdot k|p_{n-1} - p_{n-2}| \\
&\leq k^2|p_{n-1} - p_{n-2}| \\
&. \\
&. \\
&. \\
&\leq k^n|p_1 - p_0|.
\end{aligned}
$$

Therefore, if $m > n$, then

$$
\begin{aligned}
|p_m - p_n| &= |p_m - p_{m-1} + p_{m-1} - p_{m-2} + ... + p_{n+1} - p_n| \\
&\leq |p_m - p_{m-1}| + |p_{m-1} - p_{m-2}| + ... + |p_{n+1} - p_n| \\
&\leq k^{m-1}|p_1 - p_0| + k^{m-2}|p_1 - p_0| + ... + k^n|p_1 - p_0| \\
&\leq k^n|p_1 - p_0|(1 + k + k^2 + ... + k^{m-n-1})
\end{aligned}
$$

As $m \to \infty, p_m \to p$.

$$
\begin{aligned}
|p - p_n| = \lim_{n\to\infty} |p_m - p_n| &\leq k^n|p_1 - p_0|(1 + k + k^2 + ...) \\
&\leq \frac{k^n}{1-k}|p_1 - p_0|
\end{aligned}
$$

$$
\implies |p - p_n| \leq \frac{k^n}{1-k}|p_1 - p_0|
$$

■

Theorem 3. *Let g be a continuous function on the closed interval $[a, b]$ with*

$g : [a, b] \to [a, b]$ and suppose that g' is continuous on the open interval (a, b) with $|g'(x)| \leq k < 1$ for all x in (a, b). If $g'(p) \neq 0$, then for any $p_0 \in [a, b]$, the sequence $p_n = g(p_{n-1})$ converges only linearly to fixed point p.

PROOF. Since $g : [a, b] \to [a, b]$ be a continuous function and g' be a continuous on the open interval (a, b) with $|g'(x)| \leq k < 1, \quad \forall x \in (a, b)$.

Therefore, g has a fixed point p in $[a, b]$.

Now it is sufficient to show that for any starting value $p_0 \in [a, b]$ and $g'(p) \neq 0$, the sequence generated by $p_n = g(p_{n-1})$ converges linearly to p.

That is to show, if $g'(p) \neq 0$, then $\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lambda \in (0, 1)$ for some λ.

Consider,

$$\begin{aligned}
|p_{n+1} - p| &= |g(p_n) - g(p)| \\
&= |g'(\xi)(p_n - p)| \qquad \text{By Mean Value Theorem} \\
&= |g'(c_n)||p_n - p| \\
\frac{|p_{n+1} - p|}{|p_n - p|} &= |g'(c_n)|
\end{aligned}$$

where c_n is between p_n and p.

Since $p_n \to p$ it follows by squeeze theorem that $c_n \to p$. Further g' is continuous on (a, b).

$$\begin{aligned}
\lim_{n\to\infty} |g'(c_n)| &= |g'(\lim_{n\to\infty} c_n)| \\
&= |g'(p)|
\end{aligned}$$

Hence, $\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = |g'(p)|$.

$\Longrightarrow$ the sequence p_n generated by $p_n = g(p_{n-1})$ converges to p linearly with asymptotic error constant $|g'(p)|$. ■

Theorem 4. *Let g be a continuous function on the closed interval $[a, b]$ with $\alpha > 1$ continuous derivatives on the open interval (a, b). Further, let $p \in (a, b)$ be fixed point of g. If*

$$g'(p) = g''(p) = \cdots = g^{(\alpha-1)}(p) = 0,$$

but $g^{(\alpha)}(p) \neq 0$, then there exists a $\delta > 0$ such that for any $p_0 \in [p-\delta, p+\delta]$, the sequence $p_n = g(p_{n-1})$ converges to the fixed point p of order α with asymptotic error constant

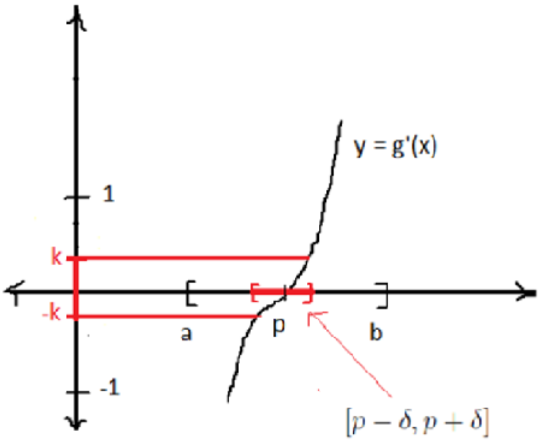
$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^\alpha} = \frac{|g^{(\alpha)}(p)|}{\alpha!}.$$

PROOF. To show: The existence of δ and the sequence $\{p_n\}$ generated by fixed point iteration scheme $g(p_n) = p_{n-1}$ converges to fixed point p.

Let $k < 1$ and $p_0 \in [p - \delta, p + \delta]$ be any real number.

Since $g'(p) = 0$ and g' is continuous.

$\Longrightarrow \exists \delta > 0$ such that $|g'(x)| \leq k < 1$ for all $x \in I = [p-\delta, p+\delta]$.



If $x \in I$ then,

$$\begin{aligned} |g(x) - p| &= |g(x) - g(p)| \\ &= |g'(\xi)(x-p)| \\ &= |g'(\xi)||x-p| \\ &\leq k|x-p| \\ &< |x-p| \\ &< \delta. \end{aligned}$$

Therefore, $g : I \to I$ is continuous mapping and $|g'(x)| \leq k < 1, \quad \forall x \in I$.
By theorem 2, the sequence generated by fixed point iteration scheme $g(p_n) = p_{n+1}$ converges to fixed point p for any $p_0 \in [p-\delta, p+\delta]$.
Now to establish the order of convergence, let $x \in I$ and expand the iteration function g into Taylor series about $x = p$:

$$g(x) = g(p) + g'(p)(x-p) + ... + \frac{g^{(\alpha-1)}(p)}{(\alpha-1)!}(x-p)^{(\alpha-1)} + \frac{g^{(\alpha)}(\xi)}{\alpha!}(x-p)^{\alpha},$$

where ξ is a number between x and p.
Since by hypothesis $g'(p) = g''(p) = \cdots = g^{(\alpha-1)}(p) = 0$.

$$\begin{aligned}
\implies g(x) &= g(p) + \frac{g^{(\alpha)}(\xi)}{\alpha!}(x-p)^{\alpha} \\
\implies g(p_n) &= p + \frac{g^{(\alpha)}(\xi)}{\alpha!}(p_n-p)^{\alpha} \\
\implies p_{n+1} - p &= \frac{g^{(\alpha)}(\xi)}{\alpha!}(p_n-p)^{\alpha} \\
\implies \frac{p_{n+1}-p}{(p_n-p)^{\alpha}} &= \frac{g^{(\alpha)}(\xi)}{\alpha!}
\end{aligned}$$

where ξ is a number between p_n and p.
Therefore, if $n \to \infty$, then $p_n \to p$ and $\xi \to p$.
Hence,

$$\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|^{\alpha}} = \frac{|g^{(\alpha)}(p)|}{\alpha!}$$

Therefore, the sequence $\{p_n\} \to p$ of order α. ■

STOPPING CONDITION

Suppose the sequence $\{p_n\}$ is generated by the fixed point iteration scheme $p_n = g(p_{n-1})$. Further, suppose that the sequence converges linearly to fixed point p.

(a) Show that

$$g'(p) \approx \frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}}.$$

(b) Show that

$$|e_n| \approx \left|\frac{g'(p)}{g'(p)-1}\right| |p_n - p_{n-1}|.$$

PROOF. (a) Consider,

$$\begin{aligned}
p_n - p_{n-1} &= g(p_{n-1}) - g(p_{n-2}) \\
&= g'(c_n)(p_{n-1} - p_{n-2}) \quad \because \text{by MVT}
\end{aligned}$$

As $n \to \infty, c_n \to p$.

$$g'(p) = \lim_{n\to\infty} \frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}}$$

Therefore,

$$g'(p) \approx \frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}}$$

(b) We know that

$$\begin{aligned}
e_n &= p_n - p \\
&= p_n - p_{n-1} + p_{n-1} - p \\
&= p_n - p_{n-1} + e_{n-1} \qquad (1)
\end{aligned}$$

Also, by linear convergence theorem we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{|e_n|}{|e_{n-1}|} &= |g'(x)| \\
\implies |e_{n-1}| &\approx \frac{|e_n|}{|g'(p)|} \qquad (2)
\end{aligned}$$

From equation (1) and (2) we get,

$$\begin{aligned}
e_n &\approx p_n - p_{n-1} + \frac{e_n}{g'(p)} \\
\implies e_n - \frac{e_n}{g'(p)} &\approx p_n - p_{n-1} \\
\implies e_n\left(1 - \frac{1}{g'(p)}\right) &\approx p_n - p_{n-1} \\
\implies e_n\left(\frac{g'(p)-1}{g'(p)}\right) &\approx p_n - p_{n-1} \\
\implies e_n &\approx \left(\frac{g'(p)}{g'(p)-1}\right)(p_n - p_{n-1})
\end{aligned}$$

Therefore,

$$|e_n| \approx \left|\frac{g'(p)}{g'(p)-1}\right| |p_n - p_{n-1}|$$

NEWTON'S METHOD

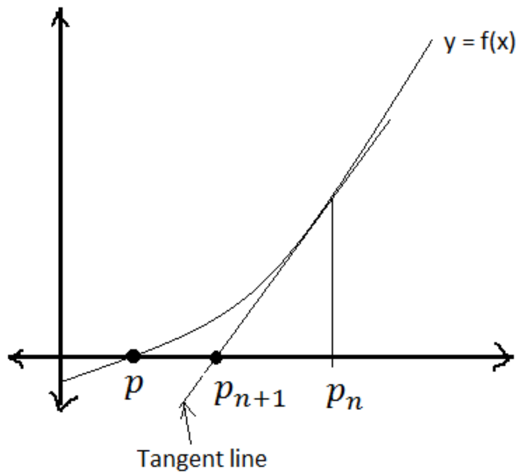
Let p_n denote most recent approximation to the root p of a function $y = f(x)$. Draw the tangent line at the point $(p_n, f(p_n))$ and take the $x-$intercept of tangent line as the next approximation, p_{n+1} to the root. The equation of tangent line at $x = p_n$ is given by

$$y - f(p_n) = f'(p_n)(x - p_n)$$

Since the point $(p_{n+1}, 0)$ lies on the tangent line hence it satisfies the equation.
Therefore,

$$\begin{aligned}
0 - f(p_n) &= f'(p_n)(p_{n+1} - p_n) \\
\implies p_{n+1} &= p_n - \frac{f(p_n)}{f'(p_n)}
\end{aligned}$$

This provides the definition for the iteration function of Newton's method.



Definition. Newton's method is fixed point iteration scheme based on the iteration function

$$g(x) = x - \frac{f(x)}{f'(x)}$$

That is, starting from an initial approximation, p_0, the sequence $\{p_n\}$ is generated by $p_n = g(p_{n-1})$.

Example 1. Find the root of $f(x) = x^3 + 2x^2 - 3x - 1$ using Newton's method.

Solution: $f(x) = x^3 + 2x^2 - 3x - 1 \Longrightarrow f'(x) = 3x^2 + 4x - 3$.

Since root of $f(x)$ lies in interval $(1, 2)$, starting with $p_0 = 1$,

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)} = 1 - \frac{-1}{4} = 1.25$$

$$p_2 = p_1 - \frac{f(p_1)}{f'(p_1)} = 1.2009345794$$

$$p_3 = p_2 - \frac{f(p_2)}{f'(p_2)} = 1.1986958411$$

$$p_4 = p_3 - \frac{f(p_3)}{f'(p_3)} = 1.1986912435$$

The approximation p_4 is correct to the digit shown and has an absolute error of roughly 1.937×10^{-11}.

Note. By the theorem 4 we see that, to show the sequence generated by fixed point iteration scheme $g(p_n) = p_{n-1}$ converges to fixed point p if

1. g is continuous on the interval I;
2. g maps I into I; and
3. $|g'(x)| \leq k < 1$ for all $x \in I$.

Theorem 5. *Let f be twice continuously differentiable function on the interval $[a, b]$ with $p \in (a, b)$ and $f(p) = 0$. Further suppose that $f'(p) \neq 0$. Then there exists a $\delta > 0$ such* *that for any $p_0 \in I = [p-\delta, p+\delta]$, the sequence $\{p_n\}$ generated by Newton's method converges to p.*

PROOF. The Newton's iteration function is given by

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Since we have given $f(p) = 0$,

$$\begin{aligned} \Longrightarrow g(p) &= p - \frac{f(p)}{f'(p)} \\ &= p \end{aligned}$$

$\Longrightarrow p$ is fixed point of $g(x)$.

Step 1. Show that g is continuous near p.

By definition of g the only possible discontinuity in g are the points at which $f'(x) = 0$. But by hypothesis $f'(p) \neq 0$ and f' is continuous.

$\Longrightarrow \exists \delta_1 > 0$ such that $f'(x) \neq 0$ for all $x \in I_1 = [p - \delta_1, p + \delta_1] \subset [a, b]$.

Step 2. Show that $|g'(x)|$ is small near p.

$$\begin{aligned} g(x) &= x - \frac{f(x)}{f'(x)} \\ \Longrightarrow g'(x) &= \frac{f(x)f''(x)}{[f(x)]^2} \end{aligned}$$

Since $f'(x) \neq 0$ for all $x \in I_1$.

$\Longrightarrow g'$ is continuous on I_1.

Furthermore, $f(p) = 0 \Longrightarrow g'(p) = 0$.

Choose $0 < k < 1$,

By similar argument applied in step 1, there exists positive constant δ, with $\delta \leq \delta_1$ such that

$$|g'(x)| \leq k < 1$$

for all $x \in I = [p - \delta, p + \delta]$.

Step 3. Show that g maps the interval I into itself.

Let $x \in I$. Then

$$\begin{aligned} |g(x) - p| &= |g(x) - g(p)| \\ &= |g'(\xi)(x - p)| \quad \text{For some } \xi \text{ between } x \text{ and } p \\ &\leq k|x - p| \\ &< \delta \end{aligned}$$

$\therefore |g(x) - p| < \delta \Longrightarrow p - \delta < g(x) < p + \delta$.

$\Longrightarrow g(x) \in I = [p - \delta, p + \delta]$.

Therefore, g is mapping from I into itself.
Therefore, by theorem 4 the sequence generated by Newton's method converges to fixed point p for any starting approximation $p_0 \in I$. ■
ORDER OF CONVERGENCE OF NEWTON'S METHOD
Let $f(x)$ is twice continuously differentiable function.
The Taylor's expansion of $f(x)$ about $x = p_n$ is,

$$f(x) = f(p_n) + f'(p_n)(x - p_n) + \frac{f''(p_n)}{2!}(x - p_n)^2$$

Suppose p be a root of $f(x)$.

$$\begin{aligned}
f(p) &= f(p_n) + f'(p_n)(p - p_n) + \frac{f''(p_n)}{2!}(p - p_n)^2 \\
0 &= f(p_n) + f'(p_n)(p - p_n) + \frac{f''(p_n)}{2!}(p - p_n)^2 \\
0 &= \frac{f(p_n)}{f'(p_n)} + (p - p_n) + \frac{f''(p_n)}{2!f'(p_n)}(p - p_n)^2 \\
-\left[\frac{f(p_n)}{f'(p_n)} + (p - p_n)\right] &= \frac{f''(p_n)}{2!f'(p_n)}(p - p_n)^2 \\
p_n - \left[\frac{f(p_n)}{f'(p_n)} + (p - p_n)\right] &= p_n + \frac{f''(p_n)}{2!f'(p_n)}(p - p_n)^2 \\
p_n - \frac{f(p_n)}{f'(p_n)} - p + p_n &= p_n + \frac{f''(p_n)}{2!f'(p_n)}(p - p_n)^2 \\
p_{n+1} - p &= \frac{f''(p_n)}{2!f'(p_n)}(p - p_n)^2 \\
\frac{|p_{n+1} - p|}{|p - p_n|^2} &= \frac{|f''(p_n)|}{2!\,|f'(p_n)|} \\
\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p - p_n|^2} &= \lim_{n\to\infty} \frac{|f''(p_n)|}{2!\,|f'(p_n)|} \\
\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p - p_n|^2} &= \frac{|f''(p)|}{2!\,|f'(p)|}
\end{aligned}$$

Therefore, $\alpha = 2$ and $\lambda = \dfrac{|f''(p)|}{2!\,|f'(p)|}$.

Therefore, the convergence of Newton's method is quadratic.
SECANT METHOD

Suppose $y = f(x)$ be a function whose root we have to find. Let p_{n-1} and p_n are initial approximations to the root of $y = f(x)$. Draw the secant line to the $y = f(x)$ passing through $(p_{n-1}, f(p_{n-1})$ and $(p_n, f(p_n))$. We will consider the x-intercept of secant line will be our next approximation to the root of $y = f(x)$. Say p_{n+1}. The equation of secant line is given by

$$y - f(p_n) = \frac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}}(x - p_n)$$

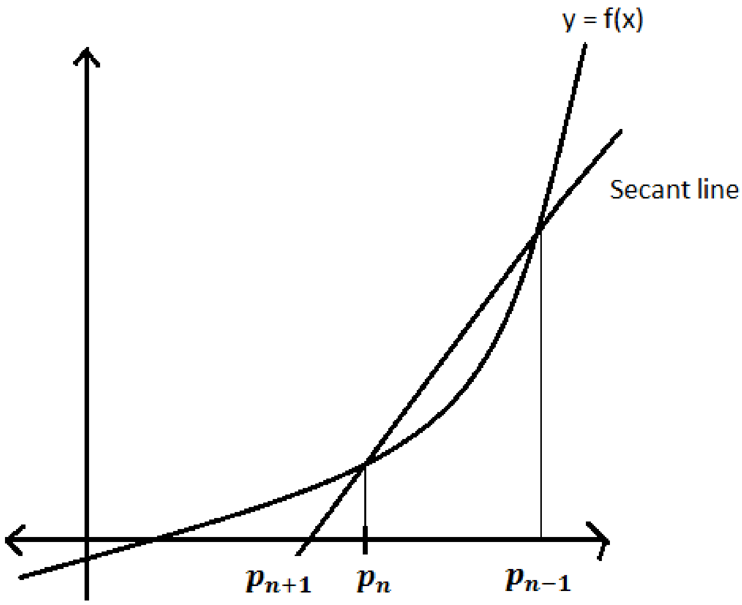
Since the point $(p_{n+1}, 0)$ lies on secant line

$$\therefore 0 - f(p_n) = \frac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}}(p_{n+1} - p_n)$$

Solving this equation for p_{n+1} we get,

$$p_{n+1} = p_n - f(p_n)\frac{p_n - p_{n-1}}{f(p_n) - f(p_n - 1)}$$

which is secant method iteration function.



Secant method for approximating zero of function: schematic for single iteration

Note.
Same formula is obtained by replacing $f'(p_n)$ in Newton's method by $\dfrac{f(p_n) - f(p_{n-1})}{p_n - p_{n-1}}$.
ORDER OF CONVERGENCE OF SECANT METHOD

Suppose $y = f(x)$ be a twice continuously differentiable function and p be a root of $y = f(x)$.
We have secant method iteration formula is

$$p_{n+1} = p_n - f(p_n)\frac{p_n - p_{n-1}}{f(p_n) - f(p_{n-1})}$$

Subtracting p from both side we get,

$$p_{n+1} - p = p_n - p - f(p_n)\frac{p_n - p_{n-1}}{f(p_n) - f(p_{n-1})} \qquad (1)$$

Express $f(p_n)$ and $f(p_{n-1})$ by using Taylor's series expansion.

$$f(p_n) = f(p) + f'(p)(p_n - p) + \frac{f''(p)}{2!}(p_n - p)^2$$

Since p is a root of $y = f(x) \Longrightarrow f(p) = 0$.

$$f(p_n) = f'(p)(p_n - p) + \frac{f''(p)}{2!}(p_n - p)^2 \qquad (2)$$

Similarly,

$$f(p_{n-1}) = f'(p)(p_{n-1} - p) + \frac{f''(p)}{2!}(p_{n-1} - p)^2 \qquad (3)$$

Subtracting equation (3) from (2)

$$\begin{aligned}
f(p_n) - f(p_{n-1}) &= f'(p)(p_n - p_{n-1}) + \frac{f''(p)}{2!}\left[(p_n - p)^2 - (p_{n-1} - p)^2\right] \\
&= f'(p)(p_n - p_{n-1}) + \frac{f''(p)}{2!}\left[p_n - 2p_n p + p^2 - p_{n-1}^2 + 2p_{n-1}p - p^2\right] \\
&= f'(p)(p_n - p_{n-1}) + \frac{f''(p)}{2!}\left[(p_n - p_{n-1}^2) - 2p(p_n - p_{n-1})\right] \\
&= f'(p)(p_n - p_{n-1}) + \frac{f''(p)}{2!}[(p_n - p_{n-1})(p_n + p_{n-1}) - 2p(p_n - p_{n-1})] \\
&= (p_n - p_{n-1})\left[f'(p) + \frac{f''(p)}{2!}(p_n + p_{n-1} - 2p)\right]
\end{aligned}$$

Substituting this value of $f(p_n) - f(p_{n-1})$ in equation (1) we get,

$$\begin{aligned}
p_{n+1} - p &= p_n - p - \left[f'(p)(p_n - p) + \frac{f''(p)}{2!}(p_n - p)^2 \right] \frac{p_n - p_{n-1}}{(p_n - p_{n-1}) \left[f'(p) + \frac{f''(p)}{2!}(p_n + p_{n-1} - 2p) \right]} \\
&= p_n - p - \frac{f'(p)(p_n - p) + \frac{f''(p)}{2!}(p_n - p)^2}{\left[f'(p) + \frac{f''(p)}{2!}(p_n + p_{n-1} - 2p) \right]} \\
&= (p_n - p) \left[1 - \frac{f'(p) + \frac{f''(p)}{2!}(p_n - p)}{\left(\frac{2f'(p) + f''(p)(p_n + p_{n-1} - 2p)}{2!} \right)} \right] \\
&= (p_n - p) \left[1 - \frac{\frac{f'(p) + f''(p)(p_n - p)}{2!}}{\left(\frac{2f'(p) + f''(p)(p_n + p_{n-1} - 2p)}{2!} \right)} \right] \\
&= (p_n - p) \left[\frac{2f'(p) + f''(p)(p_n + p_{n-1} - 2p) - (2f'(p) + f''(p)(p_n - p))}{(2f'(p) + f''(p)(p_n + p_{n-1} - 2p))} \right] \\
&= (p_n - p) \left[\frac{f''(p)\left[p_n + p_{n-1} - 2p - (p_n - p)) \right]}{(2f'(p) + f''(p)(p_n + p_{n-1} - 2p))} \right] \\
&= (p_n - p) \left[\frac{f''(p)\left[p_{n-1} - p \right]}{(2f'(p) + f''(p)(p_n + p_{n-1} - 2p))} \right]
\end{aligned}$$

As $n \to \infty \implies p_n, p_{n-1} \to p \implies p_n + p_{n-1} - 2p \to 0$

$$\lim_{n \to \infty} \frac{p_{n+1} - p}{p_n - p} \approx \frac{f''(p) e_{n-1}}{2f'(p)} \tag{4}$$

Suppose the order of convergence is α.

$$\begin{aligned}
\therefore \frac{|e_{n+1}|}{|e_n|^\alpha} &\approx \lambda \\
|e_{n+1}| &\approx \lambda |e_n|^\alpha \qquad (5)
\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{|e_n|}{|e_{n-1}|^\alpha} &\approx \lambda \\ \Longrightarrow |e_{n-1}|^\alpha &\approx \tfrac{1}{\lambda}|e_n| \\ \Longrightarrow |e_{n-1}| &\approx \left(\tfrac{1}{\lambda}\right)^{\frac{1}{\alpha}} |e_n|^{\frac{1}{\alpha}} \qquad (6)\end{aligned}$$

From equation (4), (5) and (6)

$$\begin{aligned}\frac{\lambda|e_n|^\alpha}{|e_n|} &= \frac{\left|f''(p)\left(\frac{1}{\lambda}|e_n|\right)^{\frac{1}{\alpha}}\right|}{|2f'(p)|} \\ \lambda|e_n|^{\alpha-1} &= C\cdot\left(\frac{1}{\lambda}|e_n|\right)^{\frac{1}{\alpha}}\cdot|e_n|, \text{ where, } C=\left|\frac{f''(p)}{2f'(p)}\right|\end{aligned}$$

Equating exponents of $|e_n|$ on both side we get,

$$\begin{aligned}\alpha-1 &= \frac{1}{\alpha} \\ \alpha^2-\alpha-1 &= 0 \\ \alpha &= \frac{1\pm\sqrt{5}}{2}\end{aligned}$$

But order of convergence is positive number and hence $\alpha = \dfrac{1+\sqrt{5}}{2} \approx 1.618$.
Calculate value of λ:
Equating coefficients of $|e_n|$ on both side we get,

$$\begin{aligned}\lambda &= C\left(\frac{1}{\lambda}\right)^{\frac{1}{\alpha}} \\ \lambda^{1+\frac{1}{\alpha}} &= C \\ \lambda^\alpha &= C \qquad \because \alpha-1=\frac{1}{\alpha}\Longrightarrow \alpha = 1+\frac{1}{\alpha} \\ \lambda &= C^{\frac{1}{\alpha}} \\ \lambda &= C^{1-\alpha} \qquad \because \alpha-1=\frac{1}{\alpha}\end{aligned}$$

$$\Longrightarrow \lambda = \left|\frac{f''(p)}{2f'(p)}\right|^{\alpha-1}$$

ACCELERATING CONVERGENCE
A natural question to ask whether it is possible to accelerate convergence speed of a sequence. For example can anything be done to speed up the convergence of a linearly convergent sequence ? These questions will be addressed in this section.
Aitken's Δ^2 method
Let us start by accelerating the convergence of a linearly convergent sequence. By stopping condition in fixed point iteration scheme we have

$$\begin{aligned} g'(p) &\approx \frac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}} \\ e_n &\approx \frac{g'(p)}{g'(p) - 1}\,(p_n - p_{n-1}) \end{aligned}$$

$$\begin{aligned} \Longrightarrow p_n - p &\approx \left[\frac{\dfrac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}}}{\dfrac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}} - 1}\right](p_n - p_{n-1}) \\ &\approx \left[\frac{\dfrac{p_n - p_{n-1}}{p_{n-1} - p_{n-2}}}{\dfrac{p_n - p_{n-1} - (p_{n-1} - p_{n-2})}{p_{n-1} - p_{n-2}}}\right](p_n - p_{n-1}) \\ &\approx \frac{p_n - p_{n-1}}{p_n - p_{n-1} - (p_{n-1} - p_{n-2})}\,(p_n - p_{n-1}) \\ &\approx \frac{(p_n - p_{n-1})^2}{(p_n - p_{n-1}) - (p_{n-1} - p_{n-2})} \qquad (1) \end{aligned}$$

Let Δ denote the difference operator. Therefore, $\Delta p_n = p_n - p_{n-1}$ and

$$\begin{aligned} (p_n - p_{n-1}) - (p_{n-1} - p_{n-2}) &= \Delta p_n - \Delta p_{n-1} \\ &= \Delta(p_n - p_{n-1}) \\ &= \Delta(\Delta p_n) \\ &= \Delta^2 p_n \end{aligned}$$

Substituting these values in equation (1) we get,

$$\begin{aligned} p_n - p &\approx \frac{(\Delta p_n)^2}{\Delta^2 p_n} \\ p &\approx p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n} \end{aligned}$$

Therefore, the formula for $\hat{p}_n$ can be written as

$$\hat{p}_n = p_n - \frac{(\Delta p_n)^2}{\Delta^2 p_n}$$

The sequence $\{\hat{p}_n\}$ is guaranteed to converge more rapidly than the sequence $\{p_n\}$.
Example 1. The sequence below was obtained from fixed point iteration applied to the function $g(x) = \sqrt{\frac{10}{2+x}}$, which has a unique fixed point.

$$\begin{aligned} p_0 &= 2.23606977 \\ p_1 &= 1.536450382 \\ p_2 &= 1.681574897 \\ p_3 &= 1.648098560 \\ p_4 &= 1.655643081 \\ p_5 &= 1.653933739 \\ p_6 &= 1.654320556 \end{aligned}$$

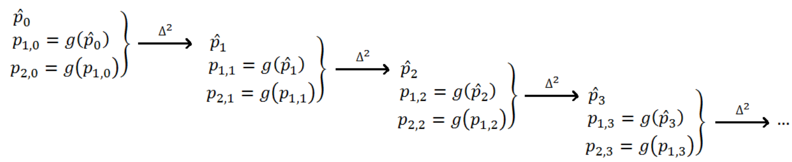
Apply Aitken's Δ^2-method to the given sequence.

Steffensen's Method

For the linearity convergence fixed point iteration schemes of the form $p_{n+1} = g(p_n)$, it is possible to accelerate convergence even further by applying a variation of the Aitken's Δ^2-method. The basis idea can be explained as follows.
Suppose the starting approximation p_0 is given and the value $p_1 = g(p_0)$ and $p_2 = g(p_1)$ are calculated. Aitken's Δ^2-method is then applied to compute $\hat{p}$. Since $\hat{p}$ is supposed to be a better approximation to the fixed point than p_2. This three steps process is then repeated: From current approximation, perform two fixed point iterations and then combine the current approximation and the two intermediate values according to the Aitken's Δ^2 formula to form the next approximation. This scheme is known as Steffensen's method. The sequence of approximation is denoted by $\hat{p}_n$, and for consistency the initial approximation is denoted by $\hat{p}_0$. Finally, $p_{1,n}$ and $p_{2,n}$ denote the intermediate values calculated using the iteration function starting with $x = \hat{p}_n$ with this notation the Aitken's Δ^2 formula takes the form

$$\hat{p}_{n+1} = p_{2,n} - \frac{(p_{2,n} - p_{1,n})^2}{p_{2,n} - 2p_{1,n} + \hat{p}_n}$$

The graphical depiction of the sequence of calculations in Steffensen's method is given below:



The sequence of approximations is denoted by the $\hat{p}$ values. The values $p_{1,n}$ and $p_{2,n}$ are intermediate values used in Aitken's Δ^2 formula.

Example. Apply the Steffensen's method to find the fixed point of the function $g(x) = e^{-x}$ starting with initial approximation $\hat{p}_0 = 0$.

Solution. We calculate $p_{1,0} = g(\hat{p}_0) = e^0 = 1, p_{2,0} = g(p_{1,0}) = e^{-1} = 0.3678794412$. Then

$$\begin{aligned}\hat{p}_1 &= p_{2,0} - \frac{(p_{2,0} - p_{1,0})^2}{p_{2,0} - 2p_{1,0} + \hat{p}_0} \\ &= 0.3678794412 - \frac{(0.3678794412 - 1)^2}{0.3678794412 - 2(1) + 0} \\ &= 0.6126998368\end{aligned}$$

Reinitializing the iteration with $\hat{p}_1$, we obtain $p_{1,1} = g(\hat{p}_1) = 0.541885888, p_{2,1} = g(p_{1,1}) = 0.5816502896$. Then

$$\begin{aligned}\hat{p}_2 &= p_{2,1} - \frac{(p_{2,1} - p_{1,1})^2}{p_{2,1} - 2p_{1,1} + \hat{p}_1} \\ &= 0.5816502896 - \frac{(0.5816502896 - 0.541885888)^2}{0.5816502896 - 2(0.541885888) + 0.6126998368} \\ &= 0.5673508577\end{aligned}$$

At this point, note that $\hat{p}_2$ is more accurate than the tenth term in the sequence generated by fixed point iteration.

The third iteration of Steffensen's method produces $\hat{p}_3 = 0.5671432948$. This value is correct to eight decimal places and has an absolute error of roughly 4.4421×10^{-9}.

This shows that, the convergence of Steffensen's method is faster than convergence of fixed point iteration scheme.

♣♣♣