Introduction to Operations Research

What is Operations Research?

Operations research is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control

Operations research is the "mathematical or scientific analysis of the systematic efficiency and performance of manpower, machinery, equipment and policies used in governmental, military or commercial operation."

History

The term operations research emerged in the year 1940, during World War II in Great Britain with the establishment of groups of scientists to analyze the strategic and tactical problems associated with military operations. The aim was to discover the most efficient uses of limited military resources by the application of quantitative techniques.

India was among the few nations which began utilizing O.R. In 1949, the first operational research unit was established at Hyderabad. In 1953 at Kolkata an O.R. unit was established in Indian statistical Institute. The objective was to use O.R. techniques in National planning and survey. In 1955, Operations Research Society of India was created.

Applications of Operations Research

1. Allocation and Distribution in Projects:

i. Optimal allocation of resources such as men, materials, machines, time and money to projects

ii. Project scheduling, monitoring and control

2. Production and Facilities Planning:

i. Factory size and location decision

ii. Estimation of number of facilities required

iii. Preparation of forecasts for the various inventory items and computation of economic order quantities and reorder levels.

iv. Scheduling and sequencing of production runs by proper allocation of machines

v. Transportation loading and unloading.

3. Marketing

i. Advertising budget allocation
ii. Product introduction timing
iii. Selection of advertising media
iv. Selection of product mix
v. Customer's preference of size, color and packaging of various products.

4. Organization Behavior

i. Selection of personnel, determination of retirement age and skills
ii. Recruitment policies and assignment of jobs
iii. Recruitment of employees
iv. Scheduling of training programs

5. Finance

i. Capital requirements, cash flow analysis
ii. Credit policies, credit risks
iii. Investment decisions
iv. Profit plan for the company

6. Research and Development

i. Product introduction planning
ii. Control of R & D policies
iii. Determination of areas for research and development
iv. Selection of projects and preparation of their budgets.

v. Reliability and control of development projects.

Phases of Operations Research

1. Definition of the problem

It defines the scope of the problem under investigation. The aim is to identify the description of alternatives, determination of the objective of the problem and specifications of limitations

2. Construction of the model

It attempts to translate the problem definition into mathematical relationships. If the resulting model fits one of the standard mathematical models, we can usually reach a solution by using available algorithms.

3. Solution of the model

An important aspect of the model solution phase is sensitivity analysis. It deals with obtaining additional information about the behavior of optimum solution when the model undergoes some parameter changes.
It is also important to study the behavior of optimum solution in the neighborhood of the estimated parameters.

4. Validation of the model

It checks whether or not the proposed model does what it purports to do- that is, does it predict adequately the behavior of the system under study? A common method for checking the model validity is to compare its output with historical output data. The model is valid if, under similar input conditions, it reasonably duplicates the past performance.

5. Implementation of the solution

Implementation of solution of a validated model involves the translation of the results into understandable operating instructions to be issued to the people who will administer the recommended system.

Tools of Operations Research

- Linear Programming
- Queuing Theory
- Inventory Control Models

- Replacement Problems
- Network Analysis
- Sequencing
- Transportation and Assignment Problems
- Integer Programming
- Game Theory

1. The Simplex Method

Formulation of LP model

There are three basic components required in LP model formulation

1. Decision variables that we seek to determine.
2. Objective that we need to optimize (maximize or minimize).
3. Constraints that the solution must satisfy.

Decision variables: The variables in a linear program are a set of quantities that need to be determined in order to solve the problem. That is, the problem is solved when the best values of the variables have been identified.

The **objective function** indicates how much each variable contributes to the value to be optimized in the problem.

The **constraints** are the restrictions or limitations on the decision variables.

Example 1: (Two variable LP model)
Reddy Mikks produces both interior and exterior paints from two raw materials M_1 and M_2. The following table provides the basic data of the problem:

	Tons of raw material per ton of		Maximum daily availability (tons)
	Exterior paint	Interior paint	
Raw material, $\boldsymbol{M_1}$	6	4	24
Raw material, $\boldsymbol{M_2}$	1	2	6
Profit per ton ($\$\mathbf{1000}$)	5	4	

A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also, the maximum daily demand for interior paints is 2 tons.
Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit.

Solution 1: Let x_1 denotes the number of tons of exterior paint produced daily and x_2 denotes number of tons of interior paint produced daily.

Given that the profits per ton of exterior and interior paints are 5 and 4 (thousand)dollars, respectively.

Total profit from exterior paint= $5x_1$

Total profit from interior paint= $4x_2$

The objective function is, Maximize $z = 5x_1 + 4x_2$

Usages of raw material M1 by exterior paint= $6x_1$tons/day

Usages of raw material M1 by interior paint= $4x_2$ tons/day

Hence usages of raw material M1 by both paints= $6x_1$+$4x_2$ tons/ day

In a similar manner, usages of raw material M2 by both paints =$x_1 + 2x_2$ tons/day

Because the daily availability of raw materials M1 and M2 are limited to 24 and 6 tons, therefore the constraints are

$$6x_1 + 4x_2 \leq 24$$

$$x_1 + 2x_2 \leq 6$$

Also the daily production of interior over exterior paint should not exceed 1 ton

Therefore, $x_2 - x_1 \leq 1$

And the maximum daily demand of interior paint is limited to 2 tons, that is, $x_2 \leq 2$

Complete Reddy Mikks model

Maximize $z = 5x_1 + 4x_2$

Subject to

$$6x_1 + 4x_2 \leq 24$$

$$x_1 + 2x_2 \leq 6$$

$$x_2 - x_1 \leq 1$$

$$x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

Example 2: For the Reddy Mikks model, construct each of the following constraints.

i. The daily demand for interior paint exceeds that of exterior paint by at least 1 ton
ii. The daily usage of raw material M2 in tons is at most 6 and at least 3.
iii. The demand for interior paint cannot be less than the demand for exterior paint.
iv. The minimum quantity that should be produced of both paints is 3 tons.
v. The proportion of interior paint to the total production of both interior and exterior paints must not exceed 0.5.

Solution 2:

i. $x_2 - x_1 \geq 1$
ii. $x_1 + 2x_2 \leq 6$ and $x_1 + 2x_2 \geq 3$
iii. $x_2 \geq x_1$
iv. $x_1 + x_2 \geq 3$
v. $\frac{x_2}{x_1+x_2} \leq 0.5$

Example 3: A company produces two products A and B. The sales volume for A is at least 80% of the total sales of both A and B. However the company cannot sell more than 100 units of A per day. Both products use one raw material, of which the maximum daily availability is 240 lb. The usage rates of the raw material are 2 lb per unit of A and 4 lb per unit of B. The profit units for A and B are \$20 and \$50 respectively. Formulate LP model.

Solution 3: Let x_1 denotes the number of units of product A produced daily and x_2 denotes number of units of product B produced daily.

Given that the profits per unit of A and B are \$20 and \$50 respectively

$$\text{Total profit from product } A = 20x_1$$

$$\text{Total profit from product } B = 50x_2$$

The objective function is, Maximize $z = 20x_1 + 50x_2$

The sales volume for A is at least 80% of the total sales of both A and B.

That is, $x_1 \geq 0.8(x_1 + x_2)$

The company cannot sell more than 100 units of A per day, i.e. $x_1 \leq 100$.

Usages of raw material by product $A = 2x_1$lb/day

Usages of raw material by product $B = 4x_2$lb/day

Hence usages of raw material M1 by both paints= $2x_1+4x_2$ lb/ day

Because the daily availability of raw material is limited to 240 lb/day only, the associated constraint is

$$2x_1+4x_2 \leq 240$$

Therefore, the complete LP model is

$$\text{Maximize } z = 20x_1 + 50x_2$$

Subject to

$$-0.2x_1 + 0.8x_2 \leq 0$$

$$x_1 \leq 100$$

$$2x_1+4x_2 \leq 240$$

$$x_1, x_2 \geq 0$$

Graphical LP Solution

The graphical procedure includes two steps:

1. Determination of the feasible solution space
2. Determination of the optimum solution from among all the feasible points in the solution space.

The **feasible solution space** of the problem represents the area in the first quadrant in which all the constraints are satisfied simultaneously.

Example 1: Reddy Mikks Model

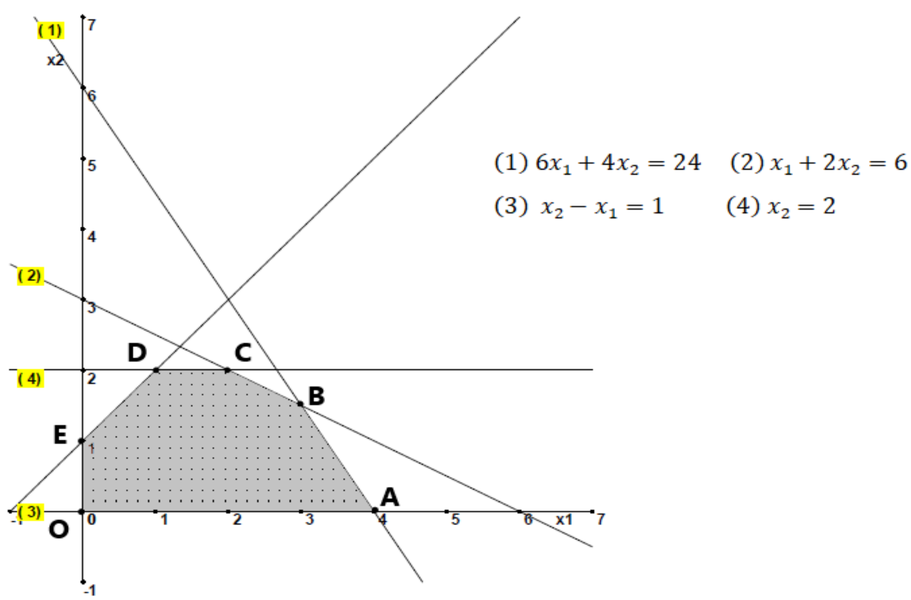
$$\text{Maximize } z = 5x_1 + 4x_2$$

Subject to

$$6x_1 + 4x_2 \leq 24$$
$$x_1 + 2x_2 \leq 6$$
$$x_2 - x_1 \leq 1$$
$$x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

Solution 1:
Step 1: Determination of feasible solution space.

Equation of Line	When $x_1 = 0$	When $x_2 = 0$
$6x_1 + 4x_2 = 24$	(0,6)	(4,0)
$x_1 + 2x_2 = 6$	(0,3)	(6,0)
$x_2 - x_1 = 1$	(0,1)	(−1,0)
$x_2 = 2$	Line parallel to x_1 axis passing through (0,2) point	



Step 2: Determination of Optimum solution

After finding the feasible solution space, find the corner points of feasible region.

Here the feasible region is shaded area in the graph whose corner points are O, A, B, C, D and E.

<u>To find coordinates of corner points:</u>

Here O is the origin, that is,(0,0).

Whereas A and E are points on the line $6x_1 + 4x_2 = 24$ and $x_2 - x_1 = 1$

when $x_2 = 0$ and $x_1 = 0$ respectively.

That is, $A \equiv (4,0)$ and $E \equiv (0,1)$

B is the point of intersection of the lines $6x_1 + 4x_2 = 24$ and $x_1 + 2x_2 = 6$. Hence by solving these two equations simultaneously, we can find the coordinates of B.

That is, $B \equiv (3,1.5)$

Similarly C and D can be found by solving corresponding equations of intersecting lines simultneously.

Hence, $C \equiv (2,2)$ and $D \equiv (1,2)$.

Now we will find the value of objective function at these corner points.

Corner Points of feasible solution space	(x_1, x_2)	Value of obj. function
O	(0,0)	0
A	(4,0)	20
B	(3,1.5)	21 (max)
C	(2,2)	18
D	(1,2)	13
E	(0,1)	4

Since the maximum value of z is at the point (3,1.5)

Hence the optimum solution is $\boldsymbol{x_1 = 3,\ x_2 = 1.5}$.

Example 2: (Diet Problem)

Ozark Farms uses at least 800 lb of special feed daily. The special feed is mixture of corn and soya bean meal with the following compositions:

	$\boldsymbol{lb}$ per $\boldsymbol{lb}$ of feedstuff		
Feedstuff	Protein	Fiber	Cost(\$/$\boldsymbol{lb}$)
Corn	6	4	24
Soya bean meal	1	2	6

The dietary requirements of the special feed are at least 30% and at most 5% fiber. Ozark Farms wishes to determine the daily minimum cost feed mix.

Solution 2:

Formulation of LP Model

Let x_1 denotes the lb of corn in the daily mix and x_2 denotes lb of soya bean meal in the daily mix.

Given that the cost per lb of corn and soya bean meal are 0.30 and 0.90 dollars respectively.

$$\text{Total cost of corn in the daily mix} = 0.30x_1$$

$$\text{Total cost of soya bean meal in the daily mix} = 0.90x_2$$

The objective function is, Minimize $z = 0.3x_1 + 0.9x_2$

Protein required for corn is $= 0.09\ x_1$lb

Protein required for soya bean meal $= 0.6x_2$ lb

Hence total protein required $= 0.09x_1 + 0.6x_2$ lb

But this quantity should equal at least 30% of the total feed mix,

That is $$0.09x_1 + 0.6x_2 \geq 0.3(x_1 + x_2).$$

Similarly, total fiber requirement is $0.02x_1 + 0.06x_2$ lb

Also the fiber requirement of at most 5% is constructed as,

$$0.02x_1 + 0.06x_2 \leq 0.05\ (x_1 + x_2).$$

And the Ozark Farms needs at least 800 lb of feed a day, the associated constraint is, $x_1 + x_2 \geq 800$.

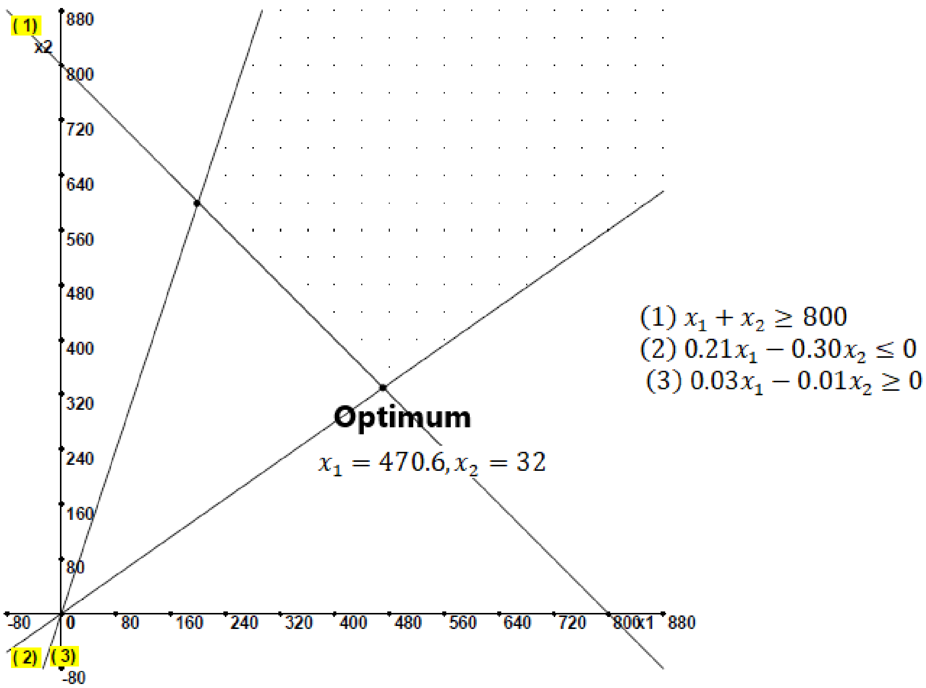
Thus the complete model is

$$\text{Minimize } z = 0.3x_1 + 0.9x_2$$

Subject to

$$x_1 + x_2 \geq 800$$
$$0.21x_1 - 0.30x_2 \leq 0$$
$$0.03x_1 - 0.01x_2 \geq 0$$
$$x_1, x_2 \geq 0$$

Graphical solution:



Example 3: ChemLabs uses raw materials I and II to produce two domestic cleaning solutions, A and$B$. The daily availabilities of raw materials I and II are 150 and 145 units, respectively. One unit of solution A consumes 0.5 unit of raw material I and 0.6 unit of raw material II and one unit of solution B uses 0.5 unit of raw material I and 0.4 unit of raw materialII. The profits per unit of one solutions A and B are $8 and $10 respectively. The daily demand for solution A lies between 30 and 150 units and that for solution B between 40 and 200 units. Find the optimal production amounts of A and$B$.

Solution 3: Let x_1 denotes the number of units of solution A and x_2 denotes number of units of solution B.

Given that the profits per unit of A and B are \$ 8 and \$ 10 respectively.

Therefore, total profit from solution A and B is $8x_1 + 10x_2$

The objective function is, Maximize $8x_1 + 10x_2$.

1. <u>Limitations on resources:</u>

Maximum daily availability of raw material I is 150 units and total requirement of raw material I is $0.5x_1 + 0.5x_2$

So the associated constraint is $0.5x_1 + 0.5x_2 \leq 150$.

Similarly, Maximum daily availability of raw material II is 145 units and total requirement of raw material II is $0.6x_1 + 0.4x_2$

So the associated constraint is $0.6x_1 + 0.4x_2 \leq 145$.

2. <u>Market Limit:</u>

The daily demand for solution A lies between 30 and 150 units

That is, $30 \leq x_1 \leq 150$.

And the daily demand of solution B lies between 40 and 200 units.

That is, $40 \leq x_2 \leq 200$.

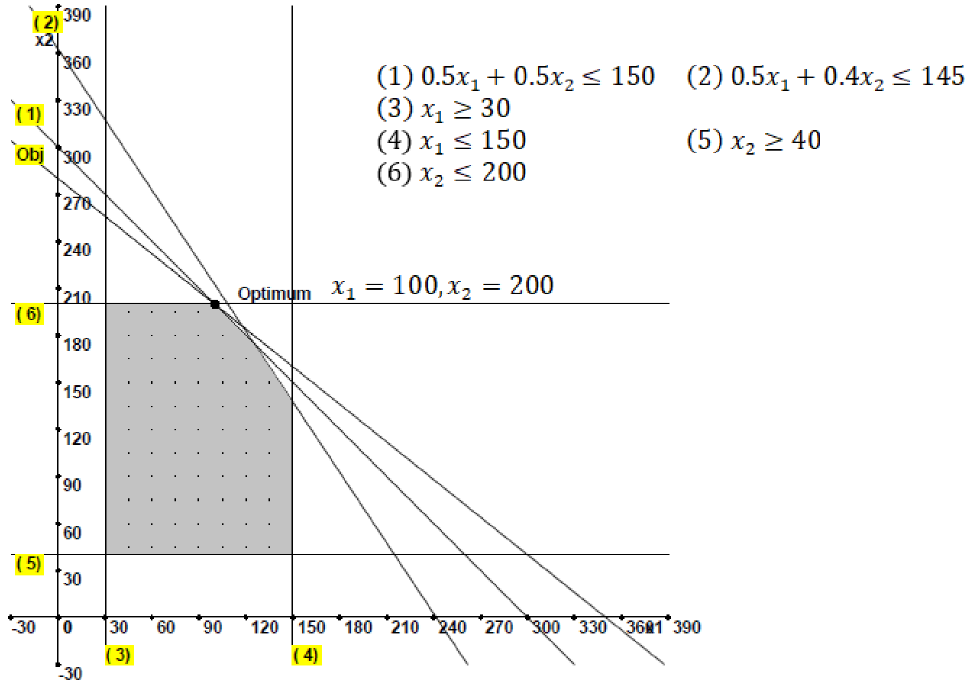
Therefore the LP model is,

$$\text{Maximize } 8x_1 + 10x_2$$

Subject to

$$0.5x_1 + 0.5x_2 \leq 150$$
$$0.6x_1 + 0.4x_2 \leq 145$$
$$30 \leq x_1 \leq 150$$
$$40 \leq x_2 \leq 200$$
$$x_1, x_2 \geq 0$$

Graphical Solution:



LP Model in Equation Form

- All the constraints (with the exception of the non-negativity of the variables) are equations with nonnegative right-hand side
- All the variables are nonnegative.
- If the right-hand side of the resulting equation is negative then the non-negativity condition is satisfied by multiplying both sides of the equation by -1.

Converting Inequalities into Equations with Nonnegative Right-Hand Side:

1. To convert $(\leq) -$ inequality to an equation, a nonnegative **slack variable** is added to the left-hand side of the constraint.
   Ex. $6x_1 + 4x_2 \leq 24$ can be converted into an equation by adding s_1 slack variable to LHS of the inequality as:
   $$6x_1 + 4x_2 + s_1 = 24$$
2. To convert $(\geq) -$ inequality to an equation, a nonnegative **surplus variable** is subtracted from the left-hand side of the constraint.

   Ex. $x_1 + x_2 \geq 800$can be converted into an equation by subtracting s_1 surplus variable from LHS of the inequality as:

$$x_1 + x_2 - s_1 = 800$$

Unrestricted Variables:

If specifically $x_i(\geq 0)$ is the workforce size in period i, then $x_{i+1}(\geq 0)$ the workforce size in period $i + 1$ can be expressed as

$$x_{i+1} = x_i + y_{i+1}$$

The variable y_{i+1} must be unrestricted in sign to allow x_{i+1} to increase or decrease relative to x_i depending on whether workers are hired or fired, respectively.

In order to convert inequalities containing unrestricted variables into an equation, substitute$y_{i+1} = y_{i+1}^{+} - y_{i+1}^{-}$, where $y_{i+1}^{+}, y_{i+1}^{-} \geq 0$.

Example 1: Write the equation form of the following LP:

$$\text{Maximize } z = 2x_1 + 3x_2$$

Subject to

$$x_1 + 3x_2 \leq 6$$

$$3x_1 + 3x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

Solution 1: The equation form of the given LP is

$$\text{Maximize } z = 2x_1 + 3x_2 + 0s_1 + 0s_2$$

Subject to

$$x_1 + 3x_2 + s_1 = 6$$

$$3x_1 + 3x_2 + s_2 = 6$$

$$x_1, x_{2,} s_1, s_2 \geq 0$$

Example 2: Write the equation form of the following LP:

$$\text{Maximize } z = 2x_1 + 3x_2 + 5x_3$$

Subject to

$$-6x_1 + 7x_2 - 9x_3 \geq 4$$

$$x_1 + x_2 + 4x_3 = 10$$

$$x_1, x_3 \geq 0, x_2 \text{ unrestricted}$$

Solution 2: The equation form of the given LP is

$$\text{Maximize } z = 2x_1 + 3x_2^+ - 3x_2^- + 5x_3 + 0s_1$$

Subject to

$$-6x_1 + 7x_2^+ - 7x_2^- - 9x_3 - s_1 = 4$$

$$x_1 + x_2^+ - x_2^- + 4x_3 = 10$$

$$x_1, x_2^+, x_2^-, x_{3,} s_1 \geq 0$$

Transition From Graphical To Algebraic Solution:

- Let $m =$ no. of equations and $n =$ no. of variables.
- In a set of $m \times n$ equations $(m < n)$, if we set $n - m$ variables equal to zero and then solve the m equations for the remaining m variables, the resulting solution, if unique, is called a basic solution and must correspond to a (feasible or infeasible) corner point of the solution space.
- The maximum no. of corner points is $\binom{n}{m} = \frac{n!}{m!(n-m)!}$
- The zero $n - m$ variables are called non basic variables.
- The remaining m variables are called basic variables and their solution obtained by solving m equations is referred to as basic solution.

Example 1: Consider Maximize $z = 2x_1 + 3x_2$

Subject to $2x_1 + x_2 \leq 4$

$$x_1 + 2x_2 \leq 5$$

$$x_1,\ x_2 \geq 0$$

Solution: The equation form of given LP is

$$\text{Maximize } z = 2x_1 + 3x_2 + 0s_1 + 0s_2$$

Subject to $2x_1 + x_2 + s_1 = 4$

$$x_1 + 2x_2 + s_2 = 5$$

$$x_1,\ x_2,\ s_1,\ s_2 \geq 0$$

The system has $m = 2$ equations and $n = 4$ variables.

Algebraically the corner points can be determined by setting $n - m = 4 - 2 = 2$ variables equal to zero the solving for remaining $m = 2$ variables.

So we have $\binom{4}{2} = \frac{4!}{2!2!} = 6$ corner points.

Non basic variables	Basic variables	Basic Solution	Feasible	Objective value, z
x_1, x_2	s_1, s_2	$s_1 = 4, s_2 = 5$	Yes	0
x_1, s_1	x_2, s_2	$x_2 = 4,\ s_2 = -3$	No	-
x_1, s_2	x_2, s_1	$x_2 = 2.5, s_1 = 1.5$	Yes	7.5
x_2, s_1	x_1, s_2	$x_1 = 2, s_2 = 3$	Yes	4
x_2, s_2	x_1, s_1	$x_1 = 5, s_1 = -6$	No	-
s_1, s_2	x_1, x_2	$\boldsymbol{x_1 = 1, x_2 = 2}$	Yes	8 (Optimum)

Selected LP Applications

1. Urban Planning

Example: (Urban Renewal Model)

The project involves two phases: (1) demolishing substandard houses to provide land for the new development and (2) building the new development. The following is the summary of the situation

1. As many as 300 substandard houses can be demolished. Each house occupies 0.25 acre lot. The cost of demolishing a condemned house is \$2000.
2. Lot sizes for new single, double, triple and quadruple- family homes (units) are 0.18, 0.28, 0.4 and 0.5 acre respectively. Streets, open space and utility easements account for 15% of available acreage.

3. In the new development the triple and quadruple units account for at least 25% of the total. Single unit must be at least 20% of all units and double units at least 10%.
4. The tax levied per unit for single, double, triple, and quadruple units is $1000, $1900, $2700 and $3400 respectively.
5. The construction cost per unit for single, double, triple, and quadruple- family homes is $50,000, $70,000, $1,30,000and $1,60,000 respectively.

Financing through a local bank can amount to a maximum of $15 million.

How many units of each type should be constructed to maximize tax collection?

Solution:

Mathematical Model:

Besides determining the number of units to be constructed of each type of housing, we also need to determine how many houses must be demolished to establish new development. Thus the variables of the problem can be defined as follows:

x_1= No. of units of single-family homes

x_2= No. of units of double-family homes

x_3= No. of units of triple-family homes

x_4= No. of units of quadruple-family homes

x_5= No. of old homes to be demolished

The objective is to maximize total tax collection from all four types of homes.

That is, Maximize $z = 1000x_1 + 1900x_2 + 2700x_3 + 3400x_4$

1. <u>Limitation on land availability:</u>

$$\begin{pmatrix}\text{Acreage used for new} \\ \text{home constructions}\end{pmatrix} \leq \begin{pmatrix}\text{Net available} \\ \text{acreage}\end{pmatrix}$$

Acreage used for new home construction $=0.18x_1 + 0.28x_2 + 0.4x_3 + 0.5x_4$

Each demolished home occupies 0.25 acre lot.

So the total available acreage$=0.25x_5$.

But out of this area allowing 15% acreage for open space, streets and easements, the net acreage available is $0.85(0.25x_5) = 0.2125x_5$.

So the resultant constraint will be

$$0.18x_1 + 0.28x_2 + 0.4x_3 + 0.5x_4 \leq 0.2125x_5$$

That is, $0.18x_1 + 0.28x_2 + 0.4x_3 + 0.5x_4 - 0.2125x_5 \leq 0$

And the no of demolished homes cannot exceed 300, that is $x_5 \leq 300$.

2. <u>Limitationson new development:</u>

$$(\text{No. of single units}) \geq (20\% \text{ of all units})$$

$$(\text{No. of double units}) \geq (10\% \text{ of all units})$$

$$(\text{No. of triple and quadruple units}) \geq (25\% \text{ of all units})$$

So the associated constraints are,

$$x_1 \geq 0.2(x_1 + x_2 + x_3 + x_4)$$

$$x_2 \geq 0.1(x_1 + x_2 + x_3 + x_4)$$

$$x_3 + x_4 \geq 0.25(x_1 + x_2 + x_3 + x_4)$$

3. <u>Limitations on budget allocation</u>

$$(\text{Construction and demolition cost}) \leq (\text{Available budget})$$

That is, $50000x_1 + 70000x_2 + 130000x_3 + 160000x_4 + 2000x_5 \leq 15000000$.

The complete LP model is

$$\text{Maximize } z = 1000x_1 + 1900x_2 + 2700x_3 + 3400x_4$$

Subject to

$$0.18x_1 + 0.28x_2 + 0.4x_3 + 0.5x_4 - 0.2125x_5 \leq 0$$

$$x_5 \leq 300$$

$$-0.8x_1 - 0.2x_2 + 0.2x_3 + 0.2x_4 \leq 0$$

$$0.1x_1 - 0.9x_2 + 0.1x_3 + 0.1x_4 \leq 0$$

$$0.25x_1 + 0.25x_2 - 0.75x_3 - 0.75x_4 \leq 0$$

$$50x_1 + 70x_2 + 130x_3 + 160x_4 + 2x_5 \leq 15000$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

2. **Investment**

Example: (Loan Policy Model)

Thriftem bank is in the process of devising a loan policy that involves a maximum of $12 million. The following table provides the pertinent data about available types of loans.

Types of Loans	Interest rate	Bad-debt ratio
Personal	0.14	0.10
Car	0.13	0.07
Home	0.12	0.03
Farm	0.125	0.05
Commercial	0.100	0.02

Bad debts are unrecoverable and produce no interest revenue.

Competition with other financial institutions requires that the bank allocate at least 40% of the funds to farm and commercial loans. To assist the housing industry in the region, home loans must equal at least 50% of the personal, car and home loans. The bank also has a stated policy of not allowing the overall ratio of bad debts on all loans to exceed 4%.

(Bad debt ratio= + Accruals for doubtful and old debts for the period
- Recovery of accruals for doubtful and old debts for the period
/ Turnover for the period.)

Solution:

Mathematical Model:

To determine amount of loan in each category, define

x_1= personal loans (in million dollars)

x_2= car loans

x_3= home loans

x_4= farm loans

x_5= commercial loans

The objective of the bank is to maximize its net return, the difference between interest revenue and lost bad debts. The interest revenue is accrued only on loans in good standing.

Thus, because 10% of personal loans are lost to bad debt, the bank will receive interest on only 90% of the loan- that is it will receive 14% interest on $0.9x_1$.

The same reasoning applies to the remaining 4 types of loans.

Hence, total interest $=0.14(0.9x_1) + 0.13(0.93x_2) + 0.12(0.97x_3) + 0.125(0.95x_4) + 0.1(0.98x_5)$

$= 0.126x_1 + 0.1209x_2 + 0.1164x_3 + 0.11875x_4 + 0.098x_5$

Bad debt $= 0.1x_1 + 0.07x_2 + 0.03x_3 + 0.05x_4 + 0.02x_5$.

Thus the objective function is expressed as,

Maximize $z =$ Total interest – bad debt $= \left(0.126x_1 + 0.1209x_2 + 0.1164x_3 + 0.11875x_4 + 0.098x_5\right) - (0.1x_1 + 0.07x_2 + 0.03x_3 + 0.05x_4 + 0.02x_5)$

$$= 0.026x_1 + 0.0509x_2 + 0.0864x_3 + 0.06875x_4 + 0.078x_5$$

1. Total funds should not exceed \$12 (million)

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 12$$

2. Farm and commercial loans equal at least 40% of all loans.

$$x_4 + x_5 \geq 0.4(x_1 + x_2 + x_3 + x_4 + x_5)$$

That is, $0.4x_1 + 0.4x_2 + 0.4x_3 - 0.6x_4 - 0.6x_5 \leq 0$

3. Home loans should be equal at least 50% of personal, car and home loans

$$x_3 \geq 0.5(x_1 + x_2 + x_3)$$

That is, $0.5x_1 + 0.5x_2 - 0.5x_3 \leq 0$

4. Bad debts should not exceed 4% of all loans

$0.1x_1 + 0.07x_2 + 0.03x_3 + 0.05x_4 + 0.02x_5 \leq 0.04(x_1 + x_2 + x_3 + x_4 + x_5)$

That is, $0.06x_1 + 0.03x_2 - 0.01x_3 + 0.01x_4 - 0.02x_5 \leq 0$

The complete LP model is

Maximize $z = 0.026x_1 + 0.0509x_2 + 0.0864x_3 + 0.06875x_4 + 0.078x_5$

Subject to

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 12$$
$$0.4x_1 + 0.4x_2 + 0.4x_3 - 0.6x_4 - 0.6x_5 \leq 0$$
$$0.5x_1 + 0.5x_2 - 0.5x_3 \leq 0$$
$$0.06x_1 + 0.03x_2 - 0.01x_3 + 0.01x_4 - 0.02x_5 \leq 0$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

3. Production planning and Inventory Control

Example: (Single-Period Production Model)

A clothing company is manufacturing parka and goose jackets, insulated pants and gloves. All products are manufactured in four different departments: cutting, sewing and packaging. The company has received firm orders for its products. The contract stipulates a penalty for undelivered items. The following table provides the pertinent data of the situation.

Department	**Time per units (in hr.)**				**Capacity(hr)**
	Parka	Goose	Pants	Gloves	
Cutting	0.30	0.30	0.25	0.15	1000
Insulating	0.25	0.35	0.30	0.10	1000
Sewing	0.45	0.50	0.40	0.22	1000
Packaging	0.15	0.15	0.1	0.05	1000
Demand	800	750	600	500	
Profit	$30	$40	$20	$10	
Penalty	$15	$20	$10	$8	

Devise an optimal production plan for the company.

Solution:

Mathematical Model:

Let x_1= No. of parka jackets

x_2= No. of goose jackets

x_3= No. of pants

x_4= No. of pairs of gloves

The company is penalized for not meeting demand. Thus the objective of the problem is to maximize the net receipts, defined as

Net receipts= Total profit - Total penalty

$$\text{Total profit} = 30x_1 + 40x_2 + 20x_3 + 10x_4$$

Total penalty is a function of shortage quantities (= demand – units supplied of each Product).

These quantities can be determined from the following demand limits:

$$x_1 \leq 800, \quad x_2 \leq 750, \quad x_3 \leq 600, \quad x_4 \leq 500$$

If a demand is not fulfilled then its constraint is satisfied as a strict inequality.

We can express the shortage of any product by defining a new nonnegative variable viz.

s_j = No. of shortages units of product j, $j = 1,2,3,4$

So the demand constraints are written as

$$x_1 + s_1 = 800$$

$$x_2 + s_2 = 750$$

$$x_3 + s_3 = 600$$

$$x_4 + s_4 = 500$$

We can now compute the shortage penalty as $15s_1 + 20s_2 + 10s_3 + 8s_4$

The remaining constraints are on the production capacity namely

$$0.30x_1 + 0.30x_2 + 0.25x_3 + 0.15x_4 \leq 1000 \quad \text{(Cutting)}$$

$$0.25x_1 + 0.35x_2 + 0.30x_3 + 0.10x_4 \leq 1000 \quad \text{(Insulating)}$$

$$0.45x_1 + 0.50x_2 + 0.40x_3 + 0.22x_4 \leq 1000 \quad \text{(Sewing)}$$
$$0.15x_1 + 0.15x_2 + 0.10x_3 + 0.05x_4 \leq 1000 \quad \text{(Packaging)}$$

The complete LP model is

Maximize $z = 30x_1 + 40x_2 + 20x_3 + 10x_4 - (15s_1 + 20s_2 + 10s_3 + 8s_4)$

Subject to

$$0.30x_1 + 0.30x_2 + 0.25x_3 + 0.15x_4 \leq 1000$$
$$0.25x_1 + 0.35x_2 + 0.30x_3 + 0.10x_4 \leq 1000$$
$$0.45x_1 + 0.50x_2 + 0.40x_3 + 0.22x_4 \leq 1000$$
$$0.15x_1 + 0.15x_2 + 0.10x_3 + 0.05x_4 \leq 1000$$
$$x_1 + s_1 = 800$$
$$x_2 + s_2 = 750$$
$$x_3 + s_3 = 600$$
$$x_4 + s_4 = 500$$
$$x_1, x_2, x_3, x_4, s_1, s_2, s_3, s_4 \geq 0$$

The Simplex Method

Iterative Nature of the Simplex Method

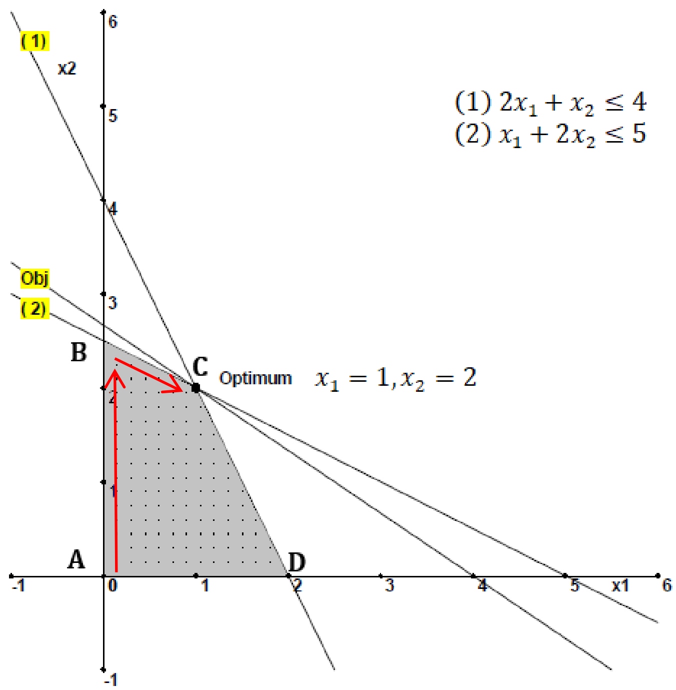
Consider the following LP:

$$\text{Maximize } z = 2x_1 + 3x_2$$

Subject to

$$2x_1 + x_2 \leq 4$$
$$x_1 + 2x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

Graphical Solution:



- The simplex method normally starts at the origin
- At the starting point, the value of the objective function is zero.
- The function shows that in increase in either x_1 or x_2 (or both) above their current zero values will improve the value of z.
- Increasing the value of one variable which has the largest rate of improvement in z at a time.
- In this example, the value of z will increase by 2 for each unit increase in x_1 and by 3 for each unit increase in x_2 . Thus we select x_2 with the largest rate of improvement.
- The value of x_2 must be increased until corner point B is reached. But B is not optimum.
- At the point B, the simplex method will then increase the value of x_1 to reach the improved corner point C, which is optimum.
- Thus the path of the simplex algorithm is defined as A$\rightarrow B \rightarrow C$.
- Each corner point along the path is associated with an iteration.
- The simplex method moves alongside the edges of the solution space.

We need to make the transition from the graphical solution to the algebraic solution by showing how the points$A$, B and C are represented by their basic and non-basic variables.

Corner Point	Basic variables	Non-basic (zero) variables
A	$s_1,\ s_2$	x_1, x_2
B	s_1, x_2	x_1,s_2
C	x_1, x_2	s_1, s_2

- From A to B, the non-basic $\boldsymbol{x_2}$ at A becomes basic at B and basic $\boldsymbol{s_2}$ at A becomes non basic at B. So here $\boldsymbol{x_2}$ is the **entering variable** (because it enters the basic solution) and $\boldsymbol{s_2}$ is the **leaving variable** (because it leaves the basic solution).
- In a similar manner, at point B, $\boldsymbol{x_1}$ enters (the basic solution) and $\boldsymbol{s_1}$ leaves, thus leading to point C.

Simplex Algorithm

Step 1: Determine a starting basic feasible solution

Step 2: Select an entering variable using the optimality condition. Stop if there is no entering variable; the last solution is optimal. Else go to step 3.

Step 3: Select a leaving variable using the feasibility condition

Step 4: Determine the new basic solution by using the appropriate Gauss-Jordan computations. Go to step 2.

Optimality Condition: The entering variable in a maximization (minimization) problem is the non-basic variable having the most negative (positive) coefficient in the z- row. Ties are broken arbitrarily. The optimum is reached at the iteration where all the z- row coefficients of the non basic variables are nonnegative (nonpositive).

Feasibility Condition: For both maximization and the minimization problems, the leaving variable is the basic variable associated with the smallest nonnegative ratio (with strictly positive denominator). Ties are broken arbitrarily.

Pivot row: The row corresponding to leaving variable is called pivot row.

Pivot Column: The column corresponding to entering variable is called pivot column.

Pivot Element: The element at the intersection of the pivot row and pivot column is called as pivot element.

Gauss- Jordan row operations:

1. **Pivot row**
   a. Replace the leaving variable in the Basic column with the entering variable.
   b. New pivot row = Current pivot row ÷ pivot element.
2. **All other rows, including *z***

New row = current row – (pivot column coefficient) × (new pivot row)

Example 1: Maximize $z = 5x_1 + 4x_2$

Subject to
$$6x_1 + 4x_2 \leq 24$$
$$x_1 + 2x_2 \leq 6$$
$$x_2 - x_1 \leq 1$$
$$x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

Solution 1: The equation form of the given LP is

$$\text{Maximize } z = 5x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

Subject to
$$6x_1 + 4x_2 + s_1 = 24$$
$$x_1 + 2x_2 + s_2 = 6$$
$$-x_1 + x_2 + s_3 = 1$$
$$x_2 + s_4 = 2$$
$$x_1,\ x_2,\ s_1,\ s_2, s_3, s_4 \geq 0$$

<u>Initial Basic Feasible Solution</u>:

The system has $m = 4$ equations and $n = 6$ variables.

setting $n - m = 6 - 4 = 2$ variables equal to zero the solving for remaining 4 variables.

Simplex iteration starts at the origin. Hence the starting solution can be found by substituting $x_1 = 0$ and $x_2 = 0$

Initial Simplex Table:

Basic	z	x_1 ↓	x_2	s_1	s_2	s_3	s_4	Solution	Ratio
z	1	-5	-4	0	0	0	0	0	
s_1	0	**6**	4	1	0	0	0	24	4 →
s_2	0	1	2	0	1	0	0	6	6
s_3	0	-1	1	0	0	1	0	1	-
s_4	0	0	1	0	0	0	1	2	-

From the table, the starting solution is given by

$z = 0, s_1 = 24, s_2 = 6,\ s_3 = 1, s_4 = 2.$

The solution is not optimal here. This solution can be improved.

So the entering variable will correspond to most negative coefficient in the objective equation.

Hence select x_1 as an entering variable.

Since the most non-negative ratio correspond to s_1 .

$\therefore s_1$ is the leaving variable.
So in the next iteration table, replace s_1 by x_1.

Gauss-Jordan Computations:

1. New x_1 –row = current pivot row÷ 6 = $\left(0,1,\frac{2}{3},\frac{1}{6},0,0,0,4\right)$

2. All other rows, including z

New row= (New x_1 –row) – (Its pivot column coefficient) × (New x_1 –row)

New z-row = (Current z-row)– (-5) × (New x_1 –row)

$$=(1,-5,-4,0,0,0,0) - 5 \times \left(0,1,\frac{2}{3},\frac{1}{6},0,0,0,4\right)$$

$$= \left(1,0,-\frac{2}{3},\frac{5}{6},0,0,0,20\right)$$

New s_2-row = (Current s_2-row)− (1) × (New x_1 –row)

$$=(0,1,2,0,1,0,0,6) - \left(0,1,\frac{2}{3},\frac{1}{6},0,0,0,4\right)$$

$$= \left(0,0,\frac{4}{3},\frac{-1}{6},1,0,0,2\right)$$

New s_3-row = (Current s_3-row)− (-1) × (New x_1 –row)

$$=(0,-1,1,0,0,1,0,1) + \left(0,1,\frac{2}{3},\frac{1}{6},0,0,0,4\right)$$

$$= \left(0,0,\frac{5}{3},\frac{1}{6},0,1,0,5\right)$$

New s_4-row = (Current s_4-row)− (0) × (New x_1 –row)

$$=(0,0,1,0,0,0,1,2) + (0)\left(0,1,\frac{2}{3},\frac{1}{6},0,0,0,4\right)$$

$$= (0,0,1,0,0,0,1,2)$$

First Iteration Table: ↓

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution	Ratio
z	1	0	-2/3	5/6	0	0	0	20	
x_1	0	1	2/3	1/6	0	0	0	4	6
s_2	0	0	4/3	-1/6	1	0	0	2	3/2 →
s_3	0	0	5/3	1/6	0	1	0	5	3
s_4	0	0	1	0	0	0	1	2	2

Since here all z row coefficients are not nonnegative.

Hence solution is not optimal.

Second Iteration Table:

Basic	z	x_1	x_2	s_1	s_2	s_3	s_4	Solution
z	1	0	0	3/4	1/2	0	0	21
x_1	0	1	0	1/4	-1/2	0	0	3
x_2	0	0	1	-1/8	3/4	0	0	3/2
s_3	0	0	0	3/8	-5/4	1	0	5/2
s_4	0	0	0	1/8	-3/4	0	1	1/2

New x_2 –row = current pivot row÷ 4/3

New z-row = (Current z-row)− (-2/3) × (New x_2 –row)

New x_1-row = (Current x_1-row)− (2/3) × (New x_2 –row)

New s_3-row = (Current s_3-row)− (5/3) × (New x_2 –row)

New s_4-row = (Current s_4-row)− (1) × (New x_2 –row)

Since here all the z row coefficients are nonnegative.

Hence the solution is optimal.

That is $\boldsymbol{x_1 = 3}$, $\boldsymbol{x_2 = 1.5}$ and $\textbf{max}\, \boldsymbol{z = 21}$

The solution can be interpreted as:

Decision variable	Optimum value	Recommendation
x_1	3	Produce 3 tons of exterior paint daily
x_2	1.5	Produce 1.5 tons of interior paint daily
z	21	Daily profit is $21,000

The solution also gives the status of the resources. A resource is designated as **scarce** if the variables of the model use the resource completely. Otherwise, the resource is **abundant.**

Resource	**Slack Value**	**Status**
Raw Material, M1	$s_1 = 0$	Scarce
Raw Material, M2	$s_2 = 0$	Scarce
Market limit	$s_3 = \frac{5}{2}$	Abundant
Demand limit	$s_4 = \frac{1}{2}$	Abundant

Example 2: Consider the following LP

$$\text{Maximize } z = 16x_1 + 15x_2$$

Subject to

$$\begin{aligned} 4x_1 + 3x_2 &\leq 24 \\ -x_1 + x_2 &\leq 1 \\ x_1 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution 2: The equation form of the given LP is

$$\text{Maximize } z = 16x_1 + 15x_2 + 0s_1 + 0s_2 + 0s_3$$

Subject to

$$\begin{aligned} 4x_1 + 3x_2 + s_1 &= 24 \\ -x_1 + x_2 + s_2 &= 1 \\ x_1 + s_3 &= 3 \\ x_1, x_2, s_1, s_2, s_3 &\geq 0 \end{aligned}$$

We write the objective equation as

$$z - 16x_1 - 15x_2 = 0$$

Initial Basic Feasible Solution

The system has $m = 3$ equations and $n = 5$ variables.

Set $n - m = 5 - 3 = 2$ variables equal to zero the solving for remaining 3 variables.

Simplex iteration starts at origin. So the associated set of nonbasic and basic variable is defined as

$$\text{Nonbasic (zero) variables:} (x_1, x_2)$$

$$\text{Basic variables: } (s_1, s_2, s_3)$$

Initial Simplex Table:

Basic	z	x_1 ↓	x_2	s_1	s_2	s_3	Solution	Ratio
z	1	-16	-15	0	0	0	0	
s_1	0	4	3	1	0	0	24	6
s_2	0	-1	1	0	1	0	1	-
s_3	0	1	0	0	0	1	3	3 →

From the table, the starting solution is given by

$z = 0, s_1 = 24, s_2 = 1, s_3 = 3.$

The solution is not optimal here. This solution can be improvised.

So the entering variable will correspond to most negative coefficient in the objective equation.

Hence select x_1 as an entering variable.

Since the most non-negative ratio correspond to s_3 .

$\therefore s_3$ is the leaving variable.
So in the next iteration table, replace s_3 by x_1.

Gauss-Jordan Computations:

1. New x_1 –row = current pivot row
2. All other rows, including z

New z-row = (Current z-row)– (-16) × (New x_1 –row)

New s_1-row = (Current s_1-row)– (4) × (New x_1 –row)

New s_2-row = (Current s_2-row)– (-1) × (New x_1 –row)

First Iteration Table: ↓

Basic	z	x_1	x_2	s_1	s_2	s_3	Solution	Ratio
z	1	0	-15	0	0	16	48	
S_1	0	0	3	1	0	-4	12	4
s_2	0	0	1	0	1	1	4	4 →
x_1	0	1	0	0	0	1	3	-

Since here all z row coefficients are not nonnegative.

Hence solution is not optimal.

Second Iteration Table:

Basic	z	x_1	x_2	s_1	s_2	s_3	Solution
z	1	0	0	0	15	31	108
S_1	0	0	0	1	-3	-7	0
x_2	0	0	1	0	1	1	4
x_1	0	1	0	0	0	1	3

New x_2 –row = current pivot row

New z-row = (Current z-row)− (-15) × (New x_2 –row)

New s_1-row = (Current s_1-row)− (3) × (New x_2 –row)

New x_1 –row = (Current x_1 –row) −(0) × (New x_2 –row)

Since here all the z row coefficients are nonnegative.

Hence the solution is optimal.

That is $\boldsymbol{x_1 = 3}$, $\boldsymbol{x_2 = 4}$ and $\mathbf{max}\, \boldsymbol{z} = \mathbf{108}$

Example 3: Consider the following LP

$$\text{Maximize } z = 3x_1 + 2x_2 + x_3$$

Subject to

$$2x_1 + x_2 + x_3 \leq 150$$
$$2x_1 + 2x_2 + 8x_3 \leq 200$$
$$2x_1 + 3x_2 + x_3 \leq 320$$
$$x_1, x_2, x_3 \geq 0$$

Solution 3: The equation form of the given LP is

$$\text{Maximize } z = 3x_1 + 2x_2 + x_3 + 0s_1 + 0s_2 + 0s_3$$

Subject to

$$2x_1 + x_2 + x_3 + s_1 = 150$$
$$2x_1 + 2x_2 + 8x_3 + s_2 = 200$$
$$2x_1 + 3x_2 + x_3 + s_3 = 320$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

We write the objective equation as

$$z - 3x_1 - 2x_2 = 0$$

Initial Basic Feasible Solution

The system has $m = 3$ equations and $n = 6$ variables.

Set $n - m = 6 - 3 = 3$ variables equal to zero the solving for remaining 3 variables.

Simplex iteration starts at origin. So the associated set of nonbasic and basic variable is defined as

Nonbasic (zero) variables:(x_1, x_2, x_3)

Basic variables: (s_1, s_2, s_3)

Initial Simplex Table:

Basic	z	x_1 ↓	x_2	x_3	s_1	s_2	s_3	Solution	Ratio
z	1	-3	-2	-1	0	0	0	0	
s_1	0	2	1	1	1	0	0	150	75 →
s_2	0	2	2	8	0	1	0	200	100
s_3	0	2	3	1	0	0	1	320	160

From the table, the starting solution is given by

$z = 0,\ s_1 = 150,\ s_2 = 200,\ s_3 = 320.$

The solution is not optimal here. This solution can be improved.

So the entering variable will correspond to most negative coefficient in the objective equation.

Hence select x_1 as an entering variable.

Since the most non-negative ratio correspond to s_1 .

∴ s_1 is the leaving variable.
So in the next iteration table, replace s_1 by x_1.

Gauss-Jordan Computations:

New x_1 –row = current pivot row÷ 2

New z-row = (Current z-row)− (-3) × (New x_1 –row)

New s_2-row = (Current s_2-row)− (2) × (New x_1 –row)

New s_3-row = (Current s_3-row)− (2) × (New x_1 –row)

First Iteration Table:

Basic	z	x_1	x_2 ↓	x_3	s_1	s_2	s_3	Solution	Ratio
z	1	0	-1/2	1/2	3/2	0	0	225	
x_1	0	1	1/2	1/2	1/2	0	0	75	150
s_2	0	0	1	7	-1	1	0	50	50 →
s_3	0	0	2	0	-1	0	1	170	135

Since here all z row coefficients are not nonnegative.

Hence solution is not optimal.

Second Iteration Table:

Basic	z	x_1	x_2	x_3	s_1	s_2	s_3	Solution
z	1	0	0	4	1	1/2	0	250
x_1	0	1	0	-3	1	-1/2	0	50
x_2	0	0	1	7	-1	1	0	50
s_3	0	0	0	-14	1	-2	1	70

New x_2 –row = current pivot row

New z-row = (Current z-row)− (-1/2) × (New x_2 –row)

New x_1 –row = (Current x_1 –row)

New s_3-row = (Current s_3-row)$-$ (2) $\times$ (New x_2 –row)

Since here all the z row coefficients are nonnegative.

Hence the solution is optimal.

That is $\boldsymbol{x_1 = 50}$, $\boldsymbol{x_2 = 50}$ $\quad$ *and* $\mathbf{max}\ \boldsymbol{z} = \mathbf{250}$

Artificial Starting Solution

- Method for solving LP models involving (=) and/or ($\geq$) constraints.

- Two methods are introduced here: the M-method and two-phase method.

Artificial Variables: Artificial variables have no meaning in a physical sense are only used as a tool for generating an initial solution to an LP problem. These variables are added to those constraints with equality (=) and greater than or equal to ($\geq$) type.

M-Method

The M-method is a method of removing artificial variable from the basis.

1. The method starts with the LP in equation form.
2. If equation i does not have a slack, an artificial variable, R_i, is added to form a starting solution.

3. As artificial variables are not part of original LP model, they are assigned a very high **penalty** in the objective function and thus forcing them (eventually) to equal zero in the optimum solution.

Penalty Rule for Artificial Variables

Given M, a sufficiently large positive value (mathematically, $M \to \infty$), the objective coefficient of an artificial variable represents an appropriate **penalty** if:

$$\text{Artificial variable objective coefficient} = \begin{cases} -M, & \text{in max problems} \\ M, & \text{in min problems} \end{cases}$$

Example 1: Consider the following LP

$$\text{Minimize } z = 4x_1 + x_2$$

Subject to

$$\begin{aligned} 3x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 &\geq 6 \\ x_1 + 2x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution 1: The equation form of the given LP is

$$\text{Minimize } z = 4x_1 + x_2 + MR_1 + MR_2$$

Subject to

$$3x_1 + x_2 + R_1 = 3$$

$$4x_1 + 3x_2 - s_2 + R_2 = 6$$

$$x_1 + 2x_2 + s_3 = 4$$

$$x_1, x_2, s_2, s_3, R_1, R_2 \geq 0$$

We write the objective equation as

$$z - 4x_1 - x_2 - MR_1 - MR_2 = 0$$

Initial Basic Feasible Solution

The system has $m = 3$ equations and $n = 6$ variables.

Set $n - m = 6 - 3 = 3$ variables equal to zero the solving for remaining 3 variables.

Simplex iteration starts at origin. So the associated set of nonbasic and basic variable is defined as

Nonbasic (zero) variables:(x_1, x_2, s_2)

Basic variables: (R_1, R_2, s_3)

Initial Simplex Table:

Basic	x_1	x_2	s_2	R_1	R_2	s_3	Solution
z	-4	-1	0	-M	-M	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
s_3	1	2	0	0	0	1	4

From the table, the starting solution is given by

$z = 0, R_1 = 3, R_2 = 6, s_3 = 4$

But this solution is not consistent. To make the z −row consistent, we use the following operation

New z-row = Old z-row + (M× $\mathbf{R_1}$ − row + $\mathbf{MR_2}$-row)

Basic	x_1 ↓	x_2	s_2	R_1	R_2	s_3	Solution	Ratio
z	-4+7M	-1+4M	-M	0	0	0	9M	
R_1	3	1	0	1	0	0	3	1 →
R_2	4	3	-1	0	1	0	6	1.5
s_3	1	2	0	0	0	1	4	4

Gauss-Jordan Computations:

1. New x_1 –row = current pivot row ÷ 3
2. All other rows, including z

New z-row = (Current z-row)− (-4+7M) × (New x_1 –row)

New R_2-row = (Current R_2-row)− (4) × (New x_1 –row)

New s_3-row = (Current s_3-row)− (1) × (New x_1 –row)

<u>First Iteration Table:</u>

Basic	x_1	x_2 ↓	s_2	R_1	R_2	s_3	Solution	Ratio
z	0	$\frac{1+5M}{3}$	-M	$\frac{4-7M}{3}$	0	0	4+2M	
x_1	1	1/3	0	1/3	0	0	1	3
R_2	0	5/3	-1	-4/3	1	0	2	6/5 →
s_3	0	5/3	0	-1/3	0	1	3	9/5

Since here all z row coefficients are not nonpositive.

Hence solution is not optimal.

Gauss-Jordan Computations:

1. New x_2 –row = current pivot row ÷ $\frac{5}{3}$
2. All other rows, including z

New z-row = (Current z-row)− $(\frac{1+5M}{3})$ × (New x_2 –row)

New x_1-row = (Current x_1-row)− (1/3) × (New x_2 –row)

New s_3-row = (Current s_3-row)− (5/3) × (New x_2 –row)

Second Iteration Table:

Basic	x_1	x_2	s_2 ↓	R_1	R_2	s_3	Solution	Ratio
z	0	0	3/15	$\frac{8-5M}{5}$	$\frac{-1-5M}{5}$	0	18/5	
x_1	1	0	1/5	3/5	-1/5	0	3/5	3
x_2	0	1	-3/5	-4/5	3/5	0	6/5	-
s_3	0	0	1	1	-1	1	1	1 →

Since here all z row coefficients are not nonpositive.

Hence solution is not optimal.

Gauss-Jordan Computations:

1. New s_2 –row = current pivot row
2. All other rows, including z

New z-row = (Current z-row)− (3/15) × (New s_2 –row)

New x_1-row = (Current x_1-row)− (1/5) × (New s_2 –row)

New x_2-row = (Current x_2-row)− (-3/5) × (New s_2 –row)

Third Iteration Table:

Basic	x_1	x_2	s_2	R_1	R_2	s_3	Solution
z	0	0	0	$\frac{7-5M}{5}$	-M	-3/15	17/5
x_1	1	0	0	2/5	0	-1/5	2/5
x_2	0	1	0	-1/5	0	3/5	9/5
s_2	0	0	1	1	-1	1	1

Since here all the z row coefficients are nonpositive.

Hence the solution is optimal.

That is $\boldsymbol{x_1 = 2/5}$, $\boldsymbol{x_2 = 9/5}$ *and* $\mathbf{min}\ \boldsymbol{z = 17/5}$

Example 2: Consider the following LP

$$\text{Maximize } z = 2x_1 + 3x_2 - 5x_3$$

Subject to

$$x_1 + x_2 + x_3 = 7$$
$$2x_1 - 5x_2 + x_3 \geq 10$$
$$x_1, x_2, x_3 \geq 0$$

Solution 2: The equation form of the given LP is

$$\text{Maximize } z = 2x_1 + 3x_2 - 5x_3 - MR_1 - MR_2$$

Subject to

$$x_1 + x_2 + x_3 + R_1 = 7$$
$$2x_1 - 5x_2 + x_3 - s_2 + R_2 = 10$$
$$x_1, x_2, x_3, s_2, R_1, R_2 \geq 0$$

We write the objective equation as

$$z - 2x_1 - 3x_2 + 5x_3 + MR_1 + MR_2 = 0$$

Initial Basic Feasible Solution

The system has $m = 2$ equations and $n = 6$ variables.

Set $n - m = 6 - 2 = 4$ variables equal to zero the solving for remaining 2 variables.

Simplex iteration starts at origin. So the associated set of nonbasic and basic variable is defined as

$$\text{Nonbasic (zero) variables:} (x_1, x_2, x_3, s_2)$$

$$\text{Basic variables: } (R_1, R_2)$$

Initial Simplex Table:

Basic	x_1	x_2	x_3	R_1	R_2	s_2	Solution
z	-2	-3	5	M	M	0	0
R_1	1	1	1	1	0	0	7
R_2	2	-5	1	0	1	-1	10

From the table, the starting solution is given by

$z = 0, R_1 = 7, R_2 = 10$

But this solution is not consistent. To make the z −row consistent, we use the following operation

New z-row = Old z-row + (-M× $\mathbf{R_1}$ − row − $\mathbf{MR_2}$-row)

Basic	x_1 ↓	x_2	x_3	R_1	R_2	s_2	Solution	Ratio
z	-2-3M	-3+4M	5-2M	0	0	M	-17M	
R_1	1	1	1	1	0	0	7	7
R_2	2	-5	1	0	1	-1	10	5 →

From the table, the starting solution is given by

$z = -17M, R_1 = 7, R_2 = 10$ which is consistent.

But here all $z -$ row coefficients are not nonnegative. Hence solution can be improved.

Gauss-Jordan Computations:

1. New x_1 –row = current pivot row ÷ 2
2. All other rows, including z

New z-row = (Current z-row)− (-2-3M) × (New x_1 –row)

New R_1-row = (Current R_1-row)− (1) × (New x_1 –row)

<u>First Iteration Table:</u>

Basic	x_1	x_2 ↓	x_3	R_1	R_2	s_2	Solution	Ratio
z	0	-8-7M/2	6-M/2	0	1+3M/2	-1-M/2	10-2M	
R_1	0	7/2	1/2	1	-1/2	1/2	2	4/7 →
x_1	1	-5/2	1/2	0	1/2	-1/2	5	-

Since here all z row coefficients are not nonnegative.

Hence solution is not optimal.

Gauss-Jordan Computations:

1. New x_2 –row = current pivot row ÷ $\frac{7}{2}$
2. All other rows, including z

New z-row = (Current z-row)− $(-8-\frac{7}{2}M)\times$ (New x_2 –row)

New x_1-row = (Current x_1-row)− (-5/2) × (New x_2 –row)

<u>Second Iteration Table</u>:

Basic	x_1	x_2	x_3	R_1	R_2	s_2	Solution
z	0	0	50/7	16/7+M	-1/7+M	1/7	102/7
x_2	0	1	1/7	2/7	-1/7	1/7	4/7
x_1	1	0	6/7	5/7	1/7	-1/7	45/7

Since here all the z row coefficients are nonnegative.

Hence the solution is optimal.

$$\boldsymbol{x_1 = 45/7 \;,\; x_2 = 4/7 \;, x_3 = 0 \;\; and \;\; \max z = 102/7}$$

Example 3: Consider the problem

$$\text{Maximize } z = 2x_1 + 4x_2 + 4x_3 - 3x_4$$

Subject to

$$x_1 + x_2 + x_3 = 4$$
$$x_1 + 4x_2 + x_4 = 8$$
$$x_1, x_2, x_3, x_4 \geq 0$$

The problem shows that x_3 and x_4 can play the role of slacks for the two equations. They differ from slacks in that they have nonzero coefficients in the objective function. They can be used as starting variable, they must be substituted out in the objective function before the simplex iterations are carried out.

Solution 3:

The system has $m = 2$ equations and $n = 4$ variables.

Set $n - m = 4 - 2 = 2$ variables equal to zero the solving for remaining 2 variables.

Simplex iteration starts at origin. So the associated set of nonbasic and basic variable is defined as

Nonbasic (zero) variables:(x_1, x_2)

Basic variables: (x_3, x_4)

Initial Simplex Table:

Basic	x_1	x_2	x_3	x_4	Solution
z	-2	-4	-4	3	0
x_3	1	1	1	0	4
x_4	1	4	0	1	8

From the table, the starting solution is given by

$z = 0, x_3 = 4, x_4 = 8$

But this solution is not consistent. To make the z −row consistent, we use the following operation

New z-row = Old z-row + ($4\times x_3 - \text{row} - 3 \times x_4$-row)

↓

Basic	x_1	x_2	x_3	x_4	Solution	Ratio
z	-1	-12	0	0	-8	
x_3	1	1	1	0	4	4
x_4	1	4	0	1	8	2 →

From the table, the starting solution is given by

$z = -8, x_3 = 4, x_4 = 2$ which is consistent.

But here all $z -$ row coefficients are not nonnegative. Hence solution can be improved.

Gauss-Jordan Computations:

1. New x_2 –row = current pivot row ÷ 4
2. All other rows, including z

New z-row = (Current z-row)− (−12) × (New x_2 –row)

New x_3-row = (Current x_3-row)− (1) × (New x_2 –row)

First Iteration Table:

Basic	x_1	x_2	x_3	x_4	Solution
z	2	0	0	3	16
x_3	3/4	0	1	-1/4	2
x_2	1/4	1	0	1/4	2

Since here all the z row coefficients are nonnegative.

Hence the solution is optimal.

That is $\boldsymbol{x_1 = 0}$, $\boldsymbol{x_2 = 2}$ *and* $\mathbf{max}\ \boldsymbol{z} = \mathbf{16}$

Two Phase Method

- Method for solving LP models involving (=) and/or (≥) constraints.

- The method solves the LP in two phases: Phase I attempts to find a starting basic feasible solution, if one is found, Phase II invoked to solve the original problem.

Steps in Two phase method:

<u>**Phase I:**</u>

- Write the equation form of given LP model and add necessary artificial variables to the constraints to find a starting basic solution.
- Find a basic solution regardless of whether the LP is maximization or minimization; always minimize the sum of artificial variables.
- If the minimum value of the sum is positive, then the LP has no feasible solution. Otherwise proceed to Phase II.

<u>**Phase II:**</u> Use the feasible solution from phase I as a starting basic feasible solution for the original problem.

Example 1: Consider the following LP

$$\text{Minimize } z = 4x_1 + x_2$$

Subject to

$$3x_1 + x_2 = 3$$

$$4x_1 + 3x_2 \geq 6$$

$$x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Solution 1: The equation form of the given LP is

$$\text{Minimize } z = 4x_1 + x_2 + 0s_2 + 0s_3$$

Subject to

$$3x_1 + x_2 = 3$$

$$4x_1 + 3x_2 - s_2 = 6$$

$$x_1 + 2x_2 + s_3 = 4$$

$$x_1, x_2, s_2, s_3 \geq 0$$

<u>**Phase I**</u>

$$\text{Minimize } r = R_1 + R_2$$

Subject to

$$3x_1 + x_2 + R_1 = 3$$

$$4x_1 + 3x_2 - s_2 + R_2 = 6$$

$$x_1 + 2x_2 + s_3 = 4$$

$$x_1, x_2, s_2, s_3, R_1, R_2 \geq 0$$

The system has $m = 3$ equations and $n = 6$ variables.

Set $n - m = 6 - 3 = 3$ variables equal to zero the solving for remaining 3 variables.

Simplex iteration starts at origin. So the associated set of nonbasic and basic variable is defined as

$$\text{Nonbasic (zero) variables:} (x_1, x_2, s_2)$$

$$\text{Basic variables: } (R_1, R_2, s_3)$$

Initial Simplex Table:

Basic	x_1	x_2	s_2	R_1	R_2	s_3	Solution
r	0	0	0	-1	-1	0	0
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
s_3	1	2	0	0	0	1	4

From the table, the starting solution is given by

$$r = 0, R_1 = 3, R_2 = 6,\ s_3 = 4$$

But this solution is not consistent. To make the r −row consistent, we use the following operation

New r-row = Old r-row + (1× R_1-row + 1× R_2-row)

↓

Basic	x_1	x_2	s_2	R_1	R_2	s_3	Solution	Ratio
r	7	4	-1	0	0	0	9	
R_1	3	1	0	1	0	0	3	1 →
R_2	4	3	-1	0	1	0	6	1.5
s_3	1	2	0	0	0	1	4	4

From the table, $r = 9,\ R_1 = 3,\ R_2 = 6, s_3 = 4$ which is consistent.

Since here all r −row coefficients are not nonpositive(≤ 0).

⇒ Solution is not optimal.

Gauss-Jordan Computations:

1. New x_1 –row = current pivot row ÷ 3
2. All other rows, including r

New r-row = (Current r-row)− (7) × (New x_1 –row)

New R_2-row = (Current R_2-row)− (4) × (New x_1 –row)

New s_3-row = (Current s_3-row)− (1) × (New x_1 –row)

First Iteration Table:

Basic	x_1	x_2 ↓	s_2	R_2	s_3	Solution	Ratio
r	0	5/3	-1	0	0	2	
x_1	1	1/3	0	0	0	1	3
R_2	0	5/3	-1	1	0	2	6/5 →
s_3	0	5/3	0	0	1	3	9/5

Since here all r row coefficients are not nonpositive.

Hence solution is not optimal.

Gauss-Jordan Computations:

1. New x_2 –row = current pivot row ÷ 5/3
2. All other rows, including r

New r-row = (Current r-row)− (5/3) × (New x_2 –row)

New x_1-row = (Current x_1-row)− (1/3) × (New x_2 –row)

New s_3-row = (Current s_3-row)− (5/3) × (New x_2 –row)

Second Iteration Table:

Basic	x_1	x_2	s_2	s_3	Solution
r	0	0	0	0	0
x_1	1	0	1/5	0	3/5
x_2	0	1	-3/5	0	6/5
s_3	0	0	1	1	1

Since here all r row coefficients are nonpositive(≤ 0).

Hence solution is optimal and $\mathbf{min}\ \boldsymbol{r} = \mathbf{0}$

⇒ Phase I is ended here and produce starting basic feasible solution for original LP

$$\boldsymbol{x_1 = 3/5 \;,\; x_2 = 6/5 \;, s_3 = 1}$$

Phase II

Use the original objective function

$$\text{Minimize } z = 4x_1 + x_2$$

We write the objective equation as

$$z - 4x_1 - x_2 = 0$$

Initial Simplex Table

Basic	x_1	x_2	s_2	s_3	Solution
z	-4	-1	0	0	0
x_1	1	0	1/5	0	3/5
x_2	0	1	-3/5	0	6/5
s_3	0	0	1	1	1

From the table, the starting solution is given by

$$z = 0, x_1 = 3/5,\; x_2 = 6/5,\; s_3 = 1$$

But this solution is not consistent. To make the z −row consistent, we use the following operation

New z-row = Old z-row + (4× x_1-row + 1× x_2-row)

↓

Basic	x_1	x_2	s_2	s_3	Solution	Ratio
z	0	0	1/5	0	18/5	
x_1	1	0	1/5	0	3/5	3
x_2	0	1	-3/5	0	6/5	-
s_3	0	0	1	1	1	1 →

From the table, $z = 18/5,\ x_1 = 3/5,\ x_2 = 6/5, s_3 = 1$ which is consistent.

Since here all z −row coefficients are not nonpositive(≤ 0).

$\Rightarrow$ Solution is not optimal.

Gauss-Jordan Computations:

1. New s_2 −row = current pivot row
2. All other rows, including z

New z-row = (Current z-row)− $(1/5) \times$ (New s_2 −row)

New x_1-row = (Current x_1-row)− $(1/5) \times$ (New s_2 −row)

New x_2-row = (Current x_2-row)− $(-3/5) \times$ (New s_2 −row)

First Iteration Table:

Basic	x_1	x_2	s_2	s_3	Solution
z	0	0	0	-1/5	17/5
x_1	1	0	0	-1/5	2/5
x_2	0	1	0	3/5	9/5
s_2	0	0	1	1	1

Since here all the z row coefficients are nonpositive.

Hence the solution is optimal.

That is $\boldsymbol{x_1 = 2/5}$, $\boldsymbol{x_2 = 9/5}$ *and* $\mathbf{min}\ \boldsymbol{z = 17/5}$

Example 2: Consider the following LP

$$\text{Maximize } z = 2x_1 + 2x_2 + 4x_3$$

Subject to

$$2x_1 + x_2 + x_3 \leq 2$$

$$3x_1 + 4x_2 + 2x_3 \geq 8$$

$$x_1, x_2, x_3 \geq 0$$

Solution 2: The equation form of the given LP is

$$\text{Maximize } z = 2x_1 + 2x_2 + 4x_3 + 0s_1 + 0s_2$$

Subject to

$$2x_1 + x_2 + x_3 + s_1 = 2$$

$$3x_1 + 4x_2 + 2x_3 - s_2 = 8$$

$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

<u>Phase I</u>

$$\text{Minimize } r = R_2$$

Subject to

$$2x_1 + x_2 + x_3 + s_1 = 2$$

$$3x_1 + 4x_2 + 2x_3 - s_2 + R_2 = 8$$

$$x_1, x_2, x_3, s_1, s_2, R_2 \geq 0$$

The system has $m = 2$ equations and $n = 6$ variables.

Set $n - m = 6 - 2 = 4$ variables equal to zero the solving for remaining 2 variables.

Simplex iteration starts at origin. So the associated set of nonbasic and basic variable is defined as

$$\text{Nonbasic (zero) variables:} (x_1, x_2, x_3, s_2)$$

$$\text{Basic variables: } (s_1, R_2)$$

Initial Simplex Table:

Basic	x_1	x_2	x_3	s_1	s_2	R_2	Solution
r	0	0	0	0	0	-1	0
s_1	2	1	1	1	0	0	2
R_2	3	4	2	0	-1	1	8

From the table, the starting solution is given by

$$r = 0, s_1 = 2, R_2 = 8$$

But this solution is not consistent. To make the r −row consistent, we use the following operation

New r-row = Old r-row + R_2-row

Basic	x_1	x_2 ↓	x_3	s_1	s_2	R_2	Solution	Ratio
r	3	4	2	0	-1	0	8	
s_1	2	1	1	1	0	0	2	2
R_2	3	4	2	0	-1	1	8	2 →

From the table, $r = 8$, $s_1 = 2$, $R_2 = 8$ which is consistent.

Since here all r −row coefficients are not nonpositive(≤ 0).

$\Rightarrow$ Solution is not optimal.

Gauss-Jordan Computations:

1. New x_2 –row = current pivot row ÷ 4
2. All other rows, including r

New r-row = (Current r-row)− (4) × (New x_2 –row)

New s_1-row = (Current s_1-row)− (1) × (New x_2 –row)

First Iteration Table:

Basic	x_1	x_2	x_3	s_1	s_2	Solution
r	0	0	0	0	0	0
s_1	5/4	0	1/2	1	1/4	0
x_2	3/4	1	1/2	0	-1/4	2

Since here all r row coefficients are nonpositive(≤ 0).

Hence solution is optimal and $\mathbf{min}\, \boldsymbol{r} = \mathbf{0}$

$\Rightarrow$ Phase I is ended here and produce starting basic feasible solution for original LP

$$\boldsymbol{x_1 = 0}, \quad \boldsymbol{x_2 = 2, x_3 = 0, s_1 = 0}$$

Phase II

Use the original objective function

Maximize $z = 2x_1 + 2x_2 + 4x_3$

We write the objective equation as

$$z - 2x_1 - 2x_2 - 4x_3 = 0$$

Initial Simplex Table

Basic	x_1	x_2	x_3	s_1	s_2	Solution
z	-2	-2	-4	0	0	0
s_1	5/4	0	1/2	1	1/4	0
x_2	3/4	1	1/2	0	-1/4	2

From the table, the starting solution is given by

$$z = 0, x_1 = 0,\ x_2 = 4,\ s_1 = 0$$

But this solution is not consistent. To make the z −row consistent, we use the following operation

New z-row = Old z-row + 2× x_2-row

↓

Basic	x_1	x_2	x_3	s_1	s_2	Solution	Ratio
z	-1/2	0	-3	0	-1/2	4	
s_1	5/4	0	1/2	1	1/4	0	0 →
x_2	3/4	1	1/2	0	-1/4	2	4

From the table, $z = 4,\ x_1 = 0,\ x_2 = 2, s_1 = 0$ which is consistent.

Since here all z −row coefficients are not nonnegative(≥ 0).

⇒ Solution is not optimal.

Gauss-Jordan Computations:

1. New x_3 –row = current pivot row ÷ 1/2
2. All other rows, including z

New z-row = (Current z-row)− (−3) × (New x_3 –row)

New x_2-row = (Current x_2-row)− (1/2) × (New x_3 –row)

First Iteration Table:

Basic	x_1	x_2	x_3	s_1	s_2	Solution
z	7	0	0	6	1	4
x_3	5/2	0	1	2	1/2	0
x_2	-1/2	1	0	-1	-1/2	2

Since here all the z row coefficients are nonnegative(≥ 0).

Hence the solution is optimal.

That is $\boldsymbol{x_1 = 0}$, $\boldsymbol{x_2 = 2}$, $\boldsymbol{x_3 = 0}$ *and* $\mathbf{max}\ \boldsymbol{z = 4}$

Special Cases In Simplex Method

1. Degeneracy
2. Alternative optima
3. Unbounded solutions
4. Infeasible (nonexisting) solutions

Degeneracy

- A tie for the minimum ratio may occur and can be broken arbitrarily.
- At least one basic variable will be zero in the next iteration.
- The condition reveals that the model has at least one redundant constraint.

Redundancy generally implies that constraints can be removed without affecting the feasible solution space.

Example 1: **(Degenerate Optimal Solution)**

$$\text{Maximize } z = 3x_1 + 9x_2$$

Subject to

$$x_1 + 4x_2 \leq 8$$

$$x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Solution 1: The equation form of the given LP is

$$\text{Maximize } z = 3x_1 + 9x_2$$

Subject to

$$x_1 + 4x_2 + s_1 = 8$$

$$x_1 + 2x_2 + s_2 = 4$$

$$x_1, x_2, s_1, s_2 \geq 0$$

We write the objective equation as

$$z - 3x_1 - 9x_2 = 0$$

Initial Basic Feasible Solution

The system has $m = 2$ equations and $n = 4$ variables.

Set $n - m = 4 - 2 = 2$ variables equal to zero the solving for remaining 2 variables.

Simplex iteration starts at origin. So the associated set of nonbasic and basic variable is defined as

Nonbasic (zero) variables: (x_1, x_2)

Basic variables: (s_1, s_2)

Initial Simplex Table:

Basic	x_1	x_2 ↓	s_1	s_2	Solution	Ratio
z	-3	-9	0	0	0	
s_1	1	4	1	0	8	2 →
s_2	1	2	0	1	4	2

From the table, the starting solution is given by

$$z = 0, s_1 = 8, s_2 = 4$$

Since here all z-row coefficients are not nonnegative(≥ 0).

$\Rightarrow$ Solution is not optimal.

Gauss-Jordan Computations:

1. New x_2 –row = current pivot row ÷ 4
2. All other rows, including z

New z-row = (Current z-row)− (-9) × (New x_2 –row)

New s_2-row = (Current s_2-row)− (2) × (New x_2 −row)

First Iteration Table:

Basic	x_1	x_2	s_1	s_2	Solution	Ratio
z	-3/4	0	9/4	0	18	
x_2	1/4	1	1/4	0	2	8
s_2	1/2	0	-1/2	1	0	0

Since here all z row coefficients are not nonnegative(≥ 0).

Hence solution is not optimal.

Gauss-Jordan Computations:

1. New x_1 −row = current pivot row ÷ $\frac{1}{2}$
2. All other rows, including z

New z-row = (Current z-row)− (-3/4) × (New x_1 −row)

New x_2-row = (Current x_2-row)− (1/4) × (New x_1 −row)

Second Iteration Table:

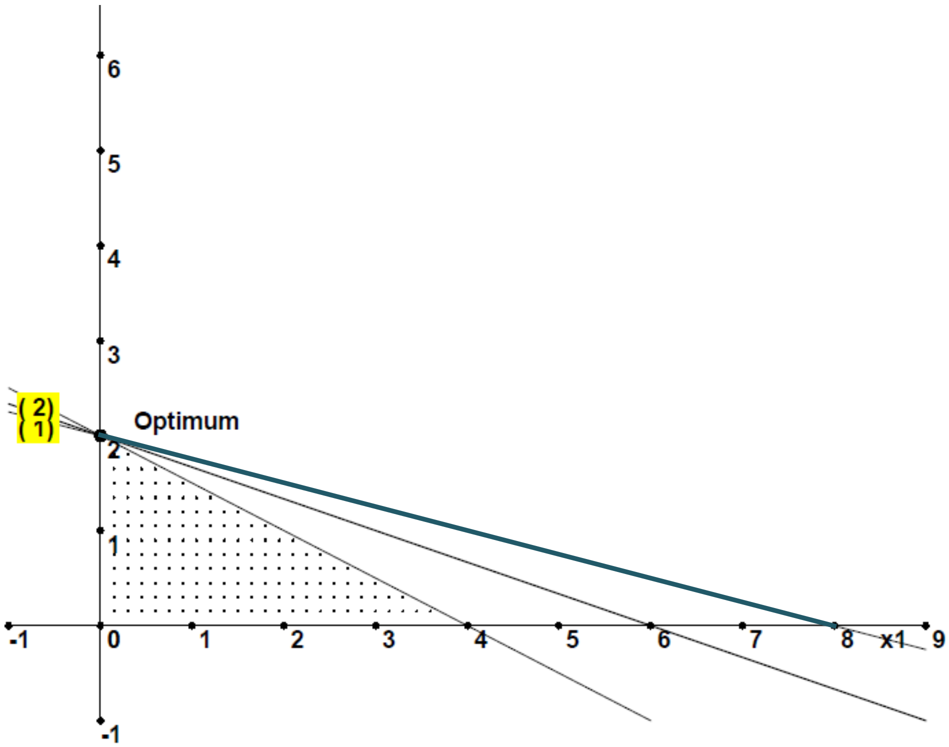
Basic	x_1	x_2	s_1	s_2	Solution
z	0	0	3/2	0	18
x_2	0	1	1/2	-1/2	2
x_1	1	0	-1	2	0

Since here all z row coefficients are nonnegative.

Hence solution is optimal.

That is $\boldsymbol{x_1 = 0}$, $\boldsymbol{x_2 = 2}$ *and* $\mathbf{max}\ \boldsymbol{z = 18}$

Graphical Solution:



Here three line passes through the optimum point $(x_1 = 0, x_2 = 2)$ and one of the constraints is redundant.

It is possible for the simplex method to enter a repetitive sequence of iterations, never improving the objective value and never satisfying the optimality condition[..\..\Documents\TORA\Problem 4, Set 3.5 A.pdf]

Alternative Optima

- The objective function is parallel to a nonredundant constraint.
- The objective function can assume the same optimal value at more than one solution point.

Example 1: **(Infinite Number of Solutions)**

$$\text{Maximize } z = 2x_1 + 4x_2$$

Subject to
$$x_1 + 2x_2 \leq 5$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Solution 1: The equation form of the given LP is

$$\text{Maximize } z = 2x_1 + 4x_2$$

Subject to

$$x_1 + 2x_2 + s_1 = 5$$
$$x_1 + x_2 + s_2 = 4$$
$$x_1, x_2, s_1, s_2 \geq 0$$

We write the objective equation as

$$z - 2x_1 - 4x_2 = 0$$

Initial Basic Feasible Solution

The system has $m = 2$ equations and $n = 4$ variables.

Set $n - m = 4 - 2 = 2$ variables equal to zero the solving for remaining 2 variables.

Simplex iteration starts at origin. So the associated set of nonbasic and basic variable is defined as

Nonbasic (zero) variables: (x_1, x_2)

Basic variables: (s_1, s_2)

Initial Simplex Table:

Basic	x_1 ↓...	x_2 ↓	s_1	s_2	Solution	Ratio
z	-2	-4	0	0	0	
s_1	1	2	1	0	5	2.5 →
s_2	1	1	0	1	4	4

From the table, the starting solution is given by

$$z = 0, s_1 = 5, s_2 = 4$$

Since here all z-row coefficients are not nonnegative(≥ 0).

$\Rightarrow$ Solution is not optimal.

Gauss-Jordan Computations:

1. New x_2 –row = current pivot row ÷ 2
2. All other rows, including z

New z-row = (Current z-row)$-$ $(-4) \times$ (New x_2 –row)

New s_2-row = (Current s_2-row)$-$ $(1) \times$ (New x_2 –row)

First Iteration Table:

Basic	x_1 ↓	x_2	s_1	s_2	Solution	Ratio
z	0	0	2	0	10	
x_2	1/2	1	1/2	0	5/2	5
s_2	1/2	0	-1/2	1	3/2	3 →

Since here all z row coefficients are nonnegative(≥ 0).

Hence solution is optimal.

That is $\boldsymbol{x_1 = 0}$, $\boldsymbol{x_2 = 5/2}$ *and* $\mathbf{max}\ \boldsymbol{z} = \mathbf{10}$

Alternate Solution:

The coefficient of nonbasic x_1 is zero, indicating that x_1 can enter the basic solution without changing the value of z, but causing a change in the value of variables.

Gauss-Jordan Computations:

1. New x_1 –row = current pivot row $\div \frac{1}{2}$

2. All other rows, including z

New z-row = Current z-row

New x_2-row = (Current x_2-row)$-$ (1/2) $\times$ (New x_1 –row)

Second Iteration Table:

Basic	x_1	x_2	s_1	s_2	Solution	Ratio
z	0	0	2	0	10	
x_2	0	1	1	-1	1	
x_1	1	0	-1	2	3	

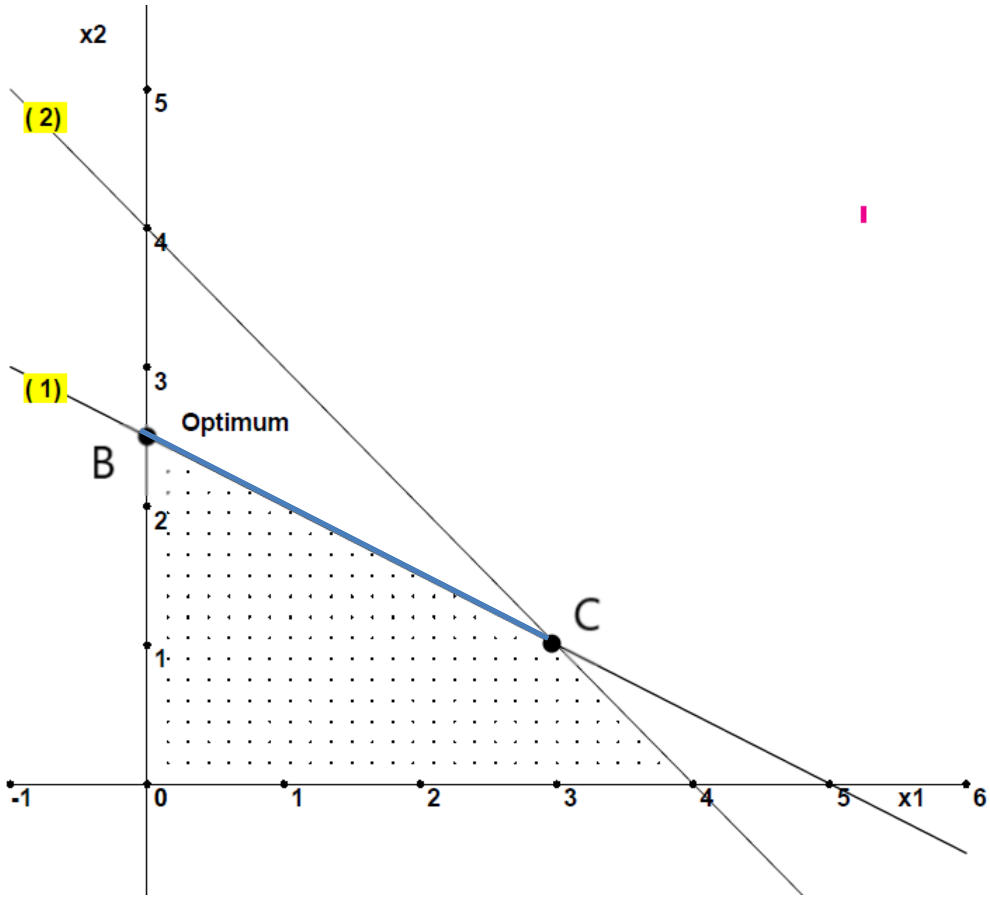
Since here all z row coefficients are nonnegative.

Hence solution is optimal.

That is $\boldsymbol{x_1 = 3}$, $\boldsymbol{x_2} = 1$ *and* $\mathbf{max}\ \boldsymbol{z} = \mathbf{10}$

Iteration 2 does just that – letting x_1 enter the basic solution and forcing s_2 to leave the basis. The new solution point occurs at C$(x_1 = 3, x_2 = 1, z = 10)$.

Graphical Solution:



The simplex method determines only the two corner points. Mathematically, we can determine all the points (x_1, x_2) on the line segment BC.

Unbounded Solution

- The values of the variables may be increased indefinitely without violating any of the constraints.
- The objective value may increase (maximization case) or decrease (minimization case) indefinitely.
- Both the solution space and the optimum objective value are unbounded.

Example 1: **(Unbounded Objective Value)**

$$\text{Maximize } z = 2x_1 + x_2$$

Subject to

$$x_1 - x_2 \leq 10$$

$$2x_1 \leq 40$$

$$x_1, x_2 \geq 0$$

Solution 1: The equation form of the given LP is

$$\text{Maximize } z = 2x_1 + x_2$$

Subject to

$$x_1 - x_2 + s_1 = 10$$

$$2x_1 + s_2 = 40$$

$$x_1, x_2, s_1, s_2 \geq 0$$

We write the objective equation as

$$z - 2x_1 - x_2 = 0$$

<u>Initial Basic Feasible Solution</u>

The system has $m = 2$ equations and $n = 4$ variables.

Set $n - m = 4 - 2 = 2$ variables equal to zero the solving for remaining 2 variables.

Simplex iteration starts at origin. So the associated set of nonbasic and basic variable is defined as

$$\text{Nonbasic (zero) variables:}(x_1, x_2)$$

$$\text{Basic variables: } (s_1, s_2)$$

<u>Initial Simplex Table</u>:

Basic	x_1 ↓	x_2	s_1	s_2	Solution	Ratio
z	-2	-1	0	0	0	
s_1	1	-1	1	0	10	10 →
s_2	2	0	0	1	40	20

From the table, the starting solution is given by

$$z = 0, s_1 = 10, s_2 = 40$$

Since here all z-row coefficients are not nonnegative(≥ 0).

$\Rightarrow$ Solution is not optimal.

Gauss-Jordan Computations:

1. New x_1 –row = current pivot row
2. All other rows, including z

New z-row = (Current z-row)$-$ $(-2) \times$ (New x_1 –row)

New s_2-row = (Current s_2-row)$-$ $(2) \times$ (New x_1 –row)

<u>First Iteration Table:</u>

Basic	x_1	x_2 ↓	s_1	s_2	Solution	Ratio
z	0	-3	2	0	20	
x_1	1	-1	1	0	10	-
s_2	0	2	-2	1	20	10 →

Since here all z row coefficients are not nonnegative(≥ 0).

Hence solution is not optimal.

Gauss-Jordan Computations:

1. New x_2 –row = current pivot row ÷ 2
2. All other rows, including z

New z-row = (Current z-row)$-$ $(-3) \times$ (New x_2 –row)

New x_1-row = (Current x_1-row)$-$ $(-1) \times$ (New x_2 –row)

Second Iteration Table:

↓

Basic	x_1	x_2	s_1	s_2	Solution
z	0	0	-1	3/2	50
x_1	1	0	0	1/2	20
x_2	0	1	-1	1/2	10

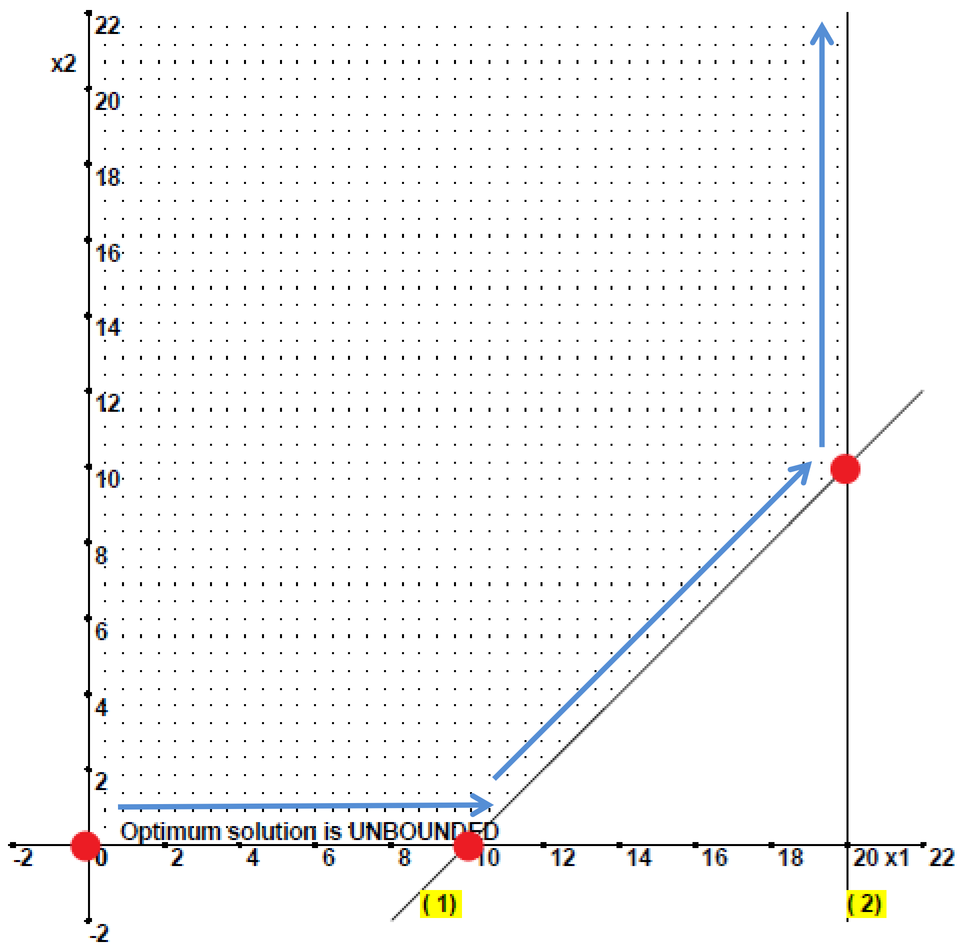
Since here all z row coefficients are not nonnegative(≥ 0).

Hence solution is not optimal.

Now s_1 is entering variable, however all the pivot column coefficients are negative. This means that there is no leaving variable.

$\Rightarrow$ Given LP has unbounded solution.

Graphical Solution:



From the graph, the objective value can be increased indefinitely in the direction of x_2.

Infeasible Solution

This situation occurs when there is at least one artificial variable have <u>positive</u> value in the optimum iteration.

$$\text{Maximize } z = 2x_1 + 5x_2$$

Subject to

$$3x_1 + 2x_2 \geq 6$$

$$2x_1 + x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

Solution: The equation form of the given LP is

$$\text{Maximize } z = 2x_1 + 5x_2 - MR_1$$

Subject to

$$3x_1 + 2x_2 - s_1 + R_1 = 6$$

$$2x_1 + x_2 + s_2 = 2$$

$$x_1, x_2, s_1, s_2, R_1 \geq 0$$

We write the objective equation as

$$z - 2x_1 - 5x_2 + MR_1 = 0$$

<u>**Initial Basic Feasible Solution**</u>

The system has $m = 2$ equations and $n = 5$ variables.

Set $n - m = 5 - 2 = 3$ variables equal to zero the solving for remaining 2 variables.

Simplex iteration starts at origin. So the associated set of nonbasic and basic variable is defined as

Nonbasic (zero) variables:(x_1, x_2, s_1)

Basic variables: (R_1, s_2)

Initial Simplex Table:

Basic	x_1	x_2	s_1	R_1	s_2	Solution
z	-2	-5	0	M	0	0
R_1	3	2	-1	1	0	6
s_2	2	1	0	0	1	2

From the table, the starting solution is given by

$$z = 0, R_1 = 6, s_2 = 2$$

But this solution is not consistent. To make the z −row consistent, we use the following operation

New z-row = Old z-row + ($-$M × R_1-row)

↓

Basic	x_1	x_2	s_1	R_1	s_2	Solution	Ratio
z	-2-3M	-5-2M	M	0	0	-6M	
R_1	3	2	-1	1	0	6	2
s_2	2	1	0	0	1	2	1 →

From the table, the starting solution is given by

$z = -6M, R_1 = 6, s_2 = 2$ which is consistent.

But here all $z -$ row coefficients are not nonnegative. Hence solution can be improved.

Gauss-Jordan Computations:

1. New x_1 –row = current pivot row ÷ 2
2. All other rows, including z

New z-row = (Current z-row)− (-2-3M) × (New x_1 –row)

New R_1-row = (Current R_1-row)− (3) × (New x_1 –row)

First Iteration Table:

Basic	x_1	x_2 ↓	s_1	R_1	s_2	Solution	Ratio
z	0	-4-M/2	M	0	1+3M/2	2-3M	
R_1	0	1/2	-1	1	-3/2	3	6
x_1	1	1/2	0	0	1/2	1	2 →

All z-row coefficients are not nonnegative.

⇒ Solution is not optimal.

Gauss-Jordan Computations:

1. New x_2 –row = current pivot row ÷ $\frac{1}{2}$
2. All other rows, including z

New z-row = (Current z-row)− $\left(-4-\frac{M}{2}\right)$ × (New x_2 –row)

New R_1-row = (Current R_1-row)− (1/2) × (New x_2 –row)

Second Iteration Table:

Basic	x_1	x_2	s_1	R_1	s_2	Solution
z	8+M	0	M	0	5+2M	10-2M
R_1	-1	0	-1	1	-2	2
x_2	2	1	0	0	1	2

All z-row coefficients are nonnegative (≥ 0)

$\Rightarrow$ Solution is optimal.

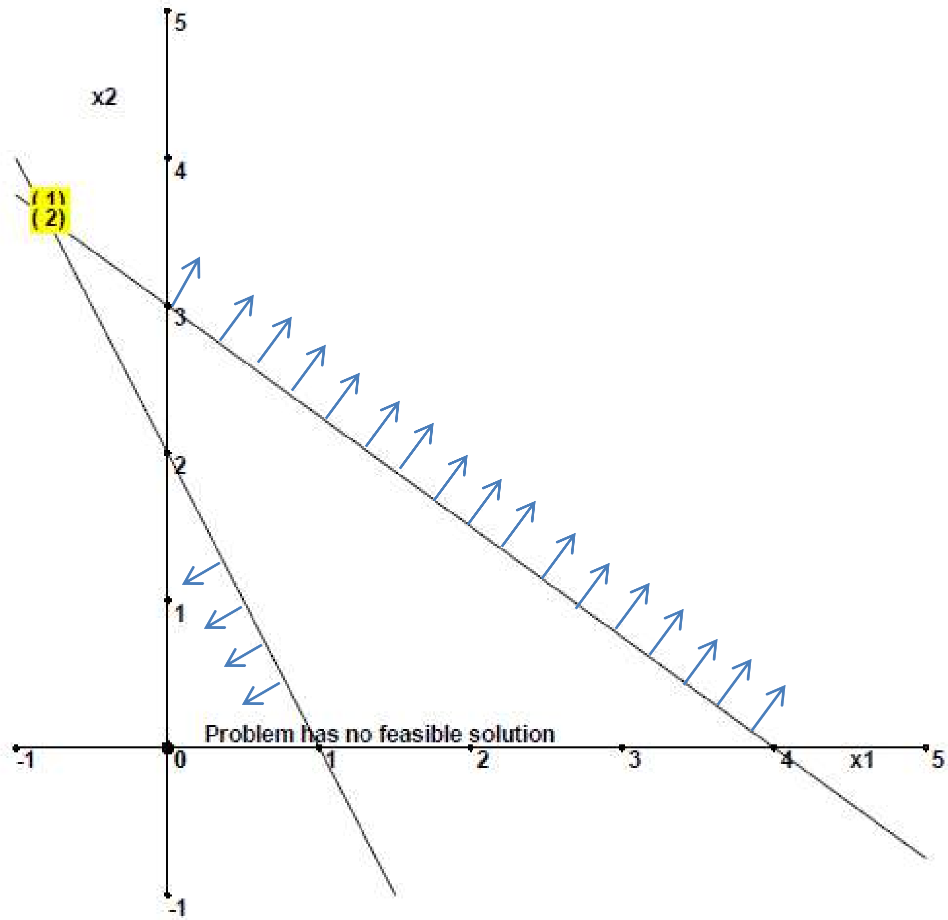
However the solution is infeasible because the artificial variable R_1 assumes a positive value.

Having positive value for the artificial variable R_1 is the same as regarding the constraints

$$3x_1 + 2x_2 \geq 6 \text{ or } 3x_1 + 2x_2 \leq 6$$

which violates the constraints of the original model.

Graphical Solution:



2. Integer Programming

A linear programming problem in which some or all of the variables must take nonnegative integer(discrete) values. There are certain decision problems where decision variables make sense only if they have integer values in the solution. When all the variables are constrained to be integers, it is called a **pure (or all)** integer programming problem. In case only some of the variables are restricted to have integer values, the problem is said to be a **mixed** integer programming problem. Capital budgeting, construction scheduling, plant location and size, routing and shipping schedule, capacity expansion etc are few problems which demonstrate the areas of application of integer programming.

Illustrative Applications

The applications generally fall into two categories: direct and transformed.

Direct category: In this category, the variables are naturally integer and may assume binary (0 or 1) or general discrete values. For example, the problem may involve determining whether or not a project is selected for execution (binary) or finding the optimal number of machines needed to perform a task (general discrete value).

Transformed category: In this category, the original problem may not involve any integer variables, is analytically intractable. In this case, auxiliary integer variables (usually binary) are used make it tractable.

1. Capital Budgeting

It deals with the decisions regarding whether or not investments should be made in individual projects. The decision is made under limited-budget considerations as well as priorities in the execution of the projects.

Example 1: (Project Selection)
Five projects are being evaluated over a 3-year planning horizon. The following table gives the expected returns for each project and the associated yearly expenditures.

	Expenditures (million $)/year			
Project	1	2	3	Returns (million $)
1	5	1	8	20
2	4	7	10	40
3	3	9	2	20
4	6	4	1	15
5	8	6	10	30
Available funds(million $)	25	25	25	

Which projects should be selected over the 3-year horizon?

IP Formulation:

The problem reduces to a "yes-no" decision for each project. Define the binary variable x_j as

$$x_j = \begin{cases} 1, & \text{if project j is selected} \\ 0, & \text{if project j is not selected} \end{cases}$$

The ILP model is

$$\text{Maximize } z = 20x_1 + 40x_2 + 20x_3 + 15x_4 + 30x_5$$

Subject to

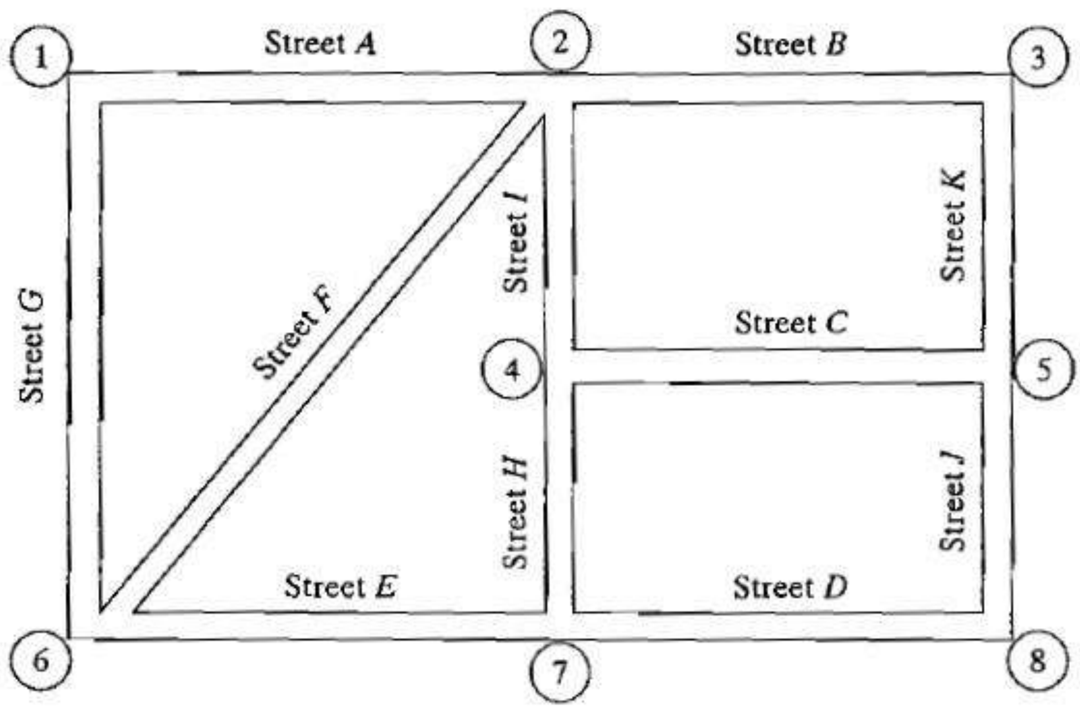
$$5x_1 + 4x_2 + 3x_3 + 7x_4 + 8x_5 \leq 25$$
$$x_1 + 7x_2 + 9x_3 + 4x_4 + 6x_5 \leq 25$$
$$8x_1 + 10x_2 + 2x_3 + x_4 + 10x_5 \leq 25$$
$$x_1, x_2, x_3, x_4, x_5 = (0,1)$$

2. Set –Covering Problem

In this class of problems, overlapping services are offered by a number of installations to a number of facilities. The objective is to determine the minimum number of installations that will cover (i.e. satisfy the service needs) of each facility. For example, water treatment plants can be constructed at various locations, with each plant serving different set of cities. The overlapping arises when a given city can receive service from more than one plant.

Example 1: (Installing Security Telephones)

To promote on-campus safety, the U of A Security Department is in the process of installing emergency telephones at selected locations. The department wants to install the minimum number of telephones, provided that each of the campus main streets is served by at least one telephone. The figure maps the principal streets A to K on campus.



It is logical to place the telephones at street intersections so that each telephone will serve at least two streets.

From figure, the layout of the streets requires a maximum of eight telephone locations.

Define

$$x_j = \begin{cases} 1, & \text{if a telephone j is installed in location j} \\ 0, & \text{if a telephone j is not installed in location j} \end{cases}$$

The constraints require installing at least one telephone on each 11 streets (A to K). Thus the model becomes

$$\text{Minimize } z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8$$

Subject to

$$x_1 + x_2 \geq 1 \quad \text{(for street A)}$$
$$x_2 + x_3 \geq 1 \quad \text{(for street B)}$$
$$x_4 + x_5 \geq 1 \quad \text{(for street C)}$$
$$x_7 + x_8 \geq 1 \quad \text{(for street D)}$$
$$x_6 + x_7 \geq 1 \quad \text{(for street E)}$$
$$x_2 + x_6 \geq 1 \quad \text{(for street F)}$$
$$x_1 + x_6 \geq 1 \quad \text{(for street G)}$$
$$x_4 + x_7 \geq 1 \quad \text{(for street H)}$$
$$x_2 + x_4 \geq 1 \quad \text{(for street I)}$$
$$x_5 + x_8 \geq 1 \quad \text{(for street J)}$$
$$x_3 + x_5 \geq 1 \quad \text{(for street K)}$$

$$x_j = (0,1), j = 1,2, \dots ,8$$

3. Fixed -Charge Problem

The fixed-charge problem deals with the situations in which the economic activity incurs two types of costs: an initial "flat" fee that must incurred to start the activity and a variable cost that is directly proportional to the level of the activity. For example, the initial tooling of a machine prior to starting production incurs a fixed setup cost regardless of how many units are manufactured. Once the setup is done, the cost of labor and material is proportional to the amount produced.

Example: (Choosing a telephone company)
I have been approached by three telephone companies to subscribe their long distance service in the U.S. Mabell will charge a flat \$16 per month plus \$0.25 a minute. Pabell will charge \$25 a month but will reduce the per minute cost to \$0.21. As for Babybell, the flat monthly charge is \$18 and the cost per minute is \$0.22. I usually make an average of 200 minutes of long distance calls a month. Assuming that I do not pay the flat monthly fees unless I make calls and that I can apportion my calls among all three companies to minimize my monthly telephone bill?

Solution: Define

$$x_1 = \text{ Mabell long distance minutes per month}$$

$$x_2 = \text{ Pabell long distance minutes per month}$$
$$x_3 = \text{ Babybell long distance minutes per month}$$
$$y_1 = 1 \text{ if } x_1 > 0 \text{ and } y_1 = 0 \text{ if } x_1 = 0$$
$$y_2 = 1 \text{ if } x_2 > 0 \text{ and } y_2 = 0 \text{ if } x_2 = 0$$
$$y_3 = 1 \text{ if } x_3 > 0 \text{ and } y_3 = 0 \text{ if } x_3 = 0$$

We can ensure that y_j will equal 1 if x_j is positive by using the constraint

$$x_j \leq My_j, j = 1,2,3$$

The value of M should be selected sufficiently large so as not restrict to the variable x_j artificially. Because I make 200 minutes of calls of a month, then $x_j \leq 200$ for all *j*, and it is safe to select $M = 200$.

The complete model is

$$\text{Minimize } z = 0.25x_1 + 0.21x_2 + 0.22x_3 + 16y_1 + 25y_2 + 18y_3$$

Subject to

$$x_1 + x_2 + x_3 = 200$$
$$x_1 \leq 200y_1$$
$$x_2 \leq 200y_2$$
$$x_3 \leq 200y_3$$
$$x_1, x_2, x_3 \geq 0$$
$$y_1, y_2, y_3 = (0,1)$$

4. Either-Or Constraints

In this section, we deal with the models in which constraints are not satisfied simultaneously (either-or) or are dependent (if-then), again using binary variable.

Example: (Job Sequencing Model)
Jobco uses single machine to process three jobs. Both the processing time and due date (in days) for each job are given in the following table. The due dates are measured from zero, the assumed start time of the first job

Job	Processing times (days)	Due date (days)	Late Penalty ($/day)
1	5	25	19
2	20	22	12
3	15	35	34

Determine the minimum late penalty sequencing for processing three jobs.

Solution: Define

$$x_j = \text{Start date in days for each job } j \text{ (measured from zero)}$$

The problem has two types of constraints: the noninterference constraints (guaranteeing that no two jobs are processed concurrently) and the due-date constraints.

<u>Noninterference constraints:</u>

Two jobs i and j with processing time p_i and p_j will not be processed concurrently if either $x_i \geq x_j + p_j$ or $x_j \geq x_i + p_i$, depending on whether job j precedes job i, or vice versa. Because all mathematical programs deal with simultaneous constraints only, we transform the either-or constraints by introducing the following auxiliary binary variable:

$$y_{ij} = \begin{cases} 1, & \text{if i precedes j} \\ 0, & \text{if j precedes i} \end{cases}$$

For M sufficiently large, the either-or constraints is converted to the following two simultaneous constraints

$$My_{ij} + (x_i - x_j) \geq p_j \text{ and } M(1 - y_{ij}) + (x_j - x_i) \geq p_i$$

The conversion guarantees that only one of the two constraints can be active at any one time.

If $y_{ij} = 0$ then the first constraint is active and the second is redundant.

If $y_{ij} = 1$ then the first constraint is redundant and the second is active.

<u>Due date Constraints:</u>

Given that d_j is the due date for job j, let s_j be unrestricted variable. Then the associated constraint is

$$x_j + p_j + s_j = d_j$$

If $s_j \geq 0$, the due date is met and if $s_j < 0$, a late penalty applies. Using the substitution $s_j = s_j^+ - s_j^-, s_j^+, s_j^- \geq 0$

The constraint becomes

$$x_j + s_j^+ - s_j^- = d_j - p_j$$

The late penalty cost is proportional to s_j^-

The model for the given problem is

$$\text{Minimize } z = 19s_1^- + 12s_2^- + 34s_3^-$$

Subject to

$$x_1 - x_2 + My_{12} \geq 20$$

$$-x_1 + x_2 - My_{12} \geq 5 - M$$

$$x_1 - x_3 + My_{13} \geq 15$$

$$-x_1 + x_3 - My_{13} \geq 5 - M$$

$$x_2 - x_3 + My_{23} \geq 15$$

$$-x_2 + x_3 - My_{23} \geq 20 - M$$

$$x_1 + s_1^+ - s_1^- = 25 - 5$$

$$x_2 + s_2^+ - s_2^- = 22 - 20$$

$$x_3 + s_3^+ - s_3^- = 35 - 15$$

$$x_1, x_2, x_3, s_1^+, s_1^-, s_1^+, s_2^+, s_2^-, s_3^+, s_3^- \geq 0$$

$$y_1, y_2, y_3 = (0,1)$$

Branch and Bound Algorithm

Step 1: Initialization

Obtain the optimal solution of the given problem ignoring integer restriction on the variables.

i. If the solution to this LP is unbounded or infeasible then the solution to the given all-integer programming problem is also unbounded or infeasible.
ii. If the solution satisfies the integer restrictions, the optimal solution to the given integer programming problem has been obtained. If one or more basic variables do not satisfy integer requirement, then go to step 2.

Let the optimal value of objective function of LP-A be Z_1. This value provides an initial upper bound on objective function value and is denoted by Z_U.

iii. Find a feasible solution by rounding off each variable value. The value of objective function so obtained is used as a lower bound and is denoted by Z_L.

Step 2: Branching step

i. Let x_k be one basic variable which does not have an integer value and also has the largest fractional value.

ii. Branch (or partition) the LP-A into two new LP sub-problems (also called nodes) LP-B and LP-C based on integer values of x_k that are immediately above and below its non-integer value. That is, it is partitioned by adding two mutually exclusive constraints:

$$x_k \leq [x_k] \text{ and } x_k \geq [x_k] + 1$$

to the original LP problem. Here $[x_k]$ is the integer portion of the current non-integer value of the variable x_k.

Step 3: Bound step

Obtain the optimal solution of sub-problems LP-B and LP-C. Let the optimal value of the objective function of LP-B be Z_2 and that of LP-C be Z_3. The best integer solution value becomes the lower bound on the integer problem objective function value (Initially this is the rounded off value). Let the lower bound be denoted by Z_L.

Step 4: Fathoming step

Examine the solution of both LP-B and LP-C

i. If a sub-problem yields an infeasible solution then terminate the branch.

ii. If a sub-problem yields a feasible solution but not an integer solution then return to step 2.

iii. If a sub-problem yields a feasible integer solution, examine the value of the objective function. If this value is equal to the upper bound, an optimal solution has been reached. But if it is not equal to the upper bound but exceeds the lower bound, this value is considered as new upper bound and return to step 2. Finally, if it is less than the lower bound, terminate this branch.

Step 5: Termination

The procedure of branching and bounding continues until no further sub-problem remains to be examined. At this stage, the integer solution corresponding to the current lower bound is the optimal all –integer programming solution.

Example 1: Solve the following integer programming problem using the branch and bound method.

$$\text{Maximize } z = 2x_1 + 3x_2$$

Subject to

$$6x_1 + 5x_2 \leq 25$$

$$x_1 + 3x_2 \leq 10$$

$$x_1, x_2 \text{ non-negative integer}$$

Solution 1: Relaxing the integer requirement, the optimal non-integer solution is obtained by graphical method as shown in fig is $x_1 = 1.92, x_2 = 2.69,$ and max $Z_1 = 11.91$

The value of Z_1 represents initial lower bound: $Z_L = 11$

Select variable x_2 for branching.

Then to divide the given problem into two sub-problems and to remove fractional part of $x_2 = 2.69$, two new constraints $x_2 \leq 2$ and $x_2 \geq 3$ are created and added to the constraints of original problem as follows:

Sub-problem B: Maximize $z = 2x_1 + 3x_2$

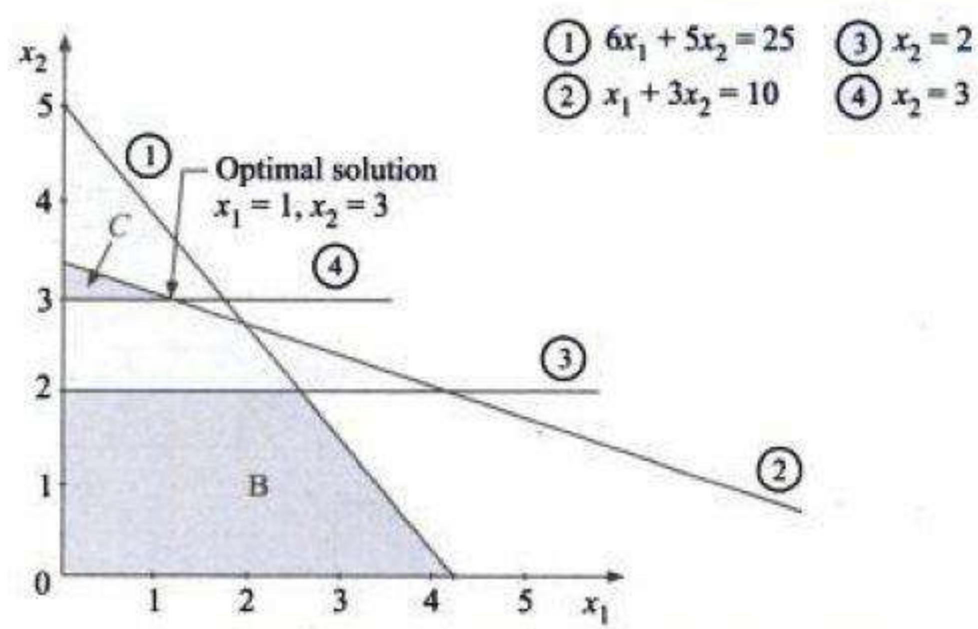
Subject to i) $6x_1 + 5x_2 \leq 25$ ii) $x_1 + 3x_2 \leq 10$ iii) $x_2 \leq 2$

$$x_1, x_2 \text{ non-negative integer}$$

Sub-problem C: Maximize $z = 2x_1 + 3x_2$

Subject to i) $6x_1 + 5x_2 \leq 25$ ii) $x_1 + 3x_2 \leq 10$ iii) $x_2 \geq 3$

x_1, x_2 non-negative integer



Solution sub-problem B

$$x_1 = 2.5,\ x_2 = 2,\ \text{and max } Z_2 = 11$$

Solution sub-problem C

$$x_1 = 1,\ x_2 = 3,\ \text{and max } Z_3 = 11 = Z_L$$

In the solution space LP sub-problem C, variables x_1 and x_2 are integers, so there is no need to branch this sub-problem further. The value max Z_L=11 is a lower bound on the maximum value of Z for future solutions.

LP sub-problem B is further subdivided into two new sub-problems D and E by taking $x_1 = 2.5$. Adding two new constraints $x_1 \leq 2$ and $x_1 \geq 3$.

Sub-problem D: Maximize $z = 2x_1 + 3x_2$

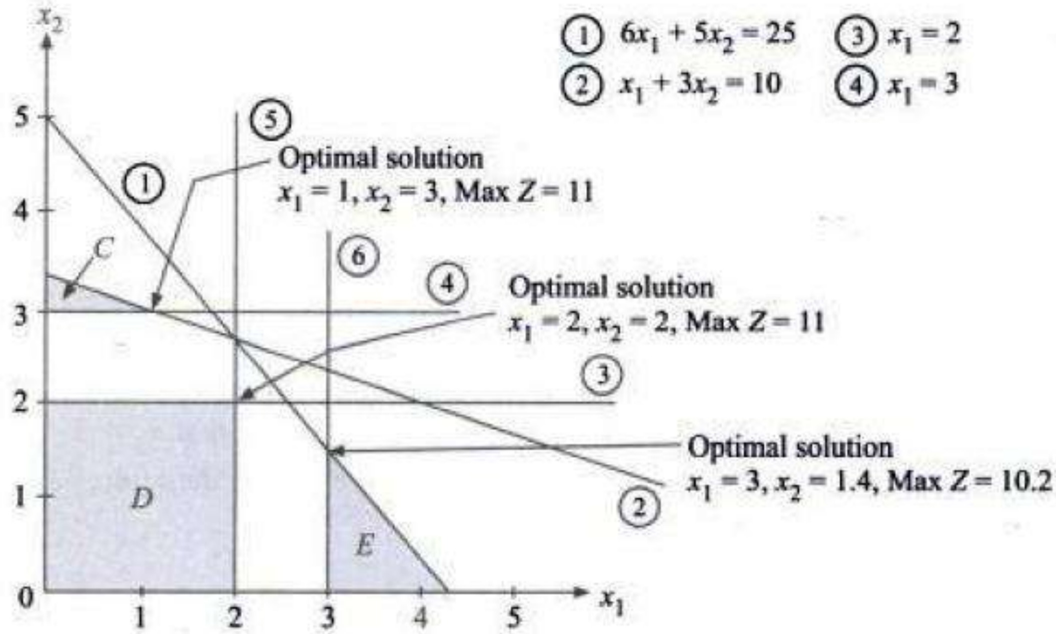
Subject to i) $6x_1 + 5x_2 \leq 25$ ii) $x_1 + 3x_2 \leq 10$ iii) $x_2 \leq 2$ iv) $x_1 \leq 2$

x_1, x_2 non-negative integer

Sub-problem E: Maximize $z = 2x_1 + 3x_2$

Subject to i) $6x_1 + 5x_2 \leq 25$ ii) $x_1 + 3x_2 \leq 10$ iii) $x_2 \leq 2$ iv) $x_1 \geq 3$

x_1, x_2 non-negative integer



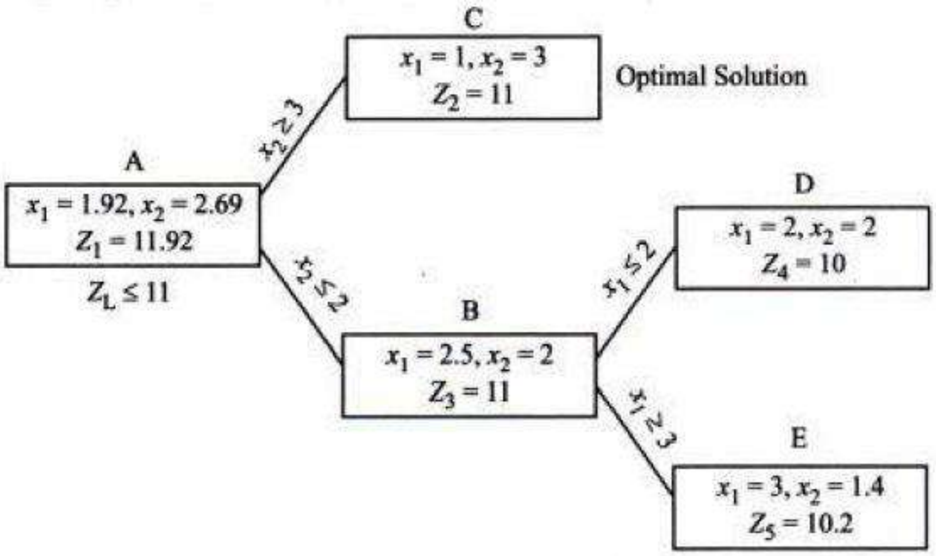
Sub-problems D and E are solved graphically.

Solution of sub-problem D: $x_1 = 2, x_2 = 2$ and max $Z_4 = 10$

Solution of sub-problem E: $x_1 = 3, x_2 = 1.4$ and max $Z_5 = 10.2$

The solution of LP sub-problem D is integer feasible but is inferior to the best available solution of LP sub-problem E. Hence the value of lower bound $Z_L = 11$ remains unchanged and sub-problem D is not considered for further division.

Since the solution of sub-problem E is non-integer, it can be further branched with x_2 as the branching variable. But the value of its objective function $(= 10.2)$ is inferior to the lower bound and hence this does not give a solution better than the one already obtained. Thus sub-problem E is also not considered for further branching. Hence, the best available solution corresponding to sub-problem C is the integer optimal solution: $x_1 = 1,\ x_2 = 3,$ and max $Z = 11$ of the given integer LP problem. The entire branch and bound procedure for the given problem is shown below:



Cutting-Plane Algorithm

Example 1: Solve the following integer programming problem using cutting plane algorithm.

$$\text{Maximize } z = 7x_1 + 10x_2$$

Subject to

$$-x_1 + 3x_2 \le 6$$

$$7x_1 + x_2 \le 35$$

$$x_1, x_2 \ge 0 \text{ and are integers}$$

Solution 1: Obtain the optimal solution to the LP problem ignoring the integer restriction by the simplex method.

The equation form of the given LP is

$$\text{Maximize } z = 7x_1 + 10x_2 + 0x_3 + 0x_4$$

Subject to

$$-x_1 + 3x_2 + x_3 = 6$$

$$7x_1 + x_2 + x_4 = 35$$

$$x_1, x_2, s_1, s_2 \ge 0$$

We write the objective equation as

$$z - 7x_1 - 10x_2 = 0$$

Initial Basic Feasible Solution

The system has $m = 2$ equations and $n = 4$ variables.

Set $n - m = 4 - 2 = 2$ variables equal to zero the solving for remaining 2 variables.

Simplex iteration starts at origin. So the associated set of nonbasic and basic variable is defined as

Nonbasic (zero) variables: (x_1, x_2)

Basic variables: (x_3, x_4)

Initial Simplex Table:

Basic	z	x_1	x_2 ↓	x_3	x_4	Solution	Ratio
z	1	-7	-10	0	0	0	
x_3	0	-1	[3]	1	0	6	2 →
x_4	0	7	1	0	1	35	35

From the table, the starting solution is given by

$z = 0, x_3 = 6, x_4 = 35$

The solution is not optimal here. This solution can be improvised.

First Iteration Table:

Basic	z	x_1 ↓	x_2	x_3	x_4	Solution	Ratio
z	1	-31/3	0	10/3	0	20	
x_2	0	-1/3	1	1/3	0	2	-
x_4	0	[22/3]	0	-1/3	1	33	9/2 →

Second Iteration Table:

Basic	z	x_1	x_2	x_3	x_4	Solution
z	1	0	0	63/22	31/22	$66\frac{1}{2}$
x_2	0	0	1	7/22	1/22	$3\frac{1}{2}$
x_1	0	1	0	-1/22	3/22	$4\frac{1}{2}$

The optimum continuous solution is

$$z = 66\tfrac{1}{2}, x_1 = 4\tfrac{1}{2}, x_2 = 3\tfrac{1}{2}, x_3 = 0, \;\; x_4 = 0.$$

The cut is developed under the assumption that all the variables (including the slacks x_3 and x_4) are integer. Note also that because the original objective coefficients are integer in example, the value of z is integer as well.

The information in the optimum tableau can be explicitly written as

$$z + \tfrac{63}{22}x_3 + \tfrac{31}{22}x_4 = 66\tfrac{1}{2} \qquad (z - \text{equation})$$

$$x_2 + \tfrac{7}{22}x_3 + \tfrac{1}{22}x_4 = 3\tfrac{1}{2} \qquad (x_2 - \text{equation})$$

$$x_1 - \tfrac{1}{22}x_3 + \tfrac{3}{22}x_4 = 4\tfrac{1}{2} \qquad (x_1 - \text{equation})$$

A constraint equation can be used as a **source row** for generating cut, provided its right hand side is fractional. Also note that the z-equation can be used as a source row because z happens to be integer as well.

Generating a cut

Factor out all the non-integer coefficients of the equation into an integer value and a fractional component, provided that the resulting fractional component is strictly positive.

The factoring of the z-equation yields

$$z + \left(2 + \frac{19}{22}\right)x_3 + \left(1 + \frac{9}{22}\right)x_4 = \left(66 + \frac{1}{2}\right)$$

Moving all the integer components to the left-hand side and all the fractional components to the right-side, we get

$$z + 2x_3 + 1x_4 - 66 = -\frac{19}{22}x_3 - \frac{9}{22}x_4 + \frac{1}{2} \qquad (1)$$

Because x_3 and x_4 are non-negative and all fractions are originally strictly positive, the right-hand side must satisfy the following inequality.

$$-\frac{19}{22}x_3 - \frac{9}{22}x_4 + \frac{1}{2} \leq \frac{1}{2} \qquad (2)$$

Because the left-hand side in equation(1), $z + 2x_3 + 1x_4 - 66$ is an integer value by construction, the right-hand side, $-\frac{19}{22}x_3 - \frac{9}{22}x_4 + \frac{1}{2}$ must also be integer. It then follows that (2) can be replaced with the inequality:

$$-\frac{19}{22}x_3 - \frac{9}{22}x_4 + \frac{1}{2} \leq 0$$

Because an integer value $\leq \frac{1}{2}$ must necessarily be ≤ 0.

The last inequality is the desired cut and it represents a necessary condition for obtaining an integer solution. It is also referred to as the **fractional cut** because all its coefficients are fractions.

x_1-row→ $x_1 + \left(-1 + \frac{21}{22}\right)x_3 + \left(0 + \frac{3}{22}\right)x_4 = \left(4 + \frac{1}{2}\right)$

The associated cut is

$$-\frac{21}{22}x_3 - \frac{3}{22}x_4 + \frac{1}{2} \leq 0$$

Similarly,

x_2-row→ $x_2 + \left(0 + \frac{7}{22}\right)x_3 + \left(0 + \frac{1}{22}\right)x_4 = \left(3 + \frac{1}{2}\right)$

The associated cut is

$$-\frac{7}{22}x_3 - \frac{1}{22}x_4 + \frac{1}{2} \leq 0$$

Any one of the three cuts given above can be used in the first iteration of the cutting-plane algorithm.

Arbitrarily selecting the cut generated from the x_2-row, we can write it in equation form as

$$-\frac{7}{22}x_3 - \frac{1}{22}x_4 + s_1 = -\frac{1}{2}, s_1 \geq 0 \qquad \text{(Cut I)}$$

This constraint is added to the LP optimum tableau as follows:

Basic	z	x_1	x_2	x_3 ↓	x_4	s_1	Solution
z	1	0	0	63/22	31/22	0	$66\frac{1}{2}$
x_2	0	0	1	7/22	1/22	0	$3\frac{1}{2}$
x_1	0	1	0	-1/22	3/22	0	$4\frac{1}{2}$
s_1	0	0	0	-7/22	-1/22	1	-1/2 →
Ratio	-	-	-	11/7	11	-	

The solution is optimal but infeasible. We apply the dual simplex method.

Basic	z	x_1	x_2	x_3	x_4	s_1	Solution
z	1	0	0	0	1	9	62
x_2	0	0	1	0	1	3	3
x_1	0	1	0	0	1/7	-1/7	$4\frac{4}{7}$
x_3	0	0	0	1	1/7	-22/7	$1\frac{4}{7}$

The last solution is still non-integer in x_1 and x_3. Let us arbitrarily select x_1 as the next source row-that is

$$x_1 + \left(0 + \frac{1}{7}\right)x_4 + \left(-1 + \frac{6}{7}\right)s_1 = \left(4 + \frac{4}{7}\right)$$

The associated cut is

$$-\frac{1}{7}x_4 - \frac{6}{7}s_1 + s_2 = -\frac{4}{7}, \qquad s_2 \geq 0 \qquad \text{(Cut II)}$$

Basic	z	x_1	x_2	x_3	x_4	s_1 ↓	s_2	Solution
z	1	0	0	0	1	9	0	62
x_2	0	0	1	0	1	3	0	3
x_1	0	1	0	0	1/7	-1/7	0	$4\frac{4}{7}$
x_3	0	0	0	1	1/7	-22/7	0	$1\frac{4}{7}$
s_2	0	0	0	0	-1/7	-6/7	1	$-\frac{4}{7}$ →
Ratio	-	-	-	-	4	4/6	-	

The tableau is optimal but infeasible. We apply the dual simplex method to recover feasibility, which yields

Basic	z	x_1	x_2	x_3	x_4	s_1	s_2	Solution
z	0	0	0	0	0	3	7	58
x_2	0	0	1	0	0	1	0	3
x_1	0	1	0	0	0	-1	1	4
x_3	0	0	0	1	0	-4	1	1
x_4	0	0	0	0	1	6	-7	4

The solution is optimal.

$x_1 = 4, x_2 = 3, z = 58$ all integer.

3. The Transportation and Assignment Models

The Transportation Model

It is a special class of linear programs that deals with shipping a commodity from sources (e.g. factories) to destinations (e.g. warehouses).

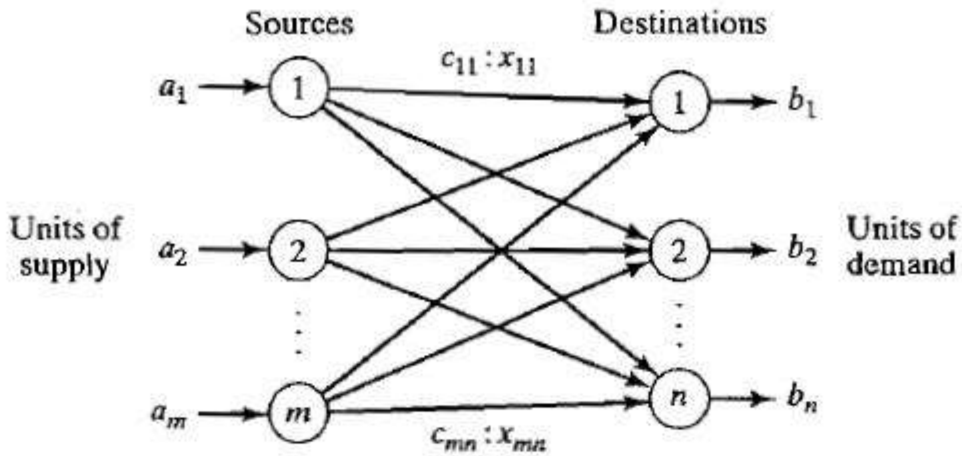
Assumptions:

- The transportation algorithm is based on the assumption that the total quantity of the item available at different sources equal to the total requirement at different destinations.
- Item can be conveniently transported from all sources to destinations.
- The unit transportation cost of the item from all sources to destinations is certainly and precisely known.
- The objective is to determine the shipping schedule that will minimize the total transportation cost while satisfying supply and demand limits.

Definition of the Transportation Model

Suppose that there are m sources and n destinations, each represented by a **node.**

The **arc** $(\boldsymbol{i}, \boldsymbol{j})$ represents the routes linking the **source** $\boldsymbol{i}$ $(i = 1,2, \dots, m)$ to **destination** $\boldsymbol{j}$ $(j = 1,2, \dots, n)$.



$\boldsymbol{C_{ij}}$: Transportation cost per unit of supply

$\boldsymbol{x_{ij}}$: No. of units shipped

$\boldsymbol{a_i}$: Number of supply units available at source i

$\boldsymbol{b_j}$: Number of demands units required at destination j

Mathematical Model of Transportation Problem

If $x_{ij} (\geq 0)$ number of the units shipped from source i to destination j, then the equivalent linear programming model will be

$$\textbf{Minimize } \boldsymbol{z = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} x_{ij}}$$

Subject to

I. **Supply Limits:**

<u>At source 1:</u>
x_{11} units will be supplied to destination 1,
x_{12} units will be supplied to destination 2,
And so on
x_{1n} units will be supplied to destination n.
And total units available at source 1 are a_1
⇒Total units to be shipped from source 1 to all n destinations are

$$x_{11} + x_{12} + \cdots . + x_{1n} = a_1$$

At source 2:

Total units to be shipped from source 1 to all n destinations are

$$x_{21} + x_{22} + \cdots. + x_{2n} = a_2$$

:

:

At source m:

Total units to be shipped from source 1 to all n destinations are

$$x_{m1} + x_{m2} + \cdots. + x_{mn} = a_m$$

So the supply limits are m equations

$$\boldsymbol{\sum_{j=1}^{n} x_{ij} = a_i, \quad (i = 1, 2, \dots, m)}$$

II. Demand Limits:

At destination 1:

Destination 1 will receive x_{11} units from source 1, x_{21} units from source 2,…, x_{i1} units from i^{th} source,…, x_{m1} units from source m.

And total units required at destination 1 are b_1

⇒Total units to be shipped from all m sources to destination 1 are

$$x_{11} + x_{21} + \cdots. + x_{m1} = b_1$$

At destination 2:

Total units to be shipped from all m sources to destination 2 are

$$x_{12} + x_{22} + \cdots. + x_{m2} = b_2$$

:

:

At destination n:

Total units to be shipped from all m sources to destination 2 are

$$x_{1n} + x_{2n} + \cdots. + x_{mn} = b_n$$

So the demand limits are n equations

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad (j = 1,2, \dots, n).$$

$$x_{ij} (\geq 0)$$

Note:https://drive.google.com/file/d/1dzbzLiS_lgC47YoHg0LyMU7gy1tAV2Lz/view?usp=sharing

1. The transportation problem will have a feasible solution if and only if $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$.
2. The equality sign of the constraints causes one of the constraints to be redundant and hence can be deleted so that the problem will have $m + n - 1$ constraints and $m \times n$ unknowns.

Tabular Form of the Transportation Model

Destinations

Sources or Origins	1	2	3	$\ldots j \ldots$	n	**Supply**
1	C_{11} x_{11}	C_{12} x_{12}	C_{13} x_{13}	C_{1j} x_{1j}	C_{1n} x_{1n}	$\boldsymbol{a_1}$
2	C_{21} x_{21}	C_{22} x_{22}	C_{23} x_{23}	C_{2j} x_{2j}	C_{2n} x_{2n}	$\boldsymbol{a_2}$
3	C_{31} x_{31}	C_{32} x_{32}	C_{33} x_{33}	C_{3j} x_{3j}	C_{3n} x_{3n}	$\boldsymbol{a_3}$
: i :	C_{i1} x_{i1}	C_{i2} x_{i2}	C_{i3} x_{i3}	C_{ij} x_{ij}	C_{in} x_{in}	$\boldsymbol{a_i}$
m	C_{m1} x_{m1}	C_{m2} x_{m2}	C_{m3} x_{m3}	C_{mj} x_{mj}	C_{mn} x_{mn}	$\boldsymbol{a_m}$
Demand	$\boldsymbol{b_1}$	$\boldsymbol{b_2}$	$\boldsymbol{b_3}$	$\boldsymbol{b_j}$	$\boldsymbol{b_n}$	

1. When total supply = total demand, the problem is called a **balanced transportation problem,** otherwise it is called an **unbalanced transportation problem.**
2. The unbalanced transportation problem can be made balanced by adding a **dummy supply row** or **dummy demand column.**
3. The cells in the transportation table that have positive allocation are called **occupied cells**; otherwise they are called **empty** or **non-occupied cells.**

Some Definitions:

Feasible Solution: A feasible solution to a transportation problem is a set of nonnegative allocations, x_{ij}, that satisfies the row and column restrictions.

Basic Feasible Solution: A feasible solution to a transportation problem is said to be a basic feasible solution if it contains no more than $m + n - 1$ nonnegative allocations, where m is the number of rows and n is the number of columns of the transportation problem

Optimal Solution: A feasible solution (not necessarily basic) that minimizes (maximizes) the transportation cost (profit) is called an optimal solution.

Non-degenerate Basic Feasible Solution: A basic feasible solution to a $(m \times n)$ transportation problem is said to be non-degenerate if, the total number of nonnegative allocations is exactly $m + n - 1$and these $m + n - 1$ allocations are in independent positions

Degenerate Basic Feasible Solution: A basic feasible solution in which the total number of nonnegative allocations is less than $m + n - 1$ is called degenerate basic feasible solution.

Formulation of Transportation Model

Example 1: MG Auto has three plants in Los Angeles, Detroit and New Orleans and two major distribution centers in Denver and Miami. The capacities of the three plants during the next quarter are 1000, 1500 and 1200 cars. The quarterly demands at the two distribution centers are 2300 and 1400 cars. The mileage chart between the plants and the distribution centers is given below:

Mileage Chart		
	Denver	Miami
Los Angeles	1000	2690
Detroit	1250	1350
New Orleans	1275	850

The trucking company in charge of transporting the cars charges 8 cents per mile per car.

Solution 1: Let x_{ij} denotes the number of cars to be transported from source i to destination j, where $i = 1,2,3$ and $j = 1,2$

The transportation cost per car on the different routes, in dollar, are given below

Transportation Cost per Car		
	Denver (1)	Miami (2)
Los Angeles (1)	\$80	\$215
Detroit (2)	\$100	\$108
New Orleans (3)	\$102	\$68

The objective is to minimize the total transportation cost.

$$\text{Minimize } z = 80x_{11} + 215x_{12} + 100x_{21} + 108x_{22} + 102x_{31} + 68x_{32}$$

Subject to

$$x_{11} + x_{12} = 1000 \qquad \text{(Los Angeles)}$$

$$x_{21} + x_{22} = 1500 \qquad \text{(Detroit)}$$

$$x_{31} + x_{32} = 1200 \qquad \text{(New Orleans)}$$

$$x_{11} + x_{21} + x_{31} = 2300 \qquad \text{(Denver)}$$

$$x_{12} + x_{22} + x_{32} = 1400 \qquad \text{(Miami)}$$

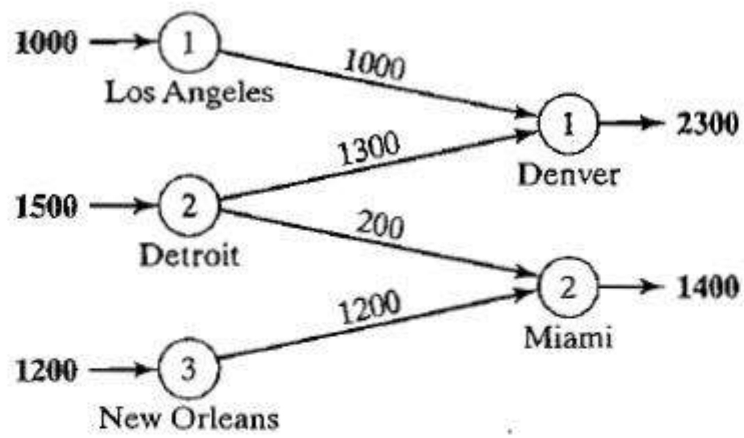
$$x_{ij} \geq 0, i = 1,2,3, j = 1,2$$

These constraints are all equations because the total supply from three sources ($= 1000 + 1500 + 1200 = 3700$ cars) equals the total demand at the two destinations ($= 2300 + 1400 = 3700$ cars)

MG Auto Transportation Model

	Denver	Miami	Supply
Los Angeles	80 x_{11}	215 x_{12}	**1000**
Detroit	100 x_{21}	108 x_{22}	**1500**
New Orleans	102 x_{31}	68 x_{32}	**1200**
Demand	**2300**	**1400**	

Optimal solution (Obtained by TORA)



The associated minimum transportation cost is computed as

$$80 \times 1000 + 100 \times 1300 + 108 \times 200 + 68 \times 1200 = \$\ 3{,}13{,}200$$

Example 2: A dairy farm has three plants located in a state. The daily milk production at plant 1, 2 and 3 are 6 million litres, 1 million litres and 10 million litres, respectively. Each day the firm must fulfill the needs of its four distribution centres. Milk requirement at distribution centres 1, 2, 3 and 4 are 7 million litres, 5 million litres, 3 million litres and 2 million litres, respectively.

Cost of shipping 1 million litres of milk from each plant to each distribution centre is given in the following table in hundreds of rupees:

		Distribution Centres			
		1	**2**	**3**	**4**
	1	2	3	11	7
Plants	**2**	1	0	6	1
	3	5	8	15	9

Formulate the mathematical model for the problem.

Solution 2: Key decision to be made is to find how much quantity of milk from which plant to which distribution centre be shipped so as to satisfy the constraints and minimize the cost.

Let x_{ij} denotes quantities of milk to be shipped from plant i to distribution centre j, where $i = 1, 2, 3$ and $j = 1, 2, 3,4$

Objective is to minimize the transportation cost.

$$\text{Minimize } z = 2x_{11} + 3x_{12} + 11x_{13} + 7x_{14} + x_{21} + 6x_{23} + x_{24}$$
$$+5x_{31} + 8x_{32} + 15x_{33} + 9x_{34}$$

Subject to

$$x_{11} + x_{12} + x_{13} + x_{14} = 6$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 1$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 10$$
$$x_{11} + x_{21} + x_{31} = 7$$
$$x_{12} + x_{22} + x_{32} = 5$$
$$x_{13} + x_{23} + x_{33} = 3$$
$$x_{14} + x_{24} + x_{34} = 2$$
$$x_{ij}(\geq 0), i = 1, 2, 3 \text{ and } j = 1, 2, 3, 4$$

Dairy Farm Transportation Model

		Distribution Centres				
		1	**2**	**3**	**4**	**Supply**
	1	2 x_{11}	3 x_{12}	11 x_{13}	7 x_{14}	**6**
Plants	**2**	1 x_{21}	0 x_{22}	6 x_{23}	1 x_{24}	**1**
	3	5 x_{31}	8 x_{32}	15 x_{33}	9 x_{34}	**10**
Demand		**7**	**5**	**3**	**2**	

The Transportation Algorithm

The steps of the transportation algorithm are exact parallels of the simplex algorithm.

Summary of the Transportation Algorithm

Step 1: Obtain an Initial Basic Feasible Solution

Determine a starting basic feasible solution using any one of the following methods listed below

- Northwest Corner Method
- Least Cost Method
- Vogel's Approximation (or penalty) Method

The initial solution must satisfy the following conditions:

i. The solution must be feasible, that is, it must satisfy all the supply and demand constraints (also called rim conditions).
ii. The number of positive allocations must be equal to $m + n - 1$, where m is the number of rows and n is the number of columns.

Step 2: Use the Optimality Condition of the simplex method to determine *entering variable* from among all the nonbasic variables. If the optimality condition is satisfied, stop. Otherwise go to step 3.

Step 3: Use the feasibility condition of the simplex method to determine the *leaving variable* from among all the current basic variables and find the new basic solution. Return to step 2.

Northwest-Corner Method

The method starts at the northwest-corner cell of the transportation table (i.e. variablex_{11}).

Step 1: Allocate as much as possible to the selected cell, and adjust the associated amounts of supply and demand by subtracting the allocated amount.

Step 2: Cross out the row or column with zero supply or zero demand to indicate that no further assignments can be made in that row or column. If both a row and column net to zero simultaneously then cross out one only, and leave a zero supply (demand) in the uncrossed-out row (column).

Step 3: If exactly one row or column is left uncrossed out, stop. Otherwise, move to the right if a column has just been crossed out or below if a row has been crossed out. Go to step 1.

Example 1: Use Northwest-corner method to find an initial basic feasible solution to the following transportation problem.

		Destinations				
		1	**2**	**3**	**4**	**Supply**
	1	19	30	50	10	**7**
Sources	**2**	70	30	40	60	**9**
	3	40	8	70	20	**18**
Demand		**5**	**8**	**7**	**14**	

Solution 1:

Here $a_1 = 7,\ a_2 = 9,\ a_3 = 18$ Supply limits

$b_1 = 5,\ b_2 = 8,\ b_3 = 7,\ b_4 = 14$ Demand limits

Total supply = 34 = total demand

⇒ Given transportation problem has a feasible solution.

		Destinations				
		1	**2**	**3**	**4**	**Supply**
	1	19 (5)	30 (2)	50	10	**7**
Sources	**2**	70	30 (6)	40 (3)	60	**9**
	3	40	8	70 (4)	20 (14)	**18**
Demand		**5**	**8**	**7**	**14**	

Cell (1,1) is northwest-corner cell. Comparing a_1 and b_1, we can allocate maximum 5 allocations in this cell, which means 5 units of commodity are to be transported from source 1 to destination 1.

However, this allocation leaves a supply of $7 - 5 = 2$ units of commodity at source 1 and zero demand at destination 1.

Cross out column 1 and move horizontally to cell(1,2).
The rim value for row 1 is 2 and for column 2 is 8.
The smaller value, i.e. 2 is placed in this cell, which leaves zero supply at source 1 and a demand of $8 - 2 = 6$ units of commodity at destination 2.
So cross out row 1 and vertically down to cell(2,2).

At cell(2,2), there is a supply of 9 units of commodity at source 2 and demand of 6 units of commodity at destination 2.
So the maximum 6 allocations can be made in this cell, which leaves a supply of $9 - 6 = 3$ units of commodity at source 2 and zero demand at destination 2.
Cross out column 2 and move horizontally to cell (2,2)

At cell(2,3), there is a supply of 3 units of commodity at source 2 and demand of 7 units of commodity at destination 3.
Allocate a maximum 3 units of commodity in this cell, which leaves zero supply at source 2 and a demand of $7 - 3 = 4$ units of commodity at destination 3.
So cross out row 2 and vertically down to cell(3,3).

At cell(3,3), there is a supply of 18 units of commodity at source 3 and demand of 4 units of commodity at destination 3.
So the maximum 4 allocations can be made in this cell, which leaves a supply of $18 - 4 = 14$ units of commodity at source 3 and zero demand at destination 3.
Cross out column 3 and move horizontally to cell (3,4)

At cell(3,4), there is a supply of 14 units of commodity at source 3 and demand of 14 units of commodity at destination 4.
Allocate 14 units of commodity in this cell, which leaves zero supply at source 3 and zero demand at destination 4.

The starting basic solution is

$$x_{11} = 5, x_{12} = 2$$

$$x_{22} = 6, x_{23} = 3$$

$$x_{33} = 4, x_{34} = 14$$

Since the no. of positive allocations are

$$m + n - 1 = 3 + 4 - 1 = 6$$

$\Rightarrow$ The solution is non-degenerate

The associated cost of the schedule is

$$z = 5 \times 19 + 2 \times 30 + 6 \times 30 + 3 \times 40 + 4 \times 70 + 14 \times 20$$

$= \text{Rs. } 1{,}015$

Example 2: Use Northwest-corner method to find an initial basic feasible solution to the following transportation problem.

	D_1	D_2	D_3	D_4	Supply
S_1	6	4	1	5	**14**
S_2	8	9	2	7	**16**
S_3	4	3	6	2	**5**
Demand	**6**	**10**	**15**	**4**	

Solution 2:

Here $a_1 = 14,\ a_2 = 16,\ a_3 = 5$ Supply limits

$b_1 = 6,\ b_2 = 10,\ b_3 = 15,\ b_4 = 4$ Demand limits

Total supply $= 35 =$ total demand

$\Rightarrow$ Given transportation problem has a feasible solution.

	D_1	D_2	D_3	D_4	Supply
S_1	6 (6)	4 (8)	1	5	**14**
S_2	8	9 (2)	2 (14)	7	**16**
S_3	4	3	6 (1)	2 (4)	**5**
Demand	**6**	**10**	**15**	**4**	

The starting basic solution is

$$x_{11} = 6, x_{12} = 8$$

$$x_{22} = 2, x_{23} = 14$$

$$x_{33} = 1, x_{34} = 4$$

Since the no. of positive allocations are $m + n - 1 = 3 + 4 - 1 = 6$

$\Rightarrow$ The solution is non-degenerate

The associated cost of the schedule is

$$z = 6 \times 6 + 8 \times 4 + 2 \times 9 + 14 \times 2 + 1 \times 6 + 4 \times 2$$

$$= \text{Rs. } 128$$

Example 3: Use Northwest-corner method to find an initial basic feasible solution to the following transportation problem.

	D_1	D_2	D_3	D_4	Capacity
P_1	4	6	8	13	**50**
P_2	13	11	10	8	**70**
P_3	14	4	10	13	**30**
P_4	9	11	13	8	**50**
Demand	**25**	**35**	**105**	**20**	

Solution 3:

Here total production capacity = 200 units
Total demand = 185 units

Surplus capacity = 15 units

Thus, production capacity and demand are not balanced in this case and we have a surplus of 15 units of the product. Therefore, we add dummy destination with associated cost coefficients are taken as zero.

Therefore, starting cost matrix becomes

	D_1	D_2	D_3	D_4	D_5	Capacity
P_1	4 (25)	6 (25)	8	13	0	**50**
P_2	13	11 (10)	10 (60)	8	0	**70**
P_3	14	4	10 (30)	13	0	**30**
P_4	9	11	13 (15)	8 (20)	0 (15)	**50**
Demand	**25**	**35**	**105**	**20**	**15**	

The starting basic solution is

$$x_{11} = 25, x_{12} = 25$$

$$x_{22} = 10, x_{23} = 60$$

$$x_{33} = 10$$

$$x_{43} = 15, x_{44} = 20, x_{45} = 15$$

Since the no. of positive allocations are

$$m + n - 1 = 4 + 5 - 1 = 8$$

$$\Rightarrow \text{The solution is non-degenerate}$$

The associated cost of the schedule is,

$$z = 25 \times 4 + 25 \times 6 + 10 \times 11 + 60 \times 10 + 30 \times 10 + 15 \times 13 + 20 \times 8 + 15 \times 0 = \text{Rs. } 1615$$

Least Cost Method/Matrix Minima Method

The method finds better starting solution by taking cheapest routes. This method consists in allocating as much as possible in the lowest cost cell/cells and then further allocation done in the cell/cells with second lowest cost and so on.

Step 1: Select the lowest cost cell in the entire transportation table and allocate as much as possible to the selected cell. Cross out the row or column with zero supply or zero demand. If a row and column are both satisfied simultaneously, then cross out one only.

In case of tie among the cost, select the cell where maximum allocation of units can be made.

Step 2: Repeat the procedure with the next lowest unit cost among the remaining rows and columns of the transportation table and allocate as much as possible to this cell. Then cross out the row or column in which either supply or demand is exhausted.

Step 3: Repeat the procedure until the entire available supply at various sources and demand at various destinations is satisfied.

Example 1: Use Least Cost Method to find an initial basic feasible solution to the following transportation problem.

		Destinations				
		1	**2**	**3**	**4**	**Supply**
	1	19	30	50	10	**7**
Sources	**2**	70	30	40	60	**9**
	3	40	8	70	20	**18**
Demand		**5**	**8**	**7**	**14**	

Solution 1: Here $a_1 = 7,\ a_2 = 9,\ a_3 = 18$ Supply limits

$b_1 = 5,\ b_2 = 8,\ b_3 = 7,\ b_4 = 14$ Demand limits

$$\text{Total supply} = 34 = \text{total demand}$$

$\Rightarrow$ Given transportation problem has a feasible solution.

		Destinations				
		1	2	3	4	Supply
	1	19	30	50	10 (7)	7
Sources	2	70 (2)	30	40 (7)	60	9
	3	40 (3)	8 (8)	70	20 (7)	18
Demand		5	8	7	14	

The Cell with the lowest unit cost (i.e.8) is (3,2) . Comparing a_3 and b_2, we can allocate maximum 8 units in this cell, which means 8 units of commodity are to be transported from source 3 to destination 2.
However, this allocation leaves a supply of $18 - 8 = 10$ units of commodity at source 3 and zero demand at destination 2.
Cross out column 2.

In the reduced table, the next lowest cost is 10 in cell(1,4). The rim value for row 1 is 7 and for column 4 is 14.
The smaller value, i.e. 7 is placed in this cell, which leaves zero supply at source 1 and a demand of $14 - 7 = 7$ units of commodity at destination 4.
So cross out row 1 and find the next lowest unit transportation cost in reduced table.

The next smallest cost is 20 in cell(3,4), there is a supply of 10 units of commodity at source 3 and demand of 7 units of commodity at destination 4.

So the maximum 7 units can be allocated in this cell, which leaves a supply of $10 - 7 = 3$ units of commodity at source 3 and zero demand at destination 4.
Cross out column 4 and check for next lowest unit transportation cost in the reduced table.

The next smallest unit cost cell is not unique. There are two cells (2,3) and (3,1) having the same unit transportation cost, i.e. 40. So allocate 7 units in cell(2,3) first because it can accommodate more units as compared to cell(3,1).
Then allocate 3 units to cell(3,1)

The starting basic solution is

$$x_{14} = 7$$

$$x_{21} = 2, x_{23} = 7$$

$$x_{31} = 3, x_{32} = 8, x_{34} = 7$$

Since the no. of positive allocations are

$$m + n - 1 = 3 + 4 - 1 = 6$$

$$\Rightarrow \text{The solution is non-degenerate}$$

The associated cost of the schedule is

$$z = 7 \times 10 + 2 \times 70 + 7 \times 40 + 3 \times 40 + 8 \times 8 + 7 \times 20$$

$$= \text{Rs. } 814$$

Example 2: Use Least Cost Method to find an initial basic feasible solution to the following transportation problem.

		Destinations				
		1	**2**	**3**	**4**	**Supply**
	1	6	4	1	5	**14**
Sources	**2**	8	9	2	7	**16**
	3	4	3	6	2	**5**
Demand		**6**	**10**	**15**	**4**	

Solution 2: Here $a_1 = 14,\ a_2 = 16,\ a_3 = 5$ Supply limits

$b_1 = 6,\ b_2 = 10,\ b_3 = 15,\ b_4 = 4$ Demand limits

Total supply $= 35 =$ total demand

$\Rightarrow$ Given transportation problem has a feasible solution.

	D_1	D_2	D_3	D_4	Supply
S_1	6	4	1 (14)	5	14
S_2	8 (6)	9 (9)	2 (1)	7	16
S_3	4	3 (1)	6	2 (4)	5
Demand	6	10	15	4	

The starting basic solution is

$$x_{13} = 14,$$

$$x_{21} = 6, x_{22} = 9, x_{23} = 1$$

$$x_{32} = 1, x_{34} = 4$$

Since the no. of positive allocations are

$$m + n - 1 = 3 + 4 - 1 = 6$$

$\Rightarrow$ The solution is non-degenerate

The associated cost of the schedule is

$$z = 1 \times 14 + 8 \times 6 + 9 \times 9 + 2 \times 1 + 3 \times 1 + 2 \times 4$$

$$= \text{Rs. } 156$$

Example 3 Find an initial basic feasible solution to the following transportation problem

	D_1	D_2	D_3	D_4	Capacity
P_1	4	6	8	13	**50**
P_2	13	11	10	8	**70**
P_3	14	4	10	13	**30**
P_4	9	11	13	8	**50**
Demand	**25**	**35**	**105**	**20**	

Solution 3:

Here total production capacity = 200 units
Total demand = 185 units
Surplus capacity = 15 units

Thus, production capacity and demand are not balanced in this case and we have a surplus of 15 units of the product. Therefore, we add dummy destination with associated cost coefficients are taken as zero.

Therefore, starting cost matrix becomes

	D_1	D_2	D_3	D_4	D_5	Capacity
P_1	4 (25)	6 (5)	8 (5)	13	0 (15)	**50**
P_2	13	11	10 (50)	8 (20)	0	**70**
P_3	14	4 (30)	10	13	0	**30**
P_4	9	11	13 (50)	8	0	**50**
Demand	**25**	**35**	**105**	**20**	**15**	

The starting basic solution is

$$x_{11} = 25, x_{12} = 5, x_{13} = 5, x_{14} = 15$$

$$x_{23} = 50, x_{24} = 20$$

$$x_{32} = 30$$

$$x_{43} = 50$$

Since the no. of positive allocations are

$$m + n - 1 = 4 + 5 - 1 = 8$$

$$\Rightarrow \text{The solution is non-degenerate}$$

The associated cost of the schedule is

$$\begin{aligned} z &= 4 \times 25 + 6 \times 5 + 8 \times 5 + 0 \times 15 + 10 \times 50 + 8 \times 20 + 4 \times 30 \\ &\quad + 13 \times 50 \\ &= \text{Rs. } 1600 \end{aligned}$$

Vogel's Approximation (or penalty) Method (VAM)

The method is preferred more than the other two methods described above.

Step 1: For each row (column), calculate a penalty by taking the difference between smallest and next smallest unit transportation cost in the same row (column). This difference indicates the penalty or extra cost that has to be paid if one fails to allocate to the cell with minimum unit transportation cost.

Step 2: Identify the row or column with the largest penalty. Allocate as much as possible to the variable with the least unit cost in the selected row or column. Adjust the supply and demand and cross out the satisfied row or column. If a row and a column are satisfied simultaneously, then cross out one only, and the remaining row (column) is assigned zero supply (demand).

In case of tie among the penalty, select the row or column where maximum allocation can be made.

Step 3: Repeat the procedure until the entire available supply at various sources and demand at various destinations is satisfied.

Example 1: Use Vogel's Approximation Method to find an initial basic feasible solution to the following transportation problem.

	Destinations 1	2	3	4	Supply
Sources 1	19	30	50	10	7
2	70	30	40	60	9
3	40	8	70	20	18
Demand	5	8	7	14	

Solution 1:

Here $a_1 = 7,\ a_2 = 9,\ a_3 = 18$ Supply limits

$b_1 = 5,\ b_2 = 8,\ b_3 = 7,\ b_4 = 14$ Demand limits

Total supply $= 34 =$ total demand

$\Rightarrow$ Given transportation problem has a feasible solution

1. **Calculate the differences for each row and column**

	D_1	D_2	D_3	D_4	Supply	Row Diff.			
S_1	19	30	50	10	7	9			
S_2	70	30	40	60	9	10			
S_3	40	8 (8)	70	20	18	12			
Demand	5	8	7	14					
Column Diff.	21	22	10	10					
		↑							

The maximum penalty, 22 occurs in columnD_2. Thus the cell (S_3, D_2) having the least unit transportation cost 8 is chosen for allocation. The maximum possible allocation in this cell is 8 units. Adjust supply and demand by subtracting allocated amount. It satisfies demand in column D_2 and leaves $18 - 8 = 10$ units of supply in row S_3.

2. **In the second round, the new row and column penalties are calculated except column D_2.**

	D_1	D_2	D_3	D_4	Supply	Row Diff.			
S_1	19 (5)	30	50	10	**7**	9	9		
S_2	70	30	40	60	**9**	10	20		
S_3	40	8 (8)	70	20	**10**	12	20		
Demand	**5**	**0**	**7**	**14**					
Column Diff.	21	22	10	10					
	21		10	10					
	↑								

The maximum penalty, 21 occurs in columnD_1. Thus the cell (S_1, D_1) having the least unit transportation cost 19 is chosen for allocation. The maximum possible allocation in this cell is 5 units. Adjust supply and demand by subtracting allocated amount. It satisfies demand in column D_1 and leaves $7 - 5 = 2$ units of supply in row S_1.

3. **In the third round, the new row and column penalties are calculated except column D_1.**

	D_1	D_2	D_3	D_4	Supply	Row Diff.			
S_1	19 (5)	30	50	10	**2**	9	9	40	
S_2	70	30	40	60	**9**	10	20	20	
S_3	40	8 (8)	70	20 (10)	**10**	12	20	50	←
Demand	**0**	**0**	**7**	**14**					
Column Diff.	21	22	10	10					
	21		10	10					
			10	10					

The maximum penalty, 50 occurs in rowS_3. Thus the cell (S_3, D_4) having the least unit transportation cost 20 is chosen for allocation. The maximum possible allocation of 10 units is made in this cell. Adjust supply and demand by subtracting allocated amount. It satisfies demand in row S_3 and leaves $14 - 10 = 4$ units of demand in columnD_4.

4. **In the fourth round, the new row and column penalties are calculated except row S_3.**

	D_1	D_2	D_3	D_4	Supply	Row Diff.			
S_1	19 (5)	30	50	10 (2)	**2**	9	9	40	40
S_2	70	30	40	60	**9**	10	20	20	20
S_3	40	8 (8)	70	20 (10)	**0**	12	20	50	
Demand	**0**	**0**	**7**	**4**					
Column Diff.	21	22	10	10					
	21		10	10					
			10	10 ↑					
			10	50					

The maximum penalty, 50 occurs in columnD_4. Thus the cell (S_1, D_4) having the least unit transportation cost 10 is chosen for allocation. The maximum possible allocation of 2 units is made in this cell. Adjust supply and demand by subtracting allocated amount. It satisfies demand in row S_1 and leaves $4 - 2 = 2$ units of demand in columnD_4.

5. **In the fifth round, the new row and column penalties are calculated except row S_1.**

	D_1	D_2	D_3	D_4	Supply	Row Penalties				
S_1	19 (5)	30	50	10 (2)	**0**	9	9	40	40	
S_2	70	30	40	60 (2)	**9**	10	20	20	20	20
S_3	40	8 (8)	70	20 (10)	**0**	12	20	50		
Demand	**0**	**0**	**7**	**2**						
Column Penalties	21	22	10	10						
	21		10	10						
			10	10						
			10	50						
			40	60						

The maximum penalty, 60 occurs in columnD_4. The maximum possible allocation of 2 units can be done in (S_2, D_4) cell only. Adjust supply and demand by subtracting allocated amount. It satisfies demand in columnD_4. and leaves $9 - 2 = 7$ units of supply in row S_2.

6. **In the last table the last allocation can only be done at the cell (S_2, D_3) of 7 units which satisfies both supply and demand limits simultaneously.**

	D_1	D_2	D_3	D_4	Supply	Row Penalties.					
S_1	19 (5)	30	50	10 (2)	**0**	9	9	40	40		
S_2	70	30	40 (7)	60 (2)	**7**	10	20	20	20	20	20
S_3	40	8 (8)	70	20 (10)	**0**	12	20	50			
Demand	**0**	**0**	**7**	**0**							
Colum Penalties	21	22	10	10							
	21		10	10							
			10	10							
			10	50 ↑							
			40	60							
			40								

In this way, we arrived at our initial basic feasible solution as shown in the table below:

	D_1	D_2	D_3	D_4
S_1	19 (5)	30	50	10 (2)
S_2	70	30	40 (7)	60 (2)
S_3	40	8 (8)	70	20 (10)

The starting basic solution is

$$x_{11} = 5, x_{14} = 2$$

$$x_{23} = 7, x_{24} = 2$$

$$x_{32} = 8, x_{34} = 10$$

Since the no. of positive allocations are

$$m + n - 1 = 3 + 4 - 1 = 6$$

$\Rightarrow$ The solution is non-degenerate

The associated cost of the schedule is

$z = 5 \times 19 + 2 \times 10 + 7 \times 40 + 2 \times 60 + 8 \times 8 + 10 \times 20 = \text{Rs. } 779$

Example 2: Use Vogel's Approximation Method to find an initial basic feasible solution to the following transportation problem.

		Destinations				
		1	**2**	**3**	**4**	**Supply**
	1	2	3	11	7	**6**
Sources	**2**	1	0	6	1	**1**
	3	5	8	15	9	**10**
Demand		**7**	**5**	**3**	**2**	

Solution 2:

Here $a_1 = 6,\ a_2 = 1,\ a_3 = 10$ Supply limits

$b_1 = 7,\ b_2 = 5,\ b_3 = 3,\ b_4 = 2$ Demand limits

Total supply $= 17 =$ total demand

$\Rightarrow$ Given transportation problem has a feasible solution.

	D_1	D_2	D_3	D_4	Supply	Row Differences					
S_1	2 (1)	3 (5)	11	7	**6**	1	1	5	-	-	-
S_2	1	0	6	1 (1)	**1**	1	-	-	-	-	-
S_3	5 (6)	8	15 (3)	9 (1)	**10**	3	3	4	4	4	5
Demand	**7**	**5**	**3**	**2**							
Column Diff.	1	3	5	6							
	3	5	4	2							
	3	-	4	2							
	5	-	15	9							
	5	-	-	9							
	5	-	-	-							

The starting basic solution is

$$x_{11} = 1, x_{12} = 5$$

$$x_{24} = 1$$

$$x_{31} = 6, x_{33} = 3, x_{34} = 1$$

Since the no. of positive allocations are

$$m + n - 1 = 3 + 4 - 1 = 6$$

$\Rightarrow$ The solution is non-degenerate

The associated cost of the schedule is

$z = 2 \times 1 + 3 \times 5 + 1 \times 1 + 5 \times 6 + 15 \times 3 + 9 \times 1 =$ Rs. 102

Test of Optimality

Method of Multipliers/ Modified Distribution Method

In the method of multipliers, we associate the multipliers u_i and v_j with row i and column j of the transportation tableau.

Step 1: For an initial basic feasible solution with $m + n - 1$ occupied cells, calculate u_i and v_j for rows and columns.

Assign zero to a particular u_i or v_j where there are maximum allocations in a row or column respectively. The complete calculation of u_i and v_j for other rows and columns by using the relation

$$c_{ij} = u_i + v_j, \text{ for all occupied cells } (i, j)$$

Step 2: For unoccupied cells, calculate the opportunity cost (the difference that indicates the per unit cost reduction that can be achieved by an allocation in the unoccupied cell) by using the relation

$$d_{ij} = c_{ij} - (u_i + v_j), \text{ for all } i \text{ and } j$$

Step 3: Examine sign of each d_{ij}

i. If $d_{ij} > 0$, then the current basic feasible solution is optimal.
ii. If $d_{ij} = 0$, then the current basic feasible solution will remain unaffected but an alternative solution exists.
iii. If one or more $d_{ij} < 0$, then an improved solution can be obtained by entering unoccupied cell (i, j) with the largest negative value of d_{ij} in the basis.

Step 4: Construct a closed path (or loop) for the unoccupied cell with largest negative opportunity cost. Start the closed path with the selected unoccupied cell and mark a plus sign(+) in this cell. Trace a path along the rows (or columns) to an occupied cell, mark the corner with a minus sign(−) and continue down the column (or row) to an occupied cell. Then mark the corner with plus sign(+) and minus sign(−) alternatively. Close the path back to the selected unoccupied cell.

Step 5: Select the smallest quantity amongst the cells marked with minus sign on the corners of closed loop. Allocate this value to the selected unoccupied cell and add it to other occupied cells marked with plus signs and subtract from the occupied cells marked with minus signs.

Step 6: Obtain a new improved solution by allocating units to the unoccupied cell according to step 5 and calculate new transportation cost.

Step 7: Revise the solution for optimality. The procedure terminates when all $d_{ij} \geq 0$ for unoccupied cells.

Remarks: The closed-loop starts and ends at the selected unoccupied cell. It consists of successive horizontal and vertical connected lines whose end points must be occupied cells, except for an end point associated with entering unoccupied cell.

Example 1: Find the optimal solution of the following transportation problem.

		Destinations				
		1	**2**	**3**	**4**	**Supply**
	1	19	30	50	10	**7**
Sources	**2**	70	30	40	60	**9**
	3	40	8	70	20	**18**
Demand		**5**	**8**	**7**	**14**	

Solution 1: We apply Vogel's approximation method to obtain an initial basic feasible solution given below: (solved)

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	19 (5)	30	50	10 (2)	7	10
S_2	70	30	40 (7)	60 (2)	9	60
S_3	40	8 (8)	70	20 (10)	18	20
Demand	5	8	7	14		
v_j	9	-12	-20	0		

Since the no. of positive allocations are $m + n - 1 = 3 + 4 - 1 = 6$

$\Rightarrow$ The solution is non-degenerate.

Thus, an optimal solution can be obtained.

IBFS: The associated cost of the schedule is

$z = 5 \times 19 + 2 \times 10 + 7 \times 40 + 2 \times 60 + 8 \times 8 + 10 \times 20 \quad = \text{Rs. } 779$

For occupied cells: Calculate u_i and v_j for rows and columns by using the relation $c_{ij} = u_i + v_j$.

Assign$v_4 = 0$ and calculate u_1, u_2 and u_3 .

$$c_{14} = u_1 + v_4 \ \Rightarrow 10 = u_1 + 0 \ \Rightarrow u_1 = 10$$
$$c_{24} = u_2 + v_4 \ \Rightarrow 60 = u_2 + 0 \ \Rightarrow u_2 = 60$$
$$c_{34} = u_3 + v_4 \ \Rightarrow 20 = u_3 + 0 \ \Rightarrow u_3 = 20$$

The values for v_1, v_2 and v_3 can also be calculated as shown below:

$$c_{11} = u_1 + v_1 \ \Rightarrow 19 = 10 + v_1 \ \Rightarrow v_1 = 9$$

$$c_{23} = u_2 + v_3 \ \Rightarrow 40 = 60 + v_3 \ \Rightarrow v_3 = -20$$

$$c_{32} = u_3 + v_2 \ \Rightarrow 8 = 20 + v_2 \ \Rightarrow v_2 = -12$$

For unoccupied cells: Calculate d_{ij} by using the relation

$$d_{ij} = c_{ij} - (u_i + v_j)$$

$$d_{12} = c_{12} - (u_1 + v_2) = 30 - (10 - 12) = 32$$
$$d_{13} = c_{13} - (u_1 + v_3) = 50 - (10 - 20) = 60$$
$$d_{21} = c_{21} - (u_2 + v_1) = 70 - (60 + 9) = 1$$
$$d_{22} = c_{22} - (u_2 + v_2) = 30 - (60 - 12) = -18$$
$$d_{31} = c_{31} - (u_3 + v_1) = 40 - (20 + 9) = 11$$
$$d_{33} = c_{33} - (u_3 + v_3) = 70 - (20 - 20) = 70$$

Since $d_{22} = -18 < 0 \Rightarrow$ Solution is not optimal

Trace the closed loop starting from the cell (S_2, D_2) in clockwise or anti-clockwise direction.

So the closed loop is traced from cell (S_2, D_2) to an occupied cell(S_3, D_2). A plus sign is placed in cell (S_2, D_2) and minus sign in cell(S_3, D_2). Now take a right angle turn and locate an occupied cell in columnD_4. An occupied cell (S_3, D_4) exists at rowS_3, and a plus sign is placed in this cell.

Continue this process and complete the closed path.

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	19 (5)	30	50	10 (2)	7	10
S_2	70	30 (+)	40 (7)	60 (2) (-)	9	60
S_3	40	8 (-) (8)	70	20 (10) (+)	18	20
Demand	5	8	7	14		
v_j	9	-12	-20	0		

Examine the occupied cells with minus sign at the corners of closed loop, and select the one that has the smallest allocation. The value of this allocation is then added to cell (S_2, D_2) and (S_3, D_4), which carry plus signs. The same value is subtracted from cells (S_2, D_4) and (S_3, D_2), which carry minus signs.

Revised Solution:

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	19 (5)	30	50	10 (2)	7	0
S_2	70	30 (2)	40 (7)	60	9	32
S_3	40	8 (6)	70	20 (12)	18	10
Demand	5	8	7	14		
v_j	19	-2	8	10		

For occupied cells: Calculate u_i and v_j for rows and columns by using the relation $c_{ij} = u_i + v_j$.

Assign$u_1 = 0$ and calculate v_1 and v_4

$$c_{11} = u_1 + v_1 \Rightarrow 19 = 0 + v_1 \Rightarrow v_1 = 19$$
$$c_{14} = u_1 + v_4 \Rightarrow 10 = 0 + v_4 \Rightarrow v_4 = 10$$

The values for u_2, u_3, v_2 and v_3 can also be calculated as shown below:

$$c_{34} = u_3 + v_4 \Rightarrow 20 = u_3 + 10 \Rightarrow u_3 = 10$$

$$c_{32} = u_3 + v_2 \Rightarrow 8 = 10 + v_2 \Rightarrow v_2 = -2$$
$$c_{22} = u_2 + v_2 \Rightarrow 30 = u_2 - 2 \Rightarrow u_2 = 32$$
$$c_{23} = u_2 + v_3 \Rightarrow 40 = 32 + v_3 \Rightarrow v_3 = 8$$

For unoccupied cells: Calculate d_{ij} by using the relation

$$d_{ij} = c_{ij} - (u_i + v_j)$$

$$d_{12} = c_{12} - (u_1 + v_2) = 30 - (0 - 2) = 32$$
$$d_{13} = c_{13} - (u_1 + v_3) = 50 - (0 + 8) = 42$$
$$d_{21} = c_{21} - (u_2 + v_1) = 70 - (32 + 19) = 19$$
$$d_{24} = c_{24} - (u_2 + v_4) = 60 - (32 + 10) = 18$$

$$d_{31} = c_{31} - (u_3 + v_1) = 40 - (10 + 19) = 11$$
$$d_{33} = c_{33} - (u_3 + v_3) = 70 - (10 + 8) = 52$$

Since $d_{ij} > 0 \Rightarrow$ Solution is optimal

Therefore, the minimum transportation cost is

$z = 5 \times 19 + 2 \times 10 + 2 \times 30 + 7 \times 40 + 6 \times 8 + 12 \times 20 =$ Rs. 743

Example 2: A company has factories at F_1, F_2 and F_3 that supply products to warehouses at W_1, W_2 and W_3. The weekly capacities of the factories are 200, 160 and 90 units, respectively. The weekly warehouse requirements are 180, 120 and 150 units, respectively. The unit shipping costs (in rupees) are as follows:

		Warehouses			
		W_1	W_2	W_3	**Supply**
	F_1	16	20	12	**200**
Factories	F_2	14	8	18	**160**
	F_3	26	24	16	**90**
Demand		**180**	**120**	**150**	

Solution 2: Here, total supply $= 450 =$ total demand

$\Rightarrow$ Given transportation problem has feasible solution

We apply Northwest-corner method to obtain an initial basic feasible solution given below:

	D_1	D_2	D_3	**Supply**
S_1	16 (180)	20 (20)	12	**200**
S_2	14	8 (100)	18 (60)	**160**
S_3	26	24	16 (90)	**90**
Demand	**180**	**120**	**150**	

Since the no. of positive allocations are $m + n - 1 = 3 + 4 - 1 = 6$

$\Rightarrow$ The solution is non-degenerate.

Thus, an optimal solution can be obtained.

<u>**IBFS:**</u> The associated cost of the schedule is

$z = 16 \times 180 + 20 \times 20 + 8 \times 100 + 18 \times 60 + 16 \times 90 \;= \text{Rs. } 6800$

For occupied cells: Calculate u_i and v_j for rows and columns by using the relation $c_{ij} = u_i + v_j$.

	D_1	D_2	D_3	Supply	u_i
S_1	16 (180)	20 (20)	12	**200**	0
S_2	14	8 (100)	18 (60)	**160**	-12
S_3	26	24	16 (90)	**90**	-14
Demand	**180**	**120**	**150**		
v_j	16	20	30		

Assign $u_1 = 0$ and calculate v_1 and v_2 .

$$c_{11} = u_1 + v_1 \;\Rightarrow 16 = 0 + v_1 \;\Rightarrow v_1 = 16$$
$$c_{12} = u_1 + v_2 \;\Rightarrow 20 = 0 + v_2 \;\Rightarrow v_2 = 20$$

The values for u_2, u_3 and v_3 can also be calculated as shown below:

$$c_{22} = u_2 + v_2 \;\Rightarrow 8 = u_2 + 20 \;\Rightarrow u_2 = -12$$

$$c_{23} = u_2 + v_3 \;\Rightarrow 18 = -12 + v_3 \;\Rightarrow v_3 = 30$$

$$c_{33} = u_3 + v_3 \;\Rightarrow 16 = u_3 + 30 \;\Rightarrow u_3 = -14$$

For unoccupied cells: Calculate d_{ij} by using the relation

$$d_{ij} = c_{ij} - (u_i + v_j)$$

$$\begin{aligned}
d_{13} &= c_{13} - (u_1 + v_3) = 12 - (0 + 30) \ = -18 \\
d_{21} &= c_{21} - (u_2 + v_1) = 14 - (-12 + 16) = 10 \\
d_{31} &= c_{31} - (u_3 + v_1) = 26 - (-14 + 16) = 24 \\
d_{32} &= c_{33} - (u_3 + v_2) = 24 - (-14 + 20) = 18
\end{aligned}$$

Since $d_{13} = -18 < 0 \Rightarrow$ Solution is not optimal.

Trace the closed loop starting from the cell (S_2, D_2) in clockwise or anti-clockwise direction.

	D_1	D_2	D_3	Supply	u_i
S_1	16 (180)	20 (−) (20)	12 ● (+)	**200**	0
S_2	14	8 (+) (100)	18 (60) (−)	**160**	-12
S_3	26	24	16 (90)	**90**	-14
Demand	**180**	**120**	**150**		
v_j	16	20	30		

Revised Solution:

	D_1	D_2	D_3	Supply	u_i
S_1	16 (180)	20	12 (20)	**200**	12
S_2	14	8 (120)	18 (40)	**160**	18
S_3	26	24	16 (90)	**90**	16
Demand	**180**	**120**	**150**		
v_j	4	-10	0		

For occupied cells: Calculate u_i and v_j for rows and columns by using the relation $c_{ij} = u_i + v_j$.

Assign $v_4 = 0$ and calculate $u_1, u_2$and u_3

$$c_{13} = u_1 + v_3 \ \Rightarrow 12 = u_1 + 0 \ \Rightarrow u_1 = 12$$
$$c_{23} = u_2 + v_3 \ \Rightarrow 18 = u_2 + 0 \ \Rightarrow u_2 = 18$$
$$c_{33} = u_3 + v_3 \ \Rightarrow 16 = u_3 + 0 \ \Rightarrow u_3 = 16$$

The values for v_1 and v_2 can also be calculated as shown below:

$$c_{11} = u_1 + v_1 \ \Rightarrow 16 = 12 + v_1 \ \Rightarrow v_1 = 4$$

$$c_{22} = u_2 + v_2 \ \Rightarrow \ 8 \ = 18 + v_2 \ \Rightarrow v_2 = -10$$

For unoccupied cells: Calculate d_{ij} by using the relation

$$d_{ij} = c_{ij} - \left(u_i + v_j\right)$$

$d_{12} = c_{12} - (u_1 + v_2) = 20 - (12 - 10) = 18$
$d_{21} = c_{21} - (u_2 + v_1) = 14 - (18 + 4) \ \ = \ -8$
$d_{31} = c_{31} - (u_3 + v_1) = 26 - (16 + 4) = 6$
$d_{33} = c_{32} - (u_3 + v_2) = 24 - (16 - 10) = 18$

Since $d_{21} = -8 < 0 \Rightarrow$ Solution is not optimal.

	D_1	D_2	D_3	Supply	u_i
S_1	16 (-) (180)	20	12 (+) (20)	**200**	12
S_2	14 (+)•	8 (120)	18 (40) (-)	**160**	18
S_3	26	24	16 (90)	**90**	16
Demand	**180**	**120**	**150**		
v_j	4	-10	0		

Revised Solution:

	D_1	D_2	D_3	Supply	u_i
S_1	16 (140)	20	12 (60)	**200**	16
S_2	14 (40)	8 (120)	18	**160**	14
S_3	26	24	16 (90)	**90**	90
Demand	**180**	**120**	**150**		
v_j	0	- 6	- 4		

For occupied cells: Calculate u_i and v_j for rows and columns by using the relation $c_{ij} = u_i + v_j$.

Assign $v_1 = 0$ and calculate u_1 and u_2

$$c_{11} = u_1 + v_1 \;\; \Rightarrow 16 = u_1 + 0 \;\Rightarrow u_1 = 16$$
$$c_{21} = u_2 + v_1 \;\; \Rightarrow 14 = u_2 + 0 \;\Rightarrow u_2 = 14$$

The values for $u_{3,}$ v_2 and v_3 can also be calculated as shown below:

$$c_{13} = u_1 + v_3 \;\; \Rightarrow 12 = 16 + v_3 \;\Rightarrow v_3 = -4$$

$$c_{33} = u_3 + v_3 \;\; \Rightarrow 16 = u_3 - 4 \;\Rightarrow u_3 = 20$$

$$c_{22} = u_2 + v_2 \;\; \Rightarrow \; 8 \; = 14 + v_2 \;\; \Rightarrow v_2 = -6$$

For unoccupied cells: Calculate d_{ij} by using the relation

$$d_{ij} = c_{ij} - \left(u_i + v_j\right)$$

$$d_{12} = c_{12} - (u_1 + v_2) = 20 - (16 - 6) = 10$$
$$d_{23} = c_{23} - (u_2 + v_3) = 18 - (14 - 4) \quad = \; 8$$
$$d_{31} = c_{31} - (u_3 + v_1) = 26 - (20 + 0) = 6$$
$$d_{33} = c_{32} - (u_3 + v_2) = 24 - (20 - 6) = 10$$

Since all $d_{ij} > 0 \Rightarrow$ Solution is optimal

Therefore, the minimum transportation cost is

$z = 16 \times 140 + 12 \times 60 + 14 \times 40 + 8 \times 120 + 16 \times 90 \;\; =$ Rs. 5920

Example 3: The following table provides all the necessary information on the availability of supply to each warehouse, the requirement of each market, and the unit transportation cost (in Rupees) from each warehouse to each market.

	Market				
	P	Q	R	S	Supply
A	6	3	5	4	22
Warehouse B	5	9	2	7	15
C	5	7	8	6	8
Demand	7	12	17	9	

The shipping clerk of the shipping agency has worked out the following schedule, based on his own experience: 12 units from A to Q, 1 unit from A to R, 8 units from A to S, 15 units from B to R, 7 units from C to P and 1 unit from C to R.

a) Check and see if the clerk has the optimal schedule.
b) Find the optimal schedule and minimum transportation cost.

Solution 3:

a) The shipping schedule determined by the clerk based on his experience is shown below:

	P	Q	R	S	Supply	u_i
A	6	3 (12)	5 (1)	4 (9)	22	0
B	5	9	2 (15)	7	15	-3
C	5 (7)	7	8 (1)	6	8	3
Demand	7	12	17	9	45	
v_j	2	3	5	4		

IBFS: The total transportation cost associated with this solution is

$z = 3 \times 12 + 5 \times 1 + 4 \times 9 + 2 \times 15 + 5 \times 7 + 8 \times 1 = \text{Rs. } 150$

b) Test for Optimality:

For occupied cells: Calculate u_i and v_j for rows and columns by using the relation $c_{ij} = u_i + v_j$.

Assign $u_1 = 0$ and calculate v_2, v_3 and v_4

$$c_{12} = u_1 + v_2 \Rightarrow 3 = 0 + v_2 \Rightarrow v_2 = 3$$
$$c_{13} = u_2 + v_3 \Rightarrow 5 = 0 + v_3 \Rightarrow v_3 = 5$$
$$c_{14} = u_3 + v_4 \Rightarrow 4 = 0 + v_4 \Rightarrow v_4 = 4$$

The values for u_2, u_3 and v_1 can also be calculated as shown below:

$$c_{23} = u_2 + v_3 \Rightarrow 2 = u_2 + 5 \Rightarrow u_2 = -3$$

$$c_{33} = u_3 + v_3 \Rightarrow 8 = u_3 + 5 \Rightarrow u_3 = 3$$

$$c_{31} = u_3 + v_1 \Rightarrow 5 = 3 + v_1 \Rightarrow v_1 = 2$$

For unoccupied cells: Calculate d_{ij} by using the relation

$$d_{ij} = c_{ij} - (u_i + v_j)$$

$$d_{11} = c_{11} - (u_1 + v_1) = 6 - (0 + 2) = 4$$
$$d_{21} = c_{21} - (u_2 + v_1) = 5 - (-3 + 2) = 6$$
$$d_{22} = c_{22} - (u_2 + v_2) = 9 - (-3 + 3) = 9$$
$$d_{24} = c_{24} - (u_2 + v_4) = 7 - (-3 + 4) = 6$$
$$d_{32} = c_{32} - (u_3 + v_2) = 7 - (3 + 3) = 1$$
$$d_{34} = c_{34} - (u_3 + v_4) = 6 - (3 + 4) = -1$$

Since $d_{34} = -1 < 0 \Rightarrow$ Solution is not optimal.

Trace the closed loop to introduce the cell (C, S) into the new transportation schedule, we get a new solution

	P	Q	R	S	Supply	u_i
A	6	3 (12)	5 (+) (1)	4 (9) (-)	22	0
B	5	9	2 (15)	7	15	-3
C	5 (7)	7	8 (-) (1)	6 (+)	8	3
Demand	7	12	17	9	45	
v_j	2	3	5	4		

Revised solution:

	P	Q	R	S	Supply	u_i
A	6	3 (12)	5 (2)	4 (8)	22	0
B	5	9	2 (15)	7	15	-3
C	5 (7)	7	8	6 (1)	8	2
Demand	7	12	17	9	45	
v_j	3	3	5	4		

For occupied cells: Calculate u_i and v_j for rows and columns by using the relation $c_{ij} = u_i + v_j$.

Assign $u_1 = 0$ and calculate $v_2, v_3$and v_4

$$
\begin{aligned}
c_{12} &= u_1 + v_2 \quad \Rightarrow 3 = 0 + v_2 \ \Rightarrow v_2 = 3 \\
c_{13} &= u_2 + v_3 \quad \Rightarrow 5 = 0 + v_3 \ \Rightarrow v_3 = 5 \\
c_{14} &= u_3 + v_4 \quad \Rightarrow 4 = 0 + v_4 \ \Rightarrow v_4 = 4
\end{aligned}
$$

The values for $u_2, u_3$and v_1 can also be calculated as shown below:

$$c_{23} = u_2 + v_3 \quad \Rightarrow 2 = u_2 + 5 \;\Rightarrow u_2 = -3$$

$$c_{34} = u_3 + v_4 \;\Rightarrow\; 6 = u_3 + 4 \;\;\Rightarrow u_3 = 2$$

$$c_{31} = u_3 + v_1 \;\Rightarrow\; 5 = 2 + v_1 \;\;\Rightarrow v_1 = 3$$

For unoccupied cells: Calculate d_{ij} by using the relation

$$d_{ij} = c_{ij} - \left(u_i + v_j\right)$$

$$\begin{aligned}
d_{11} &= c_{11} - (u_1 + v_1) = 6 - (0 + 3) = 3\\
d_{21} &= c_{21} - (u_2 + v_1) = 5 - (-3 + 3) = 5\\
d_{22} &= c_{22} - (u_2 + v_2) = 9 - (-3 + 3) = 9\\
d_{24} &= c_{24} - (u_2 + v_4) = 7 - (-3 + 4) = 6\\
d_{32} &= c_{32} - (u_3 + v_2) = 7 - (2 + 3) = 2\\
d_{33} &= c_{33} - (u_3 + v_3) = 8 - (2 + 5) = 1
\end{aligned}$$

Since all $d_{ij} > 0 \Rightarrow$ Solution is optimal

Therefore, the minimum transportation cost is

$z = 3 \times 12 + 5 \times 2 + 4 \times 8 + 2 \times 15 + 5 \times 7 + 6 \times 1 =$ Rs. 149

Degeneracy and Its Resolution

Degeneracy arises in transportation problem when the number of occupied cells are less than $m + n - 1$, where m is the number of rows and n is number of columns in transportation table.

The degeneracy in the transportation problems may occur at two stages:

a) When obtaining an initial basic feasible solution we may have less than $m + n - 1$ allocations.
b) At any stage while moving towards optimal solution. This happens when two or more occupied cells with the same minimum allocation are simultaneously unoccupied.

In such cases, the current solution cannot be improved upon because it is not possible to draw a closed path for every occupied cell. Also the values of dual multipliers u_i and v_j cannot be computed. Thus we need to remove degeneracy in order to improve solution.

How to resolve degeneracy?

- Allocate a very small quantity close to zero to one or more (if needed) unoccupied cells so as to get $m + n - 1$ number of occupied cells in independent positions (i.e. it does not form any closed-loop with adjacent cells). Denote this amount by ε (epsilon) or Δ (delta).
- In minimization transportation problem, allocate ε to unoccupied cell that have lowest transportation costs, whereas in maximization problem it should be allocated to a cell that has a high payoff value.

Case 1: Degeneracy occurs at initial stage

Consider the following transportation problem:

		Destinations				
		D_1	D_2	D_3	D_4	Supply
	S_1	10	2	20	11	15
Sources	S_2	12	7	9	20	25
	S_3	4	14	16	18	10
Demand		5	15	15	15	

Solution: Here, total supply $= 50 =$ total demand

$\Rightarrow$ The problem has a basic feasible solution.

We apply Vogel's approximation method to obtain an initial basic feasible solution given below:

	D_1	D_2	D_3	D_4	Supply	Row Penalties				
S_1	10	2 (15)	20	11	**15**	8	9	-	-	-
S_2	12	7	9 (15)	20 (10)	**25**	2	2	11	20	-
S_3	4 (5)	14	16	18 (5)	**10**	10	2	2	18	18
Demand	**5**	**15**	**15**	**15**						
Column Penalties	6	5	7	7						
	-	5	7	7						
	-	-	7	2						
	-	-	-	2						
	-	-	-	18						

Since the no. of positive allocations are 5$< m + n - 1 = 3 + 4 - 1 = 6$

$\Rightarrow$ The solution is degenerate.

Thus, an optimal solution cannot be obtained.

Allocate a very small quantity ε ($\varepsilon \to 0$) to the lowest cost cell in independent position (i.e. it does not form a closed-loop with adjacent cells)

So the new allocations are

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	10	2 (15)	20	11	**15**	-5
S_2	12	7 (ε)	9 (15)	20 (10)	**25**	0
S_3	4 (5)	14	16	18 (5)	**10**	-2
Demand	**5**	**15**	**15**	**15**		
v_j	6	7	9	20		

IBFS: The associated cost of the schedule is

$z = 2 \times 15 + 9 \times 15 + 20 \times 10 + 4 \times 5 + 18 \times 5 = \text{Rs. } 475$

For occupied cells: Calculate u_i and v_j for rows and columns by using the relation $c_{ij} = u_i + v_j$.

Assign $u_2 = 0$ and calculate $v_2,\ v_3$ and v_4 .

$$c_{22} = u_2 + v_2 \ \Rightarrow 7 = 0 + v_2 \ \Rightarrow \ v_2 = 7$$
$$c_{23} = u_2 + v_3 \ \Rightarrow 9 = 0 \ + v_3 \ \Rightarrow v_3 = 9$$
$$c_{24} = u_2 + v_4 \ \Rightarrow 20 = 0 \ + v_4 \Rightarrow v_4 = 20$$

The values for u_1, u_3 and v_1 can also be calculated as shown below: $c_{12} = u_1 + v_2 \ \Rightarrow 2 = u_1 + 7 \quad \Rightarrow u_2 = -5$

$$c_{34} = u_3 + v_4 \ \Rightarrow 18 = u_3 + 20 \ \Rightarrow u_3 = -2$$
$$c_{31} = u_3 + v_1 \ \Rightarrow 4 = -2 + v_1 \ \Rightarrow v_1 = 6$$

For unoccupied cells: Calculate d_{ij} by using the relation

$$d_{ij} = c_{ij} - \left(u_i + v_j\right)$$

$$d_{11} = c_{11} - (u_1 + v_1) = 10 - (-5 + 6) \ = 9$$
$$d_{13} = c_{13} - (u_1 + v_3) = 20 - (-5 + 9) = 16$$
$$d_{14} = c_{14} - (u_1 + v_4) = 11 - (-5 + 20) = -4$$
$$d_{21} = c_{21} - (u_2 + v_1) = 12 - (0 + 6) = 6$$
$$d_{32} = c_{32} - (u_3 + v_2) = 14 - (7 - 2) = 9$$
$$d_{33} = c_{33} - (u_3 + v_3) = 16 - (-2 + 9) = 9$$

Since $d_{14} = -4 < 0 \Rightarrow$ Solution is not optimal.

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	10	2 (-) 15	20	11 • (+)	15	-5
S_2	12	7 (+) ε	9 15	20 10 (-)	25	0
S_3	4 5	14	16	18 5	10	-2
Demand	5	15	15	15		
v_j	6	7	9	20		

Revised Solution:

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	10	2 5	20	11 10	**15**	-5
S_2	12	7 10+ε	9 15	20	**25**	0
S_3	4 5	14	16	18 5	**10**	-2
Demand	**5**	**15**	**15**	**15**		
v_j	6	7	9	20		

For occupied cells: Calculate u_i and v_j for rows and columns by using the relation $c_{ij} = u_i + v_j$.

Assign $u_2 = 0$ and calculate v_2 and v_3

$$c_{22} = u_2 + v_2 \quad \Rightarrow 7 = 0 + v_2 \ \Rightarrow v_2 = 7$$

$$c_{33} = u_3 + v_3 \quad \Rightarrow 9 = 0 + v_3 \;\Rightarrow v_3 = 9$$

The values for u_1, v_1 and v_4 can also be calculated as shown below:

$$c_{12} = u_1 + v_2 \quad \Rightarrow 2 = u_1 + 7 \Rightarrow u_1 = -5$$

$$c_{14} = u_1 + v_4 \quad \Rightarrow \; 11 \; = -5 + v_4 \quad \Rightarrow v_4 = 16$$

$$c_{34} = u_3 + v_4 \quad \Rightarrow \; 18 \; = u_3 + 16 \quad \Rightarrow u_3 = 2$$

$$c_{31} = u_3 + v_1 \quad \Rightarrow 4 = 2 + v_1 \Rightarrow v_1 = 2$$

For unoccupied cells: Calculate d_{ij} by using the relation

$$d_{ij} = c_{ij} - \left(u_i + v_j\right)$$

$$d_{11} = c_{11} - (u_1 + v_1) = 10 - (-5 + 2) = 13$$
$$d_{13} = c_{13} - (u_1 + v_3) = 20 - (-5 + 9) \quad = \; 16$$
$$d_{21} = c_{21} - (u_2 + v_1) = 12 - (0 + 2) = 10$$
$$d_{24} = c_{24} - (u_2 + v_4) = 20 - (0 + 16) = 4$$
$$d_{32} = c_{32} - (u_3 + v_2) = 14 - (2 + 7) = 5$$
$$d_{33} = c_{33} - (u_3 + v_3) = 16 - (2 + 9) = 5$$

Since all $d_{ij} > 0 \Rightarrow$ Solution is optimal

Therefore, as $\varepsilon \to 0$, the minimum transportation cost is

$z = 2 \times 5 + 11 \times 10 + 7 \times 10 + 9 \times 15 + 4 \times 5 \; =$ Rs. 435

Case 2: Degeneracy occurs at any stage

Consider the following transportation problem:

		Destinations				
		D_1	D_2	D_3	D_4	**Supply**
	S_1	10	2	20	11	**15**
Sources	S_2	12	7	9	20	**25**
	S_3	4	14	16	18	**10**
Demand		**5**	**15**	**15**	**15**	

Solution: Here, total supply = 50 = total demand

$\Rightarrow$ The problem has a basic feasible solution.

We apply Northwest-corner method to obtain an initial basic feasible solution given below:

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	10 (5)	2 (15)	20	11	**15**	-5
S_2	12	7 (5)	9 (15)	20 (10)	**25**	0
S_3	4	14	16	18 (5)	**10**	-2
Demand	**5**	**15**	**15**	**15**		
v_j	15	7	9	20		

Since the no. of positive allocations are $6 = m + n - 1 = 3 + 4 - 1$

$\Rightarrow$ The solution is non-degenerate.

Thus, an optimal solution can be obtained

<u>IBFS:</u> The associated cost of the schedule is

$$z = 10 \times 5 + 2 \times 10 + 7 \times 5 + 9 \times 15 + 20 \times 5 + 18 \times 10 \ = \text{Rs. } 520$$

For occupied cells: Calculate u_i and v_j for rows and columns by using the relation $c_{ij} = u_i + v_j$.

Assign $u_2 = 0$ and calculate $v_2,\ v_3$ and v_4 .

$$c_{22} = u_2 + v_2 \ \Rightarrow 7 = 0 + v_2 \ \Rightarrow \ v_2 = 7$$
$$c_{23} = u_2 + v_3 \ \Rightarrow 9 = 0 \ + v_3 \ \Rightarrow v_3 = 9$$
$$c_{24} = u_2 + v_4 \ \Rightarrow 20 = 0 \ + v_4 \ \Rightarrow v_4 = 20$$

The values for u_1, u_3 and v_1 can also be calculated as shown below:

$$c_{12} = u_1 + v_2 \quad \Rightarrow 2 = u_1 + 7 \quad \Rightarrow u_2 = -5$$
$$c_{34} = u_3 + v_4 \quad \Rightarrow 18 = u_3 + 20 \quad \Rightarrow u_3 = -2$$
$$c_{11} = u_1 + v_1 \quad \Rightarrow 10 = -5 + v_1 \quad \Rightarrow v_1 = 15$$

For unoccupied cells: Calculate d_{ij} by using the relation

$$d_{ij} = c_{ij} - (u_i + v_j)$$

$$d_{13} = c_{13} - (u_1 + v_3) = 20 - (-5 + 9) = 16$$
$$d_{14} = c_{14} - (u_1 + v_4) = 11 - (-5 + 20) = -4$$
$$d_{21} = c_{21} - (u_2 + v_1) = 12 - (0 + 15) = -3$$
$$d_{31} = c_{31} - (u_3 + v_1) = 4 - (-2 + 15) = -9$$
$$d_{32} = c_{32} - (u_3 + v_2) = 14 - (7 - 2) = 9$$
$$d_{33} = c_{33} - (u_3 + v_3) = 16 - (-2 + 9) = 9$$

Since all $d_{ij} \ngeq 0 \Rightarrow$ Solution is not optimal.
Select the cell with largest negative d_{ij} value as entering variable to get new solution.

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	10 (-) 5	2 15 (+)	20	11	15	-5
S_2	12	7 (-) 5	9 15	20 10 (+)	25	0
S_3	4 (+) •	14	16	18 5 (-)	10	-2
Demand	5	15	15	15		
v_j	15	7	9	20		

Revised Solution:

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	10	2 (15)	20	11	**15**	-5
S_2	12	7	9 (15)	20 (10)	**25**	0
S_3	4 (5)	14	16	18 (5)	**10**	-2
Demand	**5**	**15**	**15**	**15**		
v_j	6	7	9	20		

Here, no of allocations $= 5 < m + n - 1 = 3 + 4 - 1 = 6$

$\Rightarrow$ The solution is degenerate.

Thus, an optimal solution cannot be obtained.

Allocate a very small quantity ε ($\varepsilon \to 0$) to the lowest cost cell in independent position (i.e. it does not form a closed-loop with adjacent cells)

So the new allocations are

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	10	2 (15)	20	11	**15**	-5
S_2	12	7 (ε)	9 (15)	20 (10)	**25**	0
S_3	4 (5)	14	16	18 (5)	**10**	-2
Demand	**5**	**15**	**15**	**15**		
v_j	6	7	9	20		

For occupied cells: Calculate u_i and v_j for rows and columns by using the relation $c_{ij} = u_i + v_j$.

Assign$u_2 = 0$ and calculate $v_2,\ v_3$ and v_4 .

$$c_{22} = u_2 + v_2 \quad \Rightarrow 7 = 0 + v_2 \ \Rightarrow \ v_2 = 7$$
$$c_{23} = u_2 + v_3 \quad \Rightarrow 9 = 0\ + v_3 \quad \Rightarrow v_3 = 9$$

$$c_{24} = u_2 + v_4 \quad \Rightarrow 20 = 0 \; + v_4 \quad \Rightarrow v_4 = 20$$

The values for u_1, u_3 and v_1 can also be calculated as shown below:

$$c_{12} = u_1 + v_2 \quad \Rightarrow 2 = u_1 + 7 \quad \Rightarrow u_2 = -5$$
$$c_{34} = u_3 + v_4 \quad \Rightarrow 18 = u_3 + 20 \quad \Rightarrow u_3 = -2$$
$$c_{31} = u_3 + v_1 \quad \Rightarrow 4 = -2 + v_1 \quad \Rightarrow v_1 = 6$$

For unoccupied cells: Calculate d_{ij} by using the relation

$$d_{ij} = c_{ij} - \left(u_i + v_j\right)$$

$$d_{11} = c_{11} - (u_1 + v_1) = 10 - (-5 + 6) = 9$$
$$d_{13} = c_{13} - (u_1 + v_3) = 20 - (-5 + 9) = 16$$
$$d_{14} = c_{14} - (u_1 + v_4) = 11 - (-5 + 20) = -4$$
$$d_{21} = c_{21} - (u_2 + v_1) = 12 - (0 + 6) = 6$$
$$d_{32} = c_{32} - (u_3 + v_2) = 14 - (7 - 2) = 9$$
$$d_{33} = c_{33} - (u_3 + v_3) = 16 - (-2 + 9) = 9$$

Since $d_{14} = -4 < 0 \Rightarrow$ Solution is not optimal.

Trace the closed loop starting from the cell (S_1, D_4) in clockwise or anti-clockwise direction.

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	10	2 (-) (15)	20	11 • (+)	15	-5
S_2	12	7 (+) (ε)	9 (15)	20 (10) (-)	25	0
S_3	4 (5)	14	16	18 (5)	10	-2
Demand	5	15	15	15		
v_j	6	7	9	20		

Revised Solution:

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	10	2 (5)	20	11 (10)	**15**	-5
S_2	12	7 (10+ε)	9 (15)	20	**25**	0
S_3	4 (5)	14	16	18 (5)	**10**	-2
Demand	**5**	**15**	**15**	**15**		
v_j	6	7	9	20		

For occupied cells: Calculate u_i and v_j for rows and columns by using the relation $c_{ij} = u_i + v_j$.

Assign $u_2 = 0$ and calculate v_2 and v_3

$$c_{22} = u_2 + v_2 \;\Rightarrow 7 = 0 + v_2 \;\Rightarrow v_2 = 7$$
$$c_{33} = u_3 + v_3 \;\Rightarrow 9 = 0 + v_3 \;\Rightarrow v_3 = 9$$

The values for u_1, v_1 and v_4 can also be calculated as shown below:

$$c_{12} = u_1 + v_2 \quad \Rightarrow 2 = u_1 + 7 \Rightarrow u_1 = -5$$
$$c_{14} = u_1 + v_4 \quad \Rightarrow\ 11 = -5 + v_4 \quad \Rightarrow v_4 = 16$$
$$c_{34} = u_3 + v_4 \quad \Rightarrow\ 18 = u_3 + 16 \quad \Rightarrow u_3 = 2$$
$$c_{31} = u_3 + v_1 \quad \Rightarrow 4 = 2 + v_1 \Rightarrow v_1 = 2$$

For unoccupied cells: Calculate d_{ij} by using the relation

$$d_{ij} = c_{ij} - \left(u_i + v_j\right)$$

$$d_{11} = c_{11} - (u_1 + v_1) = 10 - (-5 + 2) = 13$$
$$d_{13} = c_{13} - (u_1 + v_3) = 20 - (-5 + 9) \quad = \ 16$$
$$d_{21} = c_{21} - (u_2 + v_1) = 12 - (0 + 2) = 10$$
$$d_{24} = c_{24} - (u_2 + v_4) = 20 - (0 + 16) = 4$$
$$d_{32} = c_{32} - (u_3 + v_2) = 14 - (2 + 7) = 5$$
$$d_{33} = c_{33} - (u_3 + v_3) = 16 - (2 + 9) = 5$$

Since all $d_{ij} > 0 \Rightarrow$ Solution is optimal

Therefore, as $\varepsilon \to 0$, the minimum transportation cost is

$z = 2 \times 5 + 11 \times 10 + 7 \times 10 + 9 \times 15 + 4 \times 5\ =$ Rs. 435

Alternate Optimal Solutions

If an unoccupied cell in an optimal solution has an opportunity cost of zero, an alternative optimal solution can be formed with another set of allocations, without increasing the total transportation cost.

Consider the following transportation problem:

	D_1	D_2	D_3	Supply
S_1	4	8	8	**76**
S_2	16	24	16	**82**
S_3	8	16	24	**77**
Demand	**72**	**102**	**41**	

Solution: Here total supply = 235 units
Total demand = 215 units
Surplus capacity = 20 units

Thus, supply and demand are not balanced in this case and we have a surplus of 20 units of the product. Therefore, we add dummy destination with associated cost coefficients are taken as zero.

Therefore, starting cost matrix becomes

	D_1	D_2	D_3	D_4	Supply
S_1	4	8	8	0	**76**
S_2	16	24	16	0	**82**
S_3	8	16	24	0	**77**
Demand	**72**	**102**	**41**	**20**	

We apply Vogel's approximation method to obtain an initial basic feasible solution given below:

	D_1	D_2	D_3	D_4	Supply	Row Penalties					
S_1	4	8 (76)	8	0	**76**	4	4	-	-	-	-
S_2	16	24 (21)	16 (41)	0 (20)	**82**	16	0	0	8	24	-
S_3	8 (72)	16 (5)	24	0	**77**	8	8	8	8	16	16
Demand	**72**	**102**	**41**	**20**							
Column Penalties	4	8	8	0							
	4	8	8	-							
	8	8	8	-							
	-	8	8	-							
	-	8	-	-							
	-	8	-	-							

Since the no. of positive allocations are $6 = m + n - 1 = 3 + 4 - 1$

$\Rightarrow$ The solution is non-degenerate.

Thus, an optimal solution can be obtained.

IBFS: The associated cost of the schedule is

$z = 8 \times 76 + 24 \times 21 + 16 \times 41 + 8 \times 72 + 16 \times 5 \ = \text{Rs. } 2424$

For occupied cells: Calculate u_i and v_j for rows and columns by using the relation $c_{ij} = u_i + v_j$.

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	4	8 (76)	8	0	76	-16
S_2	16	24 (21)	16 (41)	0 (20)	82	0
S_3	8 (72)	16 (5)	24	0	77	-8
Demand	72	102	41	20		
v_j	16	24	16	0		

Assign $u_2 = 0$ and calculate $v_2,\ v_3$ and v_4 .

$$c_{22} = u_2 + v_2 \ \Rightarrow 24 = 0 + v_2 \ \Rightarrow \ v_2 = 24$$
$$c_{23} = u_2 + v_3 \ \Rightarrow 16 = 0 \ + v_3 \ \Rightarrow v_3 = 16$$
$$c_{24} = u_2 + v_4 \ \Rightarrow 0 = 0 \ + v_4 \Rightarrow v_4 = 0$$

The values for u_1, u_3 and v_1 can also be calculated as shown below:

$$c_{12} = u_1 + v_2 \ \Rightarrow 8 = u_1 + 24 \quad \Rightarrow u_2 = -16$$
$$c_{32} = u_3 + v_2 \ \Rightarrow 16 = u_3 + 24 \ \Rightarrow u_3 = -8$$
$$c_{31} = \ u_3 + v_1 \ \Rightarrow 8 = -8 + v_1 \ \Rightarrow v_1 = 16$$

For unoccupied cells: Calculate d_{ij} by using the relation

$$d_{ij} = c_{ij} - (u_i + v_j)$$

$$d_{11} = c_{11} - (u_1 + v_1) = 4 - (-16 + 16) \ = 4$$
$$d_{13} = c_{13} - (u_1 + v_3) = 8 - (-16 + 16) = 8$$
$$d_{14} = c_{14} - (u_1 + v_4) = 0 - (-16 + 0) = 16$$

$$d_{21} = c_{21} - (u_2 + v_1) = 16 - (0 + 16) = 0$$
$$d_{33} = c_{32} - (u_3 + v_2) = 24 - (-8 + 16) = 16$$
$$d_{34} = c_{34} - (u_3 + v_4) = 0 - (-8 + 0) = 8$$

The opportunity costs in all occupied cells are positive except for (S_2, D_1) which has a zero opportunity cost.

⇒ An alternate solution exists that can be found by entering (S_2, D_1) into the basis.

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	4	8 76	8	0	76	-16
S_2	16 (+) •	24 21 (-)	16 41	0 20	82	0
S_3	8 (-) 72	16 5 (+)	24	0	77	-8
Demand	72	102	41	20		
v_j	16	24	16	0		

Revised Solution:

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	4	8 76	8	0	76	-16
S_2	16 21	24	16 41	0 20	82	0
S_3	8 51	16 26	24	0	77	-8
Demand	72	102	41	20		
v_j	16	24	16	0		

For occupied cells: Calculate u_i and v_j for rows and columns by using the relation $c_{ij} = u_i + v_j$.

Assign $u_2 = 0$ and calculate v_1, v_3 and v_4

$$c_{21} = u_2 + v_1 \;\Rightarrow 16 = 0 + v_1 \;\Rightarrow v_1 = 16$$
$$c_{23} = u_2 + v_3 \;\Rightarrow 16 = 0 + v_3 \;\Rightarrow v_3 = 16$$
$$c_{24} = u_3 + v_4 \;\Rightarrow 0 = 0 + v_4 \;\Rightarrow v_4 = 0$$

The values for u_1, v_1 and v_4 can also be calculated as shown below:

$$c_{31} = u_3 + v_1 \;\Rightarrow 8 = u_3 + 16 \Rightarrow u_3 = -8$$
$$c_{32} = u_3 + v_2 \;\Rightarrow\; 16 = -8 + v_2 \;\Rightarrow v_2 = 24$$
$$c_{12} = u_1 + v_2 \;\Rightarrow\; 8 = u_1 + 24 \;\Rightarrow u_1 = -16$$

For unoccupied cells: Calculate d_{ij} by using the relation

$$d_{ij} = c_{ij} - \left(u_i + v_j\right)$$

$$d_{11} = c_{11} - (u_1 + v_1) = 4 - (-16 + 16) = 4$$
$$d_{13} = c_{13} - (u_1 + v_3) = 8 - (-16 + 16) = 8$$
$$d_{14} = c_{14} - (u_1 + v_4) = 0 - (-16 + 0) = 16$$
$$d_{22} = c_{22} - (u_2 + v_2) = 24 - (0 + 24) = 0$$
$$d_{33} = c_{32} - (u_3 + v_2) = 24 - (-8 + 16) = 16$$
$$d_{34} = c_{34} - (u_3 + v_4) = 0 - (-8 + 0) = 8$$

Since all $d_{ij} \geq 0 \Rightarrow$ Solution is optimal.

The associated cost of the schedule is

$z = 8 \times 76 + 16 \times 21 + 16 \times 41 + 8 \times 51 + 16 \times 26 =$ Rs. 2424

The Maximization Problem

The transportation problem may involve maximization of profit. It may be solved as a maximization problem itself. However, while finding the initial basic feasible solution, allocations are to be made in highest profit cells, rather than in lowest cost cells. Also solution will be optimal when all cell evaluations are non-positive(≤ 0).

Example: A company has 3 factories manufacturing the same product and 5 sale agencies in different parts of the country. Production costs differ from factory to factory and the sale prices from agency to agency. The shipping cost per unit from each factory to each agency is known. Given the following data, find the production and distribution schedules most profitable to the company.

Factory i	Production Cost/unit (Rs.)	Max. Capacity (No. of units)
1	18	140
2	20	190
3	16	115

1	2	2	6	10	5
Factory 2	10	8	9	4	7
3	5	6	4	3	8
Agency j	1	2	3	4	5
Demand	74	94	69	39	119
Sale price (Rs.)	35	37	36	39	34

Solution1: First construct a profit matrix,

Profit/unit = sales price – production cost – shipping cost

Thus, profit/unit from factory 1 to sales agency 1 = (35 – 18 – 2) = Rs.15,

Profit/unit from factory 1 to sales agency 2 = (37 – 20 – 2) = Rs. 17,

and so on.

The resulting profit matrix is

Sales agencies

	1	2	3	4	5	Capacity
1	15	17	12	11	11	140
Factories 2	5	9	7	15	7	190
3	14	15	16	20	10	115
Demand	74	94	69	39	119	

Total production capacity = 445 units

Total demand = 395 units

∴ Surplus capacity = 50 units.

Create a dummy sale agency to take up the excess capacity of 50 units. The associated unit profit is zero in each cell.

Sales agencies

		1	2	3	4	5	6	Capacity
	1	15	17	12	11	11	0	140
Factories	2	5	9	7	15	7	0	190
	3	14	15	16	20	10	0	115
Demand		74	94	69	39	119	50	

IBFS: Since the objective is to maximize the total profit, select the highest payoff value from the table and allocate as much as possible units to selected cell.

Sales agencies

		1	2	3	4	5	6	Capacity
	1	15 (46)	17 (94)	12	11	11	0	140
Factories	2	5 (21)	9	7	15	7 (119)	0 (50)	190
	3	14 (7)	15	16 (69)	20 (39)	10	0	115
Demand		74	94	69	39	119	50	

Since the no. of positive allocations are $8 = m + n - 1 = 6 + 3 - 1$

$\Rightarrow$ The solution is non-degenerate.

Thus, an optimal solution can be obtained.

The associated profit of the schedule is

$z = 15 \times 46 + 17 \times 94 + 5 \times 21 + 7 \times 119 + 14 \times 7 + 16 \times 69 + 20 \times 39$
$= \text{Rs. } 5208$

For occupied cells: Calculate u_i and v_j for rows and columns by using the relation $c_{ij} = u_i + v_j$.

	1	2	3	4	5	6	Supply	u_i
1	15 (46)	17 (94)	12	11	11	0	140	15
2	5 (21)	9	7	15	7 (119)	0 (50)	190	5
3	14 (7)	15	16 (69)	20 (39)	10	0	115	14
Demand	74	94	69	39	119	50		
v_j	0	2	2	6	2	- 5		

Assign $v_1 = 0$ and calculate u_1, u_2 and u_3

$$c_{11} = u_1 + v_1 \;\Rightarrow 15 = u_1 + 0 \;\Rightarrow u_1 = 15$$
$$c_{21} = u_2 + v_1 \;\Rightarrow 5 = u_2 + 0 \;\Rightarrow u_2 = 5$$
$$c_{31} = u_3 + v_1 \;\Rightarrow 14 = u_3 + 0 \;\Rightarrow u_3 = 14$$

The values forv_2, v_3, v_4, v_5 and v_6 can also be calculated as shown below:

$$c_{12} = u_1 + v_2 \;\Rightarrow 17 = 15 + v_2 \Rightarrow v_2 = 2$$
$$c_{25} = u_2 + v_5 \;\Rightarrow\; 7 \;= 5 + v_5 \;\Rightarrow v_5 = 2$$
$$c_{26} = u_2 + v_6 \;\Rightarrow\; 0 \;= 5 + v_5 \;\Rightarrow v_5 = -5$$
$$c_{33} = u_3 + v_3 \;\Rightarrow 16 = 14 + v_3 \Rightarrow v_3 = 2$$
$$c_{34} = u_3 + v_4 \;\Rightarrow 20 = 14 + v_4 \Rightarrow v_4 = 6$$

For unoccupied cells: Calculate d_{ij} by using the relation

$$d_{ij} = c_{ij} - (u_i + v_j)$$

$d_{13} = c_{13} - (u_1 + v_3) = 12 - (15 + 2) = -5$
$d_{14} = c_{14} - (u_1 + v_4) = 11 - (15 + 6) = -10$
$d_{15} = c_{15} - (u_1 + v_5) = 11 - (15 + 2) = -6$
$d_{16} = c_{16} - (u_1 + v_6) = 0 - (15 - 5) = -10$

$$d_{22} = c_{22} - (u_2 + v_2) = 9 - (5 + 2) = 2$$
$$d_{23} = c_{23} - (u_2 + v_3) = 7 - (5 + 2) = 0$$
$$d_{24} = c_{22} - (u_2 + v_4) = 15 - (5 + 6) = 4$$
$$d_{32} = c_{32} - (u_3 + v_2) = 15 - (14 + 2) = -1$$
$$d_{35} = c_{35} - (u_3 + v_5) = 10 - (14 - 5) = 1$$
$$d_{36} = c_{36} - (u_3 + v_6) = 0 - (14 - 5) = -9$$

Since $d_{ij} \nleq 0 \Rightarrow$ solution is not optimal.

Select the cell with largest positive d_{ij} value as a entering variable into the basis to maximize the profit.

	1	2	3	4	5	6	Supply	u_i
1	15 (46)	17 (94)	12	11	11	0	140	15
2	5 (-) (21)	9	7	15 ● (+)	7 (119)	0 (50)	190	5
3	14 (+) (7)	15	16 (69)	20 (39) (-)	10	0	115	14
Demand	74	94	69	39	119	50		
v_j	0	2	2	6	2	- 5		

Revised solution:

	1	2	3	4	5	6	Supply	u_i
1	15 (46)	17 (94)	12	11	11	0	140	6
2	5	9	7	15 (21)	7 (119)	0 (50)	190	0
3	14 (28)	15	16 (69)	20 (18)	10	0	115	5
Demand	74	94	69	39	119	50		
v_j	9	11	11	15	7	0		

For occupied cells: Calculate u_i and v_j for rows and columns by using the relation $c_{ij} = u_i + v_j$.

Assign $u_2 = 0$ and calculate v_4, v_5 and v_6

$$c_{24} = u_2 + v_4 \;\Rightarrow 15 = 0 + v_4 \;\Rightarrow v_4 = 15$$
$$c_{25} = u_2 + v_5 \;\Rightarrow 7 = 0 + v_5 \;\Rightarrow v_5 = 7$$
$$c_{26} = u_2 + v_6 \;\Rightarrow 0 = 0 + v_6 \;\Rightarrow v_6 = 0$$

The values for u_1,u_1, v_1 , v_2 and v_3 can also be calculated as shown below:

$$c_{34} = u_3 + v_4 \;\Rightarrow\; 20 = u_3 + 15 \;\Rightarrow u_3 = 5$$
$$c_{33} = u_3 + v_3 \;\Rightarrow\; 16 = 5 + v_3 \;\Rightarrow v_3 = 11$$
$$c_{31} = u_3 + v_1 \;\Rightarrow 14 = 5 + v_1 \Rightarrow v_1 = 9$$
$$c_{11} = u_1 + v_1 \;\Rightarrow\; 15 = u_1 + 9 \;\Rightarrow u_1 = 6$$

For unoccupied cells: Calculate d_{ij} by using the relation

$$d_{ij} = c_{ij} - (u_i + v_j)$$

$$d_{13} = c_{13} - (u_1 + v_3) = 12 - (6 + 11) = -5$$
$$d_{14} = c_{14} - (u_1 + v_4) = 11 - (6 + 15) = -10$$
$$d_{15} = c_{15} - (u_1 + v_5) = 11 - (6 + 7) = -2$$
$$d_{16} = c_{16} - (u_1 + v_6) = 0 - (6 + 0) = -6$$
$$d_{21} = c_{21} - (u_2 + v_1) = 5 - (0 + 9) = -4$$
$$d_{22} = c_{22} - (u_2 + v_2) = 9 - (0 + 11) = -2$$
$$d_{23} = c_{23} - (u_2 + v_3) = 7 - (0 + 11) = -4$$
$$d_{32} = c_{32} - (u_3 + v_2) = 15 - (5 + 11) = -1$$
$$d_{35} = c_{35} - (u_3 + v_5) = 10 - (5 + 7) = -2$$
$$d_{36} = c_{36} - (u_3 + v_6) = 0 - (5 + 0) = -5$$

Since $d_{ij} < 0 \Rightarrow$ solution is optimal.

Hence, the total maximum profit,

$z = 15 \times 46 + 17 \times 94 + 15 \times 21 + 7 \times 119 + 14 \times 28 + 16 \times 69 + 20 \times 18 =$ Rs. 5292

The Assignment Model

- An assignment problem is a particular case of transportation problem where the sources are assignees (facilities) and the destinations are tasks (jobs).
- Every source has a supply of 1 and every destination has a demand of 1.
- Objective is to minimize total cost, distance, time or maximize the profit.
- There are many applications of assignment problem in allocation and scheduling, for example in assigning salesmen to different regions, vehicles and drivers to different routes, products to factories, jobs to machines etc.

Definition of the Assignment Model:

Given n facilities and n jobs and given the effectiveness of each facility for each job, the problem is to assign each facility to one and only one job so as to optimize the given measure of effectiveness.

The general assignment model with n facilities or origins and n jobs is represented in table given below:

		Jobs				
		1	**2**	$\cdots$	$\boldsymbol{n}$	**Supply**
Facilities	**1**	c_{11}	c_{12}	$\cdots$	c_{1n}	**1**
	2	c_{21}	c_{22}	$\cdots$	c_{2n}	**1**
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	**1**
	$\boldsymbol{n}$	c_{n1}	c_{n2}	$\cdots$	c_{nn}	**1**
Demand		**1**	**1**	**1**	**1**	

There is no loss of generality in assuming that the number of facilities always equals the number of jobs, because we can always add fictitious facilities or fictitious jobs to satisfy this assumption.

Solution of the Assignment Models:

Hungarian Method

Step 1: Prepare a square matrix

Step 2: Reduce the matrix

Subtract the smallest element of each row from all the elements of that row. So there will be at least one zero in each row. Examine if there is at least one zero in each column. If not, subtract the minimum element of column(s) not containing zero from all the elements of the column(s).

Step 3: Check whether an optimal assignment can be made in the reduced matrix or not. For this proceed as follows:

a) Examine rows successively until a row with exactly one unmarked zero is obtained. Make assignment to single zero by making square around it. Cross all other zeros in the same column as they will not be considered for making any more assignments in the column. Proceed in this way until all rows have been examined.
b) Now examine columns successively until a column with exactly one unmarked zero is found. Make an assignment there by making a square around it and cross any other zeros in the same row.

In case there is no row or no column containing single unmarked zero (they contain more than one unmarked zero), mark square around any unmarked zero arbitrarily and cross all other zeros in its row and column. Proceed in this manner till there is no unmarked zero left in the cost matrix.

Repeat the sub –steps (a) and (b) until one of the following two

things occur;

i. There is one assignment in each row and in each column. In this case optimal assignment can be made in the current solution, i.e., the current feasible solution is an optimal solution. The minimum number of line crossing all zeros is n, the order of the matrix.
ii. There is some rows and/or column without assignment. In this case optimal assignment cannot be made in the current solution. The minimum number of line crossing all zeros have to be obtained in this case by following step 4.

Step 4: *Find the minimum number of lines crossing all zeros.* This consists of the following sub-steps;

a) Mark (√) the rows that do not have assignments.
b) Mark (√) the columns (not already marked) that have zeros in marked rows.
c) Mark (√) the rows (not already marked) that have assignment in marked columns.

d) Repeat the sub-steps (a) and (b) till no more rows or columns can be marked.
e) Draw straight line through all unmarked rows and marked columns. This gives the minimum number of lines crossing all zeros. If this number is equal to the order of the matrix, then it is an optimal solution, otherwise go to step 5.

Step 5: Iterate towards the optimal solution. Examine the uncovered elements. Select the smallest element and subtract it from all the uncovered elements. Add this smallest element to every element that lies at the intersection of two lines. Leave the remaining elements of the matrix as such. This yields second basic feasible solution.

Step 6: Repeat the step 3 through 5 successively until the number of lines crossing all zeros becomes equal to order of the matrix. In such case every row and column will have one assignment. This indicates that an optimal solution has been obtained. The total cost associated with this solution is obtained by adding the original costs in the assigned cells.

Solution of the Assignment Model

Example 1: A departmental head has four subordinates and four tasks to be performed. The subordinates differ in efficiency and tasks differ in their intrinsic difficulty. His estimates of the times that each man would take to perform each task are given in the matrix below:

		Tasks			
		I	II	III	IV
Subordinates	A	8	26	17	11
	B	13	28	4	26
	C	38	19	18	15
	D	19	26	24	10

How should the tasks be allocated to subordinates so as to minimize the total man-hours?

Solution 1: The given matrix is a square matrix.

Reduce the matrix:

Row-wise: Select the smallest element from each row and subtract that element from all the elements of that row.

0	18	9	3
9	14	0	22
23	4	3	0
9	16	14	0

Column-wise: Subtract the minimum element of the column not containing a zero element from all the elements of that column.

0	14	9	3
9	10	0	22
23	0	3	0
9	12	14	0

Check optimality:

Substep 1: Examine rows successively until a row with exactly one unmarked zero is found. Mark (□) this zero, and cross out all other zeros in the same column. Proceed in this manner until all rows have been examined.

[0]	14	9	3
9	10	[0]	22
23	[0]	3	~~0~~
9	12	14	[0]

Here no. of assignments = order of matrix

⇒ Solution is optimal.

The tasks will be allocated to the subordinates in the following manner:

A→I, B→III, C→II, D→ IV

And the minimum hours will be taken them is,

$8 + 4 + 19 + 10 = 41$ Hours

Example 2: Four different jobs can be done on four different machines. The matrix below gives the cost in rupees of producing job i on machine j:

Machines

Jobs		M_1	M_2	M_3	M_4
	J_1	5	7	11	6
	J_2	8	5	9	6
	J_3	4	7	10	7
	J_4	10	4	8	3

How should the jobs be assigned to various machines so that the total cost is minimized?

Solution 2: The given matrix is a square matrix.

Reduce the matrix:

Row-wise: Select the smallest element from each row and subtract that element from all the elements of that row.

0	2	6	1
3	0	4	1
0	3	6	3
7	1	5	0

Column-wise: Subtract the minimum element of the column not containing a zero element from all the elements of that column.

0	2	2	1
3	0	0	1
0	3	2	3
7	1	1	0

Check optimality:

Substep 1: Examine rows successively until a row with exactly one unmarked zero is found. Mark (□) this zero, and cross out all other zeros in the same column. Proceed in this manner until all rows have been examined.

[0]	2	2	1
3	[0]	~~0~~	1
~~0~~	3	2	3
7	1	1	[0]

Here no. of assignments = 3 $\neq$ order of the matrix

$\Rightarrow$ Solution is not optimal.

Substep 2: Mark the rows which does not have an assignment. Thus the third row is marked.
Substep 3: Mark the columns which have zeros in marked rows. Thus column 1 is marked.
Substep 4: Mark rows (not already marked) which have assignments in marked columns. Thus the row 1 is marked.
Substep 5: Draw lines through all unmarked rows and through all marked columns. This gives the minimum number of lines crossing all zeros.

0	2	2	1
3	0	0	1
0	3	2	3
7	1	1	0

Select the smallest element from uncovered entries of the matrix and subtract it from all the elements that do not have a line through them. Add this smallest element to every element that lies at the intersection of two lines. Leave the remaining entries of the matrix unchanged. We get the following matrix:

0	1	1	0
4	0	0	1
0	2	1	2
8	1	1	0

Check Optimality:

0	1	1	0
4	0	0	1
0	2	1	2
8	1	1	0

Revised solution:

~~0~~	~~0~~	~~0~~	[0]
5	[0]	~~0~~	2
[0]	1	~~0~~	2
8	~~0~~	[0]	~~0~~

Here no. of assignments = order of matrix

⇒ Solution is optimal.

The tasks will be allocated to the subordinates in the following manner:

$$J_1 \rightarrow M_4, J_2 \rightarrow M_2, J_3 \rightarrow M_1, J_4 \rightarrow M_3$$

And the minimum cost will is,

$$6 + 5 + 4 + 8 = Rs. 23$$

Variations of the Assignment Problem

1. **Maximization Problem:** The maximization problem has to be changed to minimization before the Hungarian method may be applied. This transformation can be done in either of the following two ways:
   a) By subtracting all the elements from the largest element of the matrix.
   b) By multiplying the matrix elements by -1.

The Hungarian method can then be applied to this equivalent minimization problem to obtain the optimal solution.

2. **Restrictions on assignments:** Sometimes technical, space, legal or other restrictions do not permit the assignment of a particular facility to a particular job. Such problems can be solved by assigning a very heavy cost (infinite cost) i.e., ∞ or M to the corresponding cell.

3. **Alternate optimal solutions:** Sometimes, it is possible to have two or more ways to strike of all zero elements in the reduce matrix for a given problem. In such cases, there will be alternate optimal solutions with the same cost. Alternate optimal solutions offer a great flexibility to the management since it can select the one which is most suitable to its requirement.

Example 1: (Maximization Problem)

A company has a team of four salesmen and there are four districts where the company wants to start its business. After taking into account the capabilities of salesmen and the nature of districts, the company estimates that the profit per day in rupees for each salesman in each district is as below:

		District			
		1	2	3	4
Salesman	A	16	10	14	11
	B	14	11	15	15
	C	15	15	13	12
	D	13	12	14	15

Find the assignment of salesmen to various districts which will yield maximum profit?

Solution 1: Given problem is of maximization type, change it to minimization type before solving it by Hungarian method. This is achieved by subtracting all the elements of the matrix from the largest element in it.

Resultant matrix:

0	6	2	5
2	5	1	1
1	1	3	4
3	4	2	1

Reduce the matrix:
Row-wise: Subtract the minimum element of each row from all the elements of that row.

0	6	2	5
1	4	0	0
0	0	2	3
2	3	1	0

All the columns in above matrix contain at least one zero.

Check optimality:

[0]	6	2	5
1	4	[0]	~~0~~
~~0~~	[0]	2	3
2	3	1	[0]

Here no. of assignments = order of matrix

⇒ Solution is optimal.

Assignment policy shall be:

A→1, B→3, C→2, D→ 4

And the maximum profit per day,

$$16 + 15 + 15 + 15 = Rs. 61$$

Example 2: (Restrictions on assignments)

Four new machines M_1, M_2, M_3 and M_4 are to be installed in a machine shop. There are five vacant spaces A, B, C, D and E available. Because of the limited space, machine M_2 cannot be placed at C and M_3 cannot be placed at A. The assignment cost of machine i to place j in rupees is shown below:

Machines \ Places	A	B	C	D	E
M_1	4	6	10	5	6
M_2	7	4	-	5	4
M_3	-	6	9	6	2
M_4	9	3	7	2	3

Find the optimal assignment schedule?

Solution 2: The given matrix is a square matrix. Here we add a dummy machine, M_5 and associate zero cost with the corresponding cells. As machine M_2 cannot be placed at C and M_3 cannot be placed at A, we assign infinite cost in cells (M_2, C) and(M_3, A).

Resultant matrix:

		Places				
		A	B	C	D	E
Machines	M_1	4	6	10	5	6
	M_2	7	4	∞	5	4
	M_3	∞	6	9	6	2
	M_4	9	3	7	2	3
	M_5	0	0	0	0	0

Reduce the matrix:

Row-wise: Select the smallest element from each row and subtract that element from all the elements of that row.

0	2	6	1	2
3	0	∞	1	0
∞	4	7	4	0
7	1	5	0	1
0	0	0	0	0

All the columns in above matrix contain at least one zero.

Check optimality:

[0]	2	6	1	2
3	[0]	∞	1	~~0~~
∞	4	7	4	[0]
7	1	5	[0]	1
~~0~~	~~0~~	[0]	~~0~~	~~0~~

Here no. of assignments = 5 = order of the matrix

⇒ Solution is optimal.

The optimal assignment of various machines is as follows:

$M_1 \rightarrow A, M_2 \rightarrow B, M_3 \rightarrow E, M_4 \rightarrow$ D and place C will remain vacant.

Total assignment cost = 4 + 4 + 2 + 2 = Rs. 12

Example 3: (Alternate optimal solutions)

Find the optimum schedule to minimize the total cost.

	I	II	III	IV
A	2	3	4	5
B	4	5	6	7
C	7	8	9	8
D	3	5	8	4

Solution 3: The given matrix is a square matrix.

Reduce the matrix:

Row-wise: Select the smallest element from each row and subtract that element from all the elements of that row.

0	1	2	3
0	1	2	3
0	1	2	1
0	2	5	1

Column-wise: Subtract the minimum element of the column not containing a zero element from all the elements of that column.

0	0	0	2
0	0	0	2
0	0	0	0
0	1	3	0

There is no row or column with single zero.

⇒ Alternate optimal solutions exist.

Check optimality:

~~0~~	~~0~~	[0]	2
~~0~~	[0]	~~0~~	2
~~0~~	~~0~~	~~0~~	[0]
[0]	1	3	~~0~~

OR

[0]	~~0~~	~~0~~	2
~~0~~	[0]	~~0~~	2
~~0~~	~~0~~	[0]	~~0~~
~~0~~	1	3	[0]

OR

~~0~~	[0]	~~0~~	2
~~0~~	~~0~~	[0]	2
[0]	~~0~~	~~0~~	~~0~~
~~0~~	1	3	[0]

OR

~~0~~	~~0~~	[0]	2
~~0~~	[0]	~~0~~	2
[0]	~~0~~	~~0~~	~~0~~
~~0~~	1	3	[0]

OR

~~0~~	~~0~~	[0]	2
[0]	~~0~~	~~0~~	2
~~0~~	[0]	~~0~~	~~0~~
~~0~~	1	3	[0]

4. Network Models

Project

Definition: A project is defined as a combination of interrelated activities which must be executed in a certain order before the entire task is completed.

Each project has three basic requirements:

i. It should be completed without any delay.

ii. It should use as small man power and other resources as possible.

iii. It should involve as small investment as possible.

The project management involves the following three phases:

i. Project planning

ii. Project scheduling

iii. Project controlling

Out of three phases, the first two are accomplished before the start of the actual project. The third phase comes into operation during the execution of the project.

Project Planning:

1. Manpower resources
2. Equipment resources
3. Financial resources
4. Material resources
5. Time and Space resources

Project scheduling:

1. Start and finishing times of each activity and the earliest and latest times at which events can occur.
2. Critical activities that require special attention.

3. Allocating resources- men, machines, materials etc to each activity.
4. Slacks and floats for non-critical activities.
5. Various constraints due to limitation of resources.

Project Controlling:

1. Setting standards and targets with regard to time and cost of the project.
2. Reviewing the progress by comparing the work accomplished to the work scheduled at different stages of time and finding the deviations.
3. Evaluating the effect of deviations on the project plan.
4. Updating the project schedule.
5. The corrective measures to rectify the deviations from the plan should be suggested. This requires decision-making with regard to scheduling of resources, scheduling of jobs, crashing of projects etc.

Network Diagrams

Activity: An activity is represented by an arrow, the tail of which represents the start and the head, the finish of the activity.

Event/Node: The beginning and end points of activity are called as events or nodes. It is represented by a circle.

Path: An unbroken chain of activity arrows connecting the initial event to some other event is called a path.

Project Management Techniques

- **Network Techniques:**

CPM and PERT are the two most widely used techniques. These are the network based methods designed to assist the project management in planning, scheduling and control of the project. The objective is to provide analytic means for scheduling the activities.

The **CPM** assumes **deterministic** activity durations and **PERT** assumes **probabilistic** durations.

Network Construction

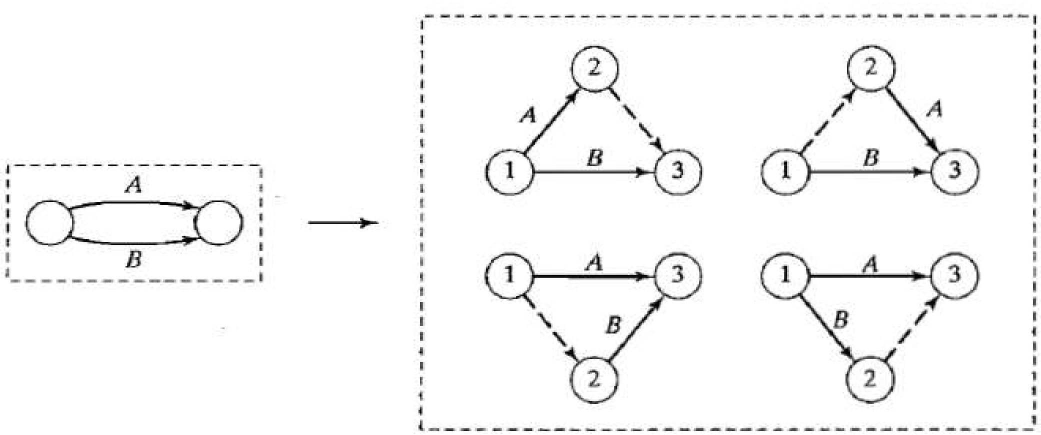
Each activity of the project is represented by an arc pointing in the direction of project.

- Rule 1: Each activity is represented by one and only arc.
- Rule 2: Each activity must be identified by two distinct nodes.

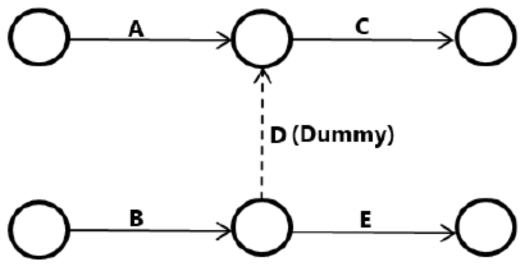
Dummy activity: An activity which only determines the dependency of one activity on the other, but does not consume any time is called a dummy activity.

- Adding dummy activity:

Use of dummy activity to produce unique representation of concurrent activities



The start of activity C depends upon the completion of activities A and B and the start of activity E depends only on the completion of activity B.

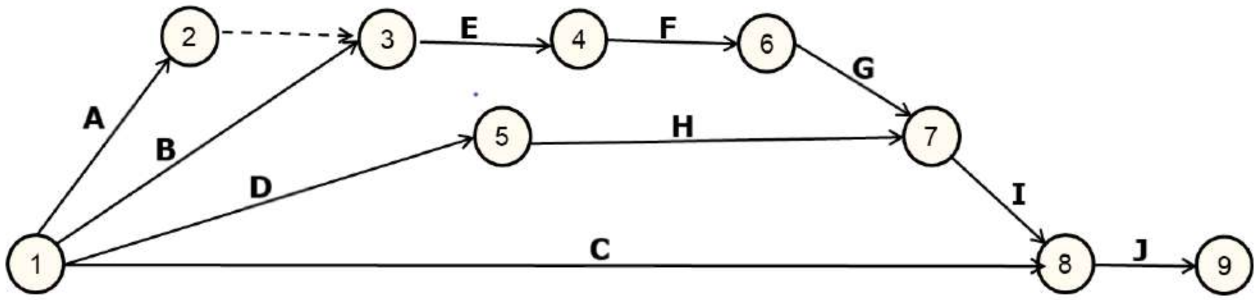


Example 1: A publisher has a contract with an author to publish a textbook. The activities associated with the production of the textbook are given below. The author is required to submit to the publisher a hard copy and computer file of the manuscript. Develop the associated network for the project.

Activity	Predecessor (s)
A: Manuscript proofreading by editor	-
B: Sample pages preparation	-
C: Book cover design	-
D: Artwork preparation	-
E: Author's approval of edited manuscript and sample pages	A, B
F: Book formatting	E
G: Author's review of formatted pages	F
H: Author's review of artwork	D
I: Production of printing plates	G, H
J: Book production and binding	C, I

Solution 1:

Network Diagram:

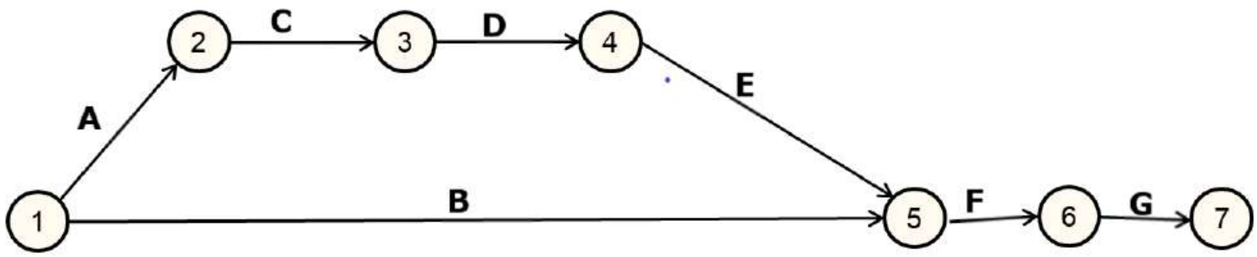


Example 2: A company is in the process of preparing a budget for launching a new product. The following table provides the associated activities. Construct a project network.

Activity	Predecessor(s)
A: Forecast sales volume	-
B: Study competitive market	-
C: Design item and facilities	A
D: Prepare production schedule	C
E: Estimate cost of production	D
F: Set sales price	B, E
G: Prepare budget	E, F

Solution 2:

Network Diagram:

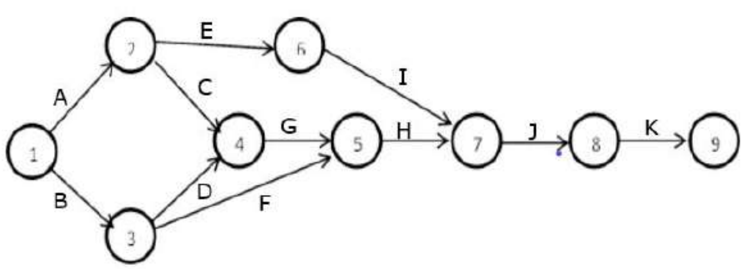


Example 3: Draw a network for the simple project of erection of steel works for a shed. The various activities of the project are as under:

Activity	Description	Preceded by
A	Erect site workshop	-
B	Fence site	-
C	Bend reinforcement	A
D	Dig foundation	B
E	Fabricate steel work	A
F	Install concrete pillars	B
Activity	**Description**	**Preceded by**
G	Place reinforcement	C, D
H	Concrete foundation	G, F
I	Erect steel work	E
J	Paint steel work	H, I
K	Give finishing touch	J

Solution 3:

Network Diagram:

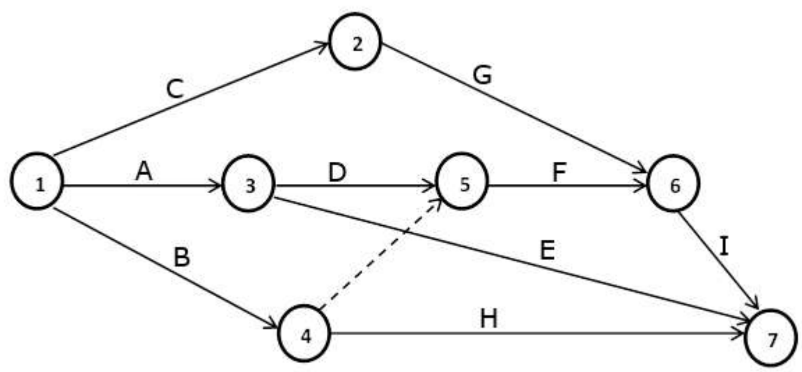


Example 4: A project consists of a series of tasks labeled A, B,…, H, I with the following relationships (W<X , Y means X and Y cannot start until W is completed; X, Y<W means W cannot start until both X and Y are completed). With this notation construct the network diagram having the following constraints:

$$A < D, E; B, D < F; C < G; \ B < H; \ F, G < I$$

Solution 4:

Network Diagram:

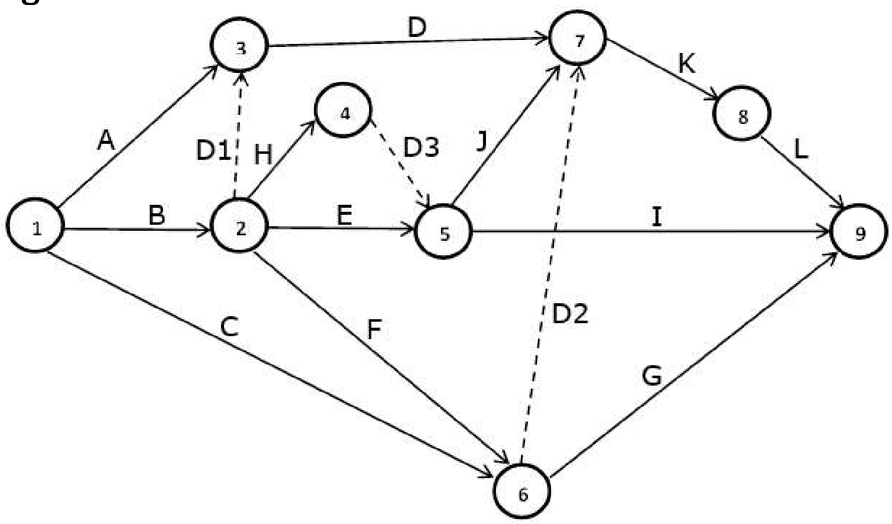


Example 5: Construct the project network comprised of activities A to L with the following precedence relationships:

- **a)** A, B and C, the first activities of the project, can be executed concurrently.
- **b)** A and B precede D.
- **c)** B precedes E, F and H.
- **d)** F and C precede G.
- **e)** E and H precede I and J.
- **f)** C, D, F and J precede K.
- **g)** K precedes L.
- **h)** I, G and L are the terminal activities of the project.

Solution 5:

Network Diagram:



Critical Path (CPM) Computations

CPM is generally used for simple, repetitive types of projects for which the activity times and costs are certainly and precisely known. It is deterministic rather than probabilistic model.

CPM is the construction of the time schedule for the project. These computations produce the following information:

1. Total duration needed to complete the project.
2. Classification of the activities of the project as critical and noncritical.

A **noncritical** activity allows some scheduling slack, so that the start time of the activity can be advanced or delayed within limits without affecting the completion date of the entire project.

Critical activities are the ones which must be started and completed on schedule or else the project may get delayed.

Critical Path is the path containing critical activities and is the longest path in terms of duration.

Critical Path Analysis: The critical path of a network gives the shortest time in which the whole project can be completed.

- Calculate the time schedule for each activity.
- Calculate the time schedule for the completion of the entire project.
- Identify the critical activities and find the critical path.

Notations:

$\square_j$= Earliest occurrence time of event j

$\triangle_j$= Latest occurrence time of event j

D_{ij}= Duration of activity (i, j)

The earliest and latest occurrences of event j are relative to the start and completion dates of the entire project.

The critical path method involves two passes: Forward pass and backward pass.

Forward Pass: The forward pass determines the earliest occurrence times of the events.

Backward Pass: The backward pass determines the latest occurrence times of the events.

Forward Pass (Earliest Occurrence times, $\square$)
The computations start at node 1 and advance recursively to end node n.
Initial step: Set $\square_1 = 0$ to indicate that the project starts at time 0.

General step j : Given that nodes $p, q, \ldots$, and v are linked directly to node j by incoming activities $(p, j), (q, j), \ldots$, and $(v, j)$and that the earliest occurrence times of events (nodes) have already been computed, then the earliest occurrence time of event j is computed as

$$\square_j = \max\{\square_p + D_{pj}, \square_q + D_{qj}, \ldots, \square_v + D_{vj}\}$$

The forward pass is complete when $\square_n$ at node n has been computed.

Backward Pass (Latest Occurrence times,$\triangle$): Following the completion of the forward pass, the backward pass computations start at node n and end at node 1.
Initial step: Set $\triangle_n = \square_n$ to indicate that the earliest and latest occurrences of the last node of the project are the same.

General step j : Given that nodes $p, q, \ldots$, and v are linked directly to node j by outgoing activities $(j, p), (j, q), \ldots$, and $(j, v)$and that the latest occurrence times of events (nodes) have already been computed, then the latest occurrence time of event j is computed as

$$\triangle_j = \min\{\triangle_p - D_{jp}, \triangle_q - D_{jq}, \ldots, \triangle_v - D_{jv}\}$$

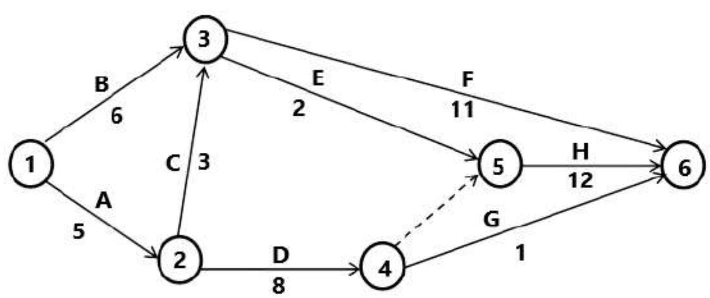
The backward pass is complete when $\triangle_1$ at node 1 has been computed. At this point, $\triangle_1 = \square_1 (= 0)$.

An activity (i, j) will be critical if it satisfies three conditions

1. $\triangle_i = \square_i$
2. $\triangle_j = \square_j$
3. $\triangle_j - \triangle_i = \square_j - \square_i = D_{ij}$

The three conditions state that the earliest and latest occurrence times of end nodes i and j are equal and the duration D_{ij} fits tightly in the specified time span.

Example 1: Determine the critical path for the following project network. All the durations are in days.



Solution 1:

Forward Pass

Node 1: Set $\square_1 = 0$

Node 2: $\square_2 = \square_1 + D_{12} = 0 + 5 = 5$

Node 3: $\square_3 = \max\{\square_1 + D_{13}, \square_2 + D_{23}\} = \max\{0 + 6, 5 + 3\} = 8$

Node 4: $\square_4 = \square_2 + D_{24} = 5 + 8 = 13$

Node 5: $\square_5 = \max\{\square_3 + D_{35}, \square_4 + D_{45}\} = \max\{8+2, 13+0\} = 13$

Node 6: $\square_6 = \max\{\square_3 + D_{36}, \square_4 + D_{46}, \square_5 + D_{56}\}$

$= \max\{8+11, 13+1, 13+12\} = 25$

The computations show that the project can be completed in 25 days.

Backward Pass

Node 6: Set $\triangle_6 = \square_6 = 25$

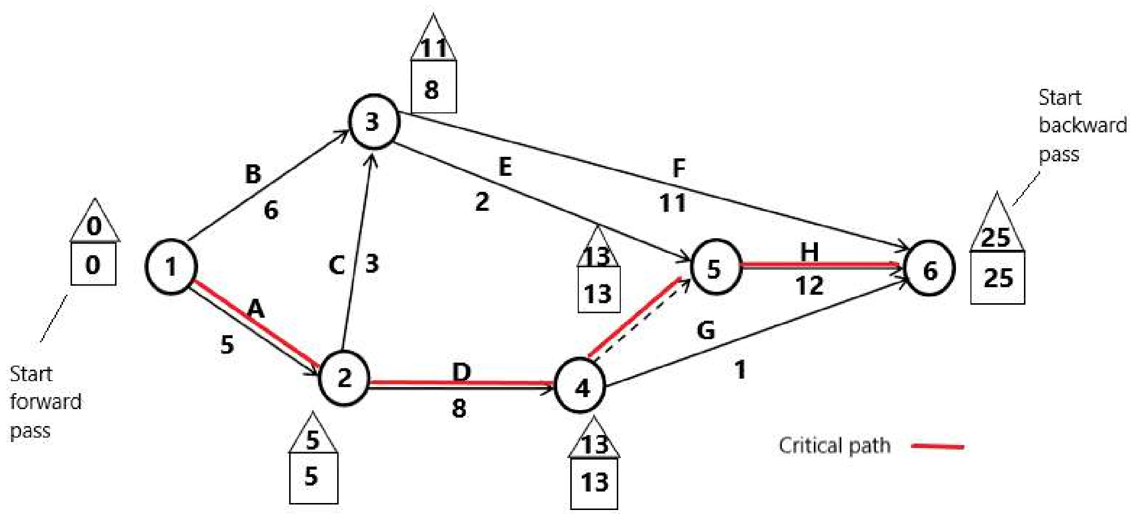
Node 5: $\triangle_5 = \triangle_6 - D_{56} = 25 - 12 = 13$

Node 4: $\triangle_4 = \min\{\triangle_6 - D_{46}, \triangle_5 - D_{45}\} = \min\{25-1, 13-0\} = 13$

Node 3: $\triangle_3 = \min\{\triangle_5 - D_{35}, \triangle_6 - D_{36}\} = \min\{13-2, 25-11\} = 11$

Node 2: $\triangle_2 = \min\{\triangle_3 - D_{23}, \triangle_4 - D_{24}\} = \min\{11-3, 13-8\} = 5$

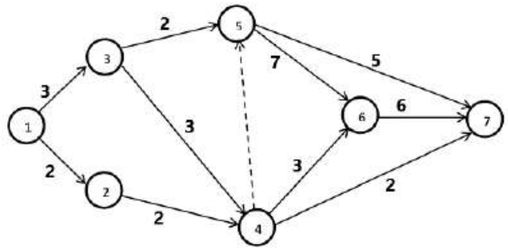
Node 1: $\triangle_1 = \min\{\triangle_2 - D_{12}, \triangle_3 - D_{13}\} = \min\{5-5, 11-6\} = 0$



The critical path is $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 6$, which, as should be expected, spans the network from start (node 1) to finish (node 6). The sum of the durations of the critical activities equals the duration of the project (=25 days).

Observe that activity (4,6) satisfies the first two conditions for a critical activity but not the third($\square_6 - \square_4 \neq D_{46}$). Hence the activity is not critical.

Example 2: Determine the critical path for the following project network.



Solution 2:

Forward Pass

Node 1: Set $\square_1 = 0$

Node 2: $\square_2 = \square_1 + D_{12} = 0 + 2 = 2$

Node 3: $\square_3 = \square_1 + D_{13} = 0 + 3 = 3$

Node 4: $\square_4 = \max\{\square_2 + D_{24}, \square_3 + D_{34}\} = \max\{2 + 2, 3 + 3\} = 6$

Node 5: $\square_5 = \max\{\square_3 + D_{35}, \square_4 + D_{45}\} = \max\{3 + 2, 6 + 0\} = 6$

Node 6: $\square_6 = \max\{\square_4 + D_{46}, \square_5 + D_{56}\} = \max\{6 + 3, 6 + 7\} = 13$

Node 7: $\square_7 = \max\{\square_4 + D_{47}, \square_5 + D_{57}, \square_6 + D_{67}\}$

$= \max\{6 + 2, 6 + 5, 13 + 6\} = 19$

The computations show that the project can be completed in 19 days.

Backward Pass

Node 7: Set $\triangle_7 = \square_7 = 19$

Node 6: $\triangle_6 = \triangle_7 - D_{67} = 19 - 6 = 13$

Node 5: $\triangle_5 = \min\{\triangle_6 - D_{56}, \triangle_7 - D_{57}\} = \min\{13 - 7, 19 - 5\} = 6$

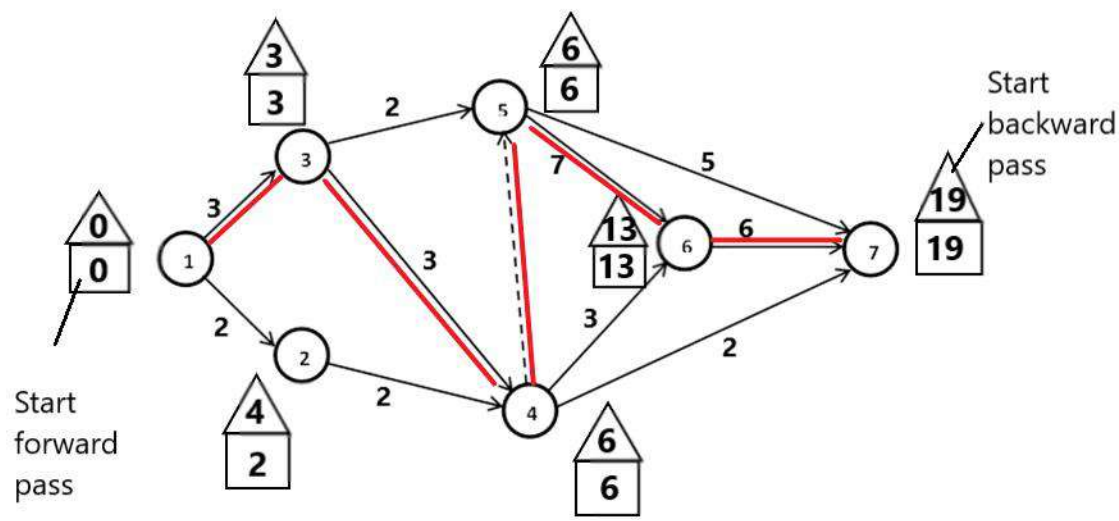
Node 4: $\triangle_4 = \min\{\triangle_5 - D_{45}, \triangle_6 - D_{46}, \triangle_7 - D_{47}\}$

$= \min\{6 - 0, 13 - 3, 19 - 2\} = 6$

Node 3: $\triangle_3 = \min\{\triangle_4 - D_{34}, \triangle_5 - D_{35}\} = \min\{6 - 3, 6 - 2\} = 3$

Node 2: $\triangle_2 = \triangle_4 - D_{24} = 6 - 2 = 4$

Node 1: $\triangle_1 = \min\{\triangle_2 - D_{12}, \triangle_3 - D_{13}\} = \min\{4 - 2, 3 - 3\} = 0$



The critical path is1 → 3 → 4 → 5 → 6 → 7, which, as should be expected, spans the network from start (node 1) to finish (node 7). The sum of the durations of the critical activities equals the duration of the project (=19 days).

Determination of the Floats

Floats are the slack times available within the allotted span of the noncritical activity.

The Total Float (TF_{ij}) for an activity (i, j) is the excess of the time span defined from the *earliest* occurrence of event i to the *latest* occurrence of event j over the duration of (i, j). That is,

$$TF_{ij} = \Delta_j - \square_i - D_{ij}$$

The Free Float $(\boldsymbol{FF_{ij}})$ for an activity (i, j) is the excess of the time span defined from the *earliest* occurrence of event i to the ear*liest* occurrence of event j over the duration of (i, j). that is,

$$FF_{ij} = \square_j - \square_i - D_{ij}$$

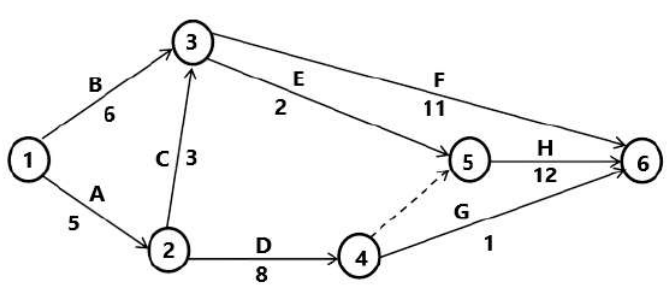
By definition, $FF_{ij} \leq TF_{ij}$.

Red-Flagging Rule: For a noncritical activity (i, j)

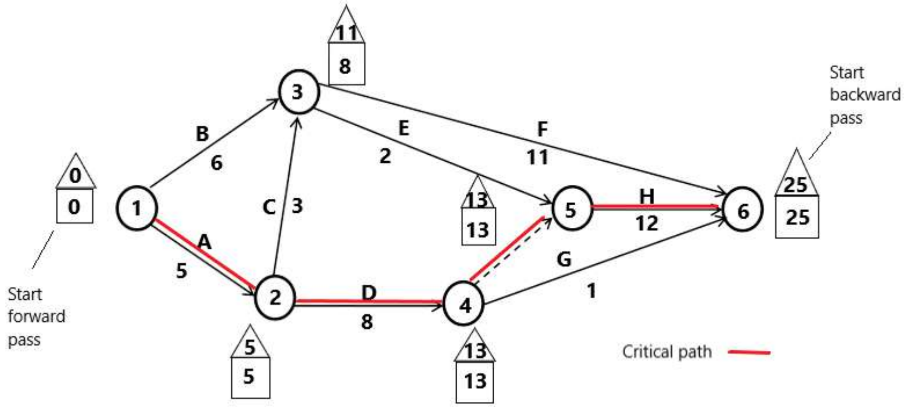
a) If $FF_{ij} = TF_{ij}$, then the activity can be scheduled anywhere within its $(\square_j, \Delta_j)$ span without causing schedule conflict.
b) If $FF_{ij} < TF_{ij}$, then the activity can be delayed by at most FF_{ij} relative to its earliest start time $(\square_i)$ without causing schedule conflict. Any delay larger than FF_{ij} (but not more thanTF_{ij}) must be coupled with an equal delay relative to $\square_j$ in the start time of all the activities leaving node$j$.

The implication of the rule is that a noncritical activity (i, j) will be red-flagged if $FF_{ij} < TF_{ij}$. This red flag is important only if we decide to delay the start of the activity past its earliest start time, $\square_i$, in which case we must pay attention to the start times of the activities leaving node j to avoid schedule conflicts.

Example 1: Determine the critical path for the following project network. Also compute the floats for the noncritical activities. All the durations are in days.



Solution 1: CPM Computations



The critical path is $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 6$, which, as should be expected, spans the network from start (node 1) to finish (node 6). The sum of the durations of the critical activities equals the duration of the project (=25 days).

Computations of Floats: For noncritical activities

Noncritical activity	Duration	Total Float ($\boldsymbol{TF}$)	Free Float($\boldsymbol{FF}$)
B $(\mathbf{1},\mathbf{3})$	6	$11-0-6=5$	$8-0-6=2$
C $(\mathbf{2},\mathbf{3})$	3	$11-5-3=3$	$8-5-3=0$
E $(\mathbf{3},\mathbf{5})$	2	$13-8-2=3$	$13-8-2=3$
F $(\mathbf{3},\mathbf{6})$	11	$25-8-11=6$	$25-8-11=6$
G $(\mathbf{4},\mathbf{6})$	1	$25-13-1=11$	$25-13-1=11$

The red-flag activities are B and C because their $FF < TF$.

The remaining activities E, F and G have $FF = TF$, and hence may be scheduled anywhere between their earliest start and latest completion times.

Significance of Red-flag activities:

a) For activity B, $TF = 5$ days. This activity can start as early as time 0 or as late as time 5. But its $FF = 2$, days starting B anywhere between time 0 and time 2 will have no effect on the succeeding activities E and F. If however, activity B must start at time 2+d ($\leq$ 5), then the start times of the immediately succeeding activities E and F must be pushed forward past their earliest start time (= 8) by at least d.

b) For activity C, $FF = 0$ days. This means that any delay in starting C past its earliest start time (=5) must be coupled with at least an equal delay in the start of its successor activities E and F.

PERT Networks

PERT (Program Evaluation and Review Technique)

In PERT analysis, time duration of each activity is a random variable characterized by some probability distribution- usually a β-distribution. To estimate the parameters of the β-distribution (the mean and variance), the PERT is based on three estimates:

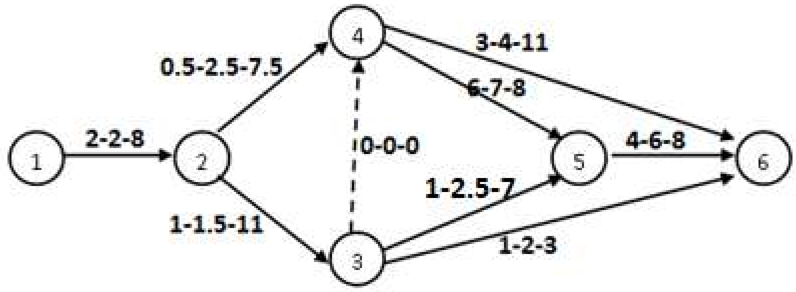
1. **The Optimistic Time Estimate($\boldsymbol{a}$ or $\boldsymbol{t_0}$):** The shortest possible time required for the completion of an activity, if all goes extremely well. No provisions are made for delays or setbacks while estimating this time.
2. The **Most Likely Time Estimate($\boldsymbol{m}$ or $\boldsymbol{t_m}$):** The time an activity will take if cc executed under normal conditions. It is modal value.
3. **The Pessimistic Time Estimate($\boldsymbol{b}$ or $\boldsymbol{t_p}$):** The maximum possible time the activity will take if everything goes bad.

The range (a, b) encloses all possible estimates of the duration of an activity. The estimate m lies somewhere in the range (a, b). Based on the estimates, the average duration time, $\overline{D}$, is approximated as:

$$\overline{D} = \frac{a + 4m + b}{6}$$

By taking $\overline{D}$ as a single time estimate, CPM calculations are directly applied to find the expected project duration or project length.

Example 1: Consider the network shown below. For each activity, the three time estimates a, m and b are given along the arrows in the $a - m - b$ order. Determine the critical path. Find expected duration of the project.

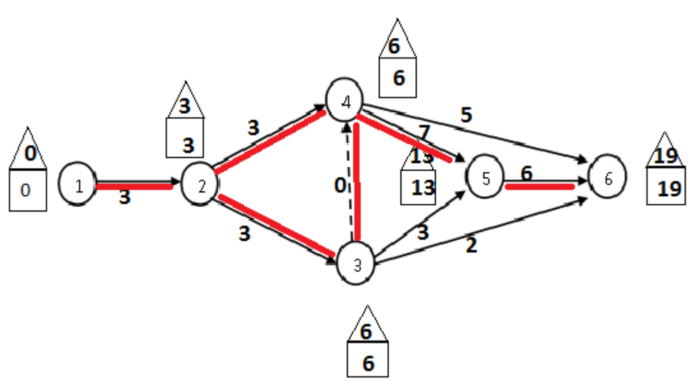


Solution 1:

Expected times

Activity	a	m	b	$\overline{D} = \frac{a+4m+b}{6}$
$(1,2)$	2	2	8	3
$(2,3)$	1	1.5	11	3
$(2,4)$	0.5	2.5	7.5	3
$(3,4)$	0	0	0	0
$(3,5)$	1	2.5	7	3
$(3,6)$	1	2	3	2
$(4,5)$	6	7	8	7
$(4,6)$	3	4	11	5
$(5,6)$	4	6	8	6

CPM Computations:



Expected duration of the project is 19 days.

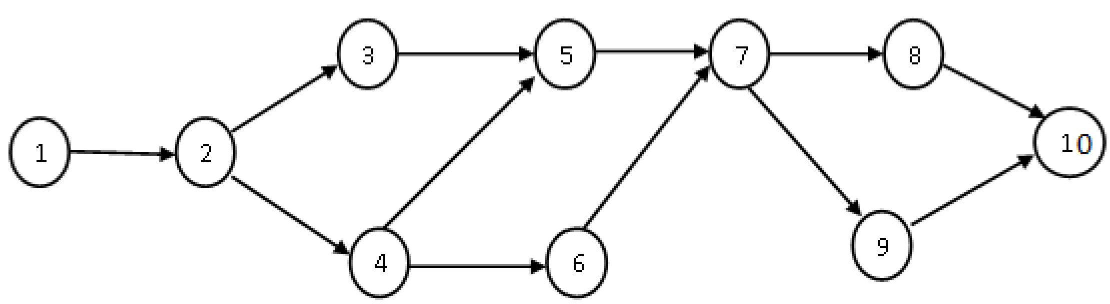
Example 2: Project schedule has the following characteristics:

Activity	a	m	b
$(1,2)$	1	2	3
$(2,3)$	1	2	3
$(2,4)$	1	3	5
$(3,5)$	3	4	5
$(4,5)$	2	3	4
$(4,6)$	3	5	7
$(5,7)$	4	5	6
$(6,7)$	6	7	8
$(7,8)$	2	4	6
$(7,9)$	4	6	8
$(8,10)$	1	2	3
$(9,10)$	3	5	7

a) Construct the project network.
b) Find the expected duration for each activity.
c) Find the critical path and expected project length.

Solution 2:

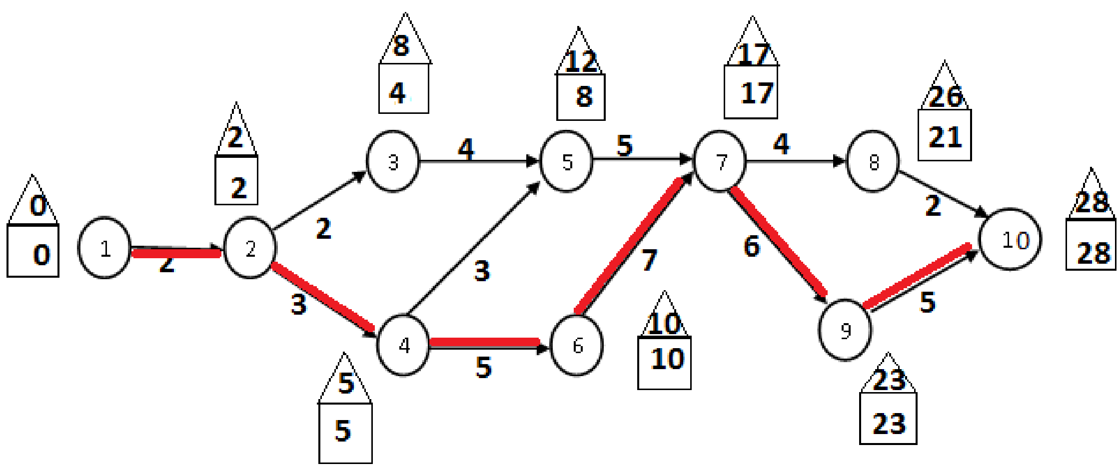
a) Network Diagram:



b) Expected Duration of each activity:

Activity	a	m	b	$\overline{D} = \frac{a+4m+b}{6}$
$(1,2)$	1	2	3	2
$(2,3)$	1	2	3	2
$(2,4)$	1	3	5	3
$(3,5)$	3	4	5	4
$(4,5)$	2	3	4	3
$(4,6)$	3	5	7	5
$(5,7)$	4	5	6	5
$(6,7)$	6	7	8	7
$(7,8)$	2	4	6	4
$(7,9)$	4	6	8	6
$(8,10)$	1	2	3	2
$(9,10)$	3	5	7	5

c) Critical Path and Expected project length:



Expected duration of the project is 28 days.

5. Game Theory

Game theory is a mathematical theory that deals with the general features of competitive situations. In such situations, a decision made by one decision-maker affects the decision made by one or more of the remaining Decision-makers and the final outcome depend upon the decision of all parties. This theory is applicable to a wide variety of situations such as two players struggling to win at chess, candidates fighting an election, two enemies planning war tactics, launching advertisement campaigns by companies marketing competing product.

Definitions:

Game: It is a competitive situation in which two or more intelligent and rational opponents with conflicting objectives are involved, which results in some gain (+ve, -ve or zero)

Player: Each participant or competitor playing a game is called a player.

Play: A play of the game is said to occur when each player chooses one of his courses of action.

Strategy: The strategy for a player is the list of all possible courses of action that will take for every payoff (outcome) that might arise

Pure Strategy: It is the decision rule which is always used by the player to select particular strategy.

Mixed Strategy: Courses of action that are to be selected on a particular occasion with some fixed probability are called mixed strategies.

Optimal Strategy: The particular strategy by which player optimizes gains or losses without knowing the competitor's strategies is called optimal strategy.

Value of Game: The expected outcome when players follow their optimal strategy is called the value of game.

Characteristics of Games

A competitive game has the following characteristics:

1) There is finite number of participants or competitors.

2) Each participant has available to him a list of possible courses of action. The list may not be same for each participant.
3) Each participant knows all the possible choices available to others but does not know which of them is going to be chosen by them.
4) A play is said to occur when each of the participants chooses one of the courses of action available to him. The choices are assumed to be made simultaneously so that no participant knows the choices made by others until he has decided his own.
5) Every combination of courses of action determines an outcome which results in gains to the participants. The gain may be positive, negative or zero. Negative gain is called a loss.
6) The gain of participant depends not only on his own actions but also those of others.
7) The gains (payoffs) for each and every play are fixed and specified in advance are known to each player.

Game Models:
The game models are based on the factors like number of participants, the sum of gains or losses and the number of strategies available etc.

1) **Number of persons:** If the number of participants is two, the game is called two-person game; for number greater than two, it is called n-person game.
2) **Sum of payoffs:** If the sum of gains and losses to the players is zero, the game is called zero-sum game; otherwise it is called non zero-sum game.
3) **Number of strategies:** If the number of strategies (Choices) is finite, the game is called a finite game; otherwise it is called infinite game.

Payoff matrix: The payoffs in terms of gains or losses, when players select their particular strategies can be represented in the form of a matrix is called the payoff matrix.

Designating the two players A and B with m and n strategies, respectively, the game is usually represented by the payoff matrix to player A as

$$\begin{array}{c|cccc|} & B_1 & B_1 & \cdots & B_n \\ \hline A_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ A_2 & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_m & a_{m1} & a_{m2} & \cdots & a_{mn} \\ \hline \end{array}$$

The representation indicates that if A uses strategy i and B uses strategy$j$, then the payoff to A is a_{ij}, which means that the payoff to B is $-a_{ij}$.

Note: For A's payoff matrix, player A is always gainer whereas player B is loser. So Player A attempts to maximize gains and player B attempts to minimize losses.

Optimal Solution of Two-Person Zero-Sum Games

I] When saddle point exists

Maximin Principle: For player A, the minimum value in each row represents the least gain to him, if he chooses his particular strategies. There are written in the matrix by row minima. He will then select the strategy that gives the largest gain among row minima values. This choice of player A is called the maximin principle and the corresponding gain is called maximin value of the game.

Minimax Principle: For player B, the maximum value in each column represents the maximum loss to him, if he chooses his particular strategies. There are written in the matrix by column maxima. He will then select the strategy that gives the minimum loss among column maxima values. This choice of player B is called the minimax principle and the corresponding loss is called minimax value of the game.

Saddle Point: If the maximin value equals the minimax value then the game is said to have a saddle point.

Note: 1. maximin value $\leq$ value of the game $\leq$ minimax value

2. The game is said to be fair if maximin value $=$ minimax value $= 0$, and is said to be strictly determinable, if

$$\text{maximin value} = \text{minimax value} \neq 0$$

Example 1: Two companies, A and B, sell two brands of flu medicine. Company A advertises in radio(A_1), television(A_2), and newspapers (A_3). Company B, in addition to using radio(B_1), television(B_2), and newspapers(B_3), also mails brochures(B_4). Depending on the effectiveness of each advertising campaign, one company can capture a portion of the market from the other. The following matrix summarizes the percentage of market captured or lost by company A.

	B_1	B_2	B_3	B_4
A_1	8	-2	9	-3
A_2	6	5	6	8
A_3	-2	4	-9	5

Solution 1: The given payoff matrix is

	B_1	B_2	B_3	B_4	row minima	
A_1	8	-2	9	-3	-3	
A_2	6	5	6	8	5	← Maximin value
A_3	-2	4	-9	5	-9	
Column maxima	8	5	9	8		
		↑ Minimax value				

Here maximin value = minimax value = 5

⇒ Saddle point exists.

That is, the game has a pure strategy solution.

The optimal solution of the game calls for selecting strategies A_2 and B_2, which means that both companies should use television advertising.

Value of the game = 5%,

The payoff will be in favor of company A, because its market share will increase by 5%.

Example 2: For what value of λ, the game with the following payoff matrix is strictly determinable?

		Player B		
		B_1	B_2	B_3
Player A	A_1	λ	6	2
	A_2	-1	λ	-7
	A_3	-2	4	λ

Solution 2: Ignoring whatever the value of λ may be, the given payoff matrix represents

	B_1	B_2	B_3	row minima	
A_1	λ	6	2	2	← Maximin value
A_2	-1	λ	-7	-7	
A_3	-2	4	λ	-2	
Column maxima	-1	6	2		
	↑ Minimax value				

Here maximin value = 2 and minimax value = -1

∴ Value of the game lies between -1 and 2, that is, $-1 \leq v \leq 2$.

∴ For strictly determinable game since maximin value = minimax value, we must have $-1 \leq \lambda \leq 2$.

Example 3: Find the ranges of values of p and q which will render the entry (2,2) a saddle point for the game

		Player B		
		B_1	B_2	B_3
Player A	A_1	2	4	5
	A_2	10	7	q
	A_3	4	p	6

Solution 3: First ignoring the values of p and q, determine the maximin and minimax values of the given payoff matrix as follows:

	B_1	B_2	B_3	row minima
A_1	2	4	5	2
A_2	10	7	q	7 ← Maximin value
A_3	4	p	6	4
Column maxima	10	7	6 ↑ Minimax value	

Since the entry (2,2) is saddle point for the game.

$\Rightarrow$ Maximin value=minimax value =7

This imposes condition on value of p as $p \leq 7$ and on q as $q \geq 7$.

Hence the range of p and q will be $p \leq 7, q \geq 7$.

Solution of Games without Saddle Point

A game without saddle point is solved by choosing strategies with fixed probabilities. To solve such games the players must determine the optimal mixture of strategies to find a saddle point.

The optimal mixture of strategies for each player may be determined by assigning to each strategy its probability of being chosen. The strategies, so determined are called mixed strategies.

A mixed strategy game can be solved by following different methods:

1) Algebraic method
2) Arithmetic method
3) Rule of Dominance
4) Graphical method
5) LPP method

1) Algebraic Method (For 2×2 games):

Consider 2×2 payoff matrix

	B_1	B_2	
A_1	a_{11}	a_{12}	x_1
A_2	a_{21}	a_{22}	x_2
	y_1	y_2	

Let x_1 and x_2 be the probabilities with which player A plays his A_1 and A_2 strategies respectively. Similarly player B plays his strategies B_1 and B_2 with the probabilities y_1 and y_2 respectively.

$$x_1 + x_2 = 1 \text{ and } y_1 + y_2 = 1$$

The probabilities and expected value of game are given by following formulas:

$$x_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} \quad \text{and } x_2 = 1 - x_1$$

$$y_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} \quad \text{and } y_2 = 1 - y_1$$

$$v = \frac{a_{11}a_{22}-a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$$

Example 1: Solve the following game by algebraic method:

	B_1	B_2
A_1	5	2
A_2	3	4

Solution 1: Given payoff matrix is

	B_1	B_2	Row minima
A_1	5	2	2
A_2	3	4	[3]
Col maxima	5	(4)	

Here maximin value =3   and minimax value = 4

$\Rightarrow$ Saddle point does not exist.

$\therefore$ The game has a mixed strategies solution.

Let x_1 and x_2 be the probabilities with which player A plays his A_1 and A_2 strategies respectively. Similarly player B plays his strategies B_1 and B_2 with the probabilities y_1 and y_2 respectively.

$$x_1 + x_2 = 1 \text{ and } y_1 + y_2 = 1$$

	B_1	B_2	Prob.
A_1	5	2	x_1
A_2	3	4	x_2
Prob.	y_1	y_2	

$$x_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{4-3}{(5+4)-(2+3)} = 1/4 \qquad x_2 = 1 - x_1 = 3/4$$

$$y_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{4-2}{(5+4)-(2+3)} = 1/2 \qquad y_2 = 1 - y_1 = 1/2,$$

$$v = \frac{a_{11}a_{22}-a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{20-6}{(5+4)-(2+3)} = 7/2$$

Optimal strategy for player A is $\left(\frac{1}{4}, \frac{3}{4}\right)$

Optimal strategy for player B is $\left(\frac{1}{2}, \frac{1}{2}\right)$

And value of the game for A is 7/2.

2) Arithmetic Method (For 2×2 games):

Step 1: Subtract two digits in column 1 and write their difference under column 2, ignoring sign.

Step 1: Subtract two digits in column 2 and write their difference under column 1, ignoring sign.

Step 3: Similarly proceed for the two rows.

These values are called as oddments. They represent the frequencies with which the players must use their courses of action in their optimum strategies.

Example 1: In a game of matching coins, player A wins Rs. 1 if they match (HH or TT) and loses Rs.1 when there are one head and one tail. Determine the payoff matrix, best strategies for each player and the value of game to A.

Solution 1: The payoff matrix for A will be

		Player B		
		H	T	Row minima
Player A	H	1	-1	-1
	T	-1	1	-1
	Col maxima	1	1	

Here maximin value = -1 and minimax value = 1

$\Rightarrow$ Saddle point does not exist.

$\therefore$ The game has a mixed strategies solution.

		Player B			
		H	T		Prob.
Player A	H	1	-1	2	$\frac{2}{2+2} = 1/2$
	T	-1	1	2	$\frac{2}{2+2} = 1/2$
		2	2		
	Prob.	$\frac{2}{2+2} = 1/2$	$\frac{2}{2+2} = 1/2$		

Optimal strategy for player A is $\left(\frac{1}{2}, \frac{3}{2}\right)$

Optimal strategy for player B is $\left(\frac{1}{2}, \frac{1}{2}\right)$

A's expected payoff

i. If B selects B_1 strategy, $v = \frac{1}{2} \times 1 + \frac{1}{2} \times (-1) = 0$

ii. If B selects B_2 strategy, $v = \frac{1}{2} \times (-1) - \frac{1}{2} \times 1 = 0$

3) Rule of Dominance:

1) For player B, if each element in column, say C_r, is greater than or equal to the corresponding element in another column, say C_s in the payoff matrix, then the column C_r is dominated by column C_s and therefore column C_r can be deleted from the payoff matrix.

2) For player A, if each element in a row, say R_r, is less than or equal to the corresponding element in another row, say R_s in the payoff matrix, then the row C_r is dominated by row C_s and therefore row C_r can be deleted from the payoff matrix.

3) A strategy say, k can also be dominated if it is inferior to an average of two or more other pure strategies. In this case, if the domination is strict, then strategy k can be deleted. If strategy k dominates the convex linear combination of some other pure strategies, then one of the pure strategies involved in the combination may be deleted.

The domination will be decided as rules 1 and 2 above.

Example 1: Reduce the following game by dominance and find the value of the game

Player B

		I	II	III	IV
Player A	I	3	2	4	0
	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

Solution 1: Given payoff matrix is

	I	II	III	IV	Row minima
I	3	2	4	0	0
II	3	4	2	4	[2]
III	4	2	4	0	0
IV	0	4	0	8	0
Col maxima	4	4	4	8	

Here maximin value $= 2 \neq$ minimax value $= 4$

$\Rightarrow$ Saddle point does not exist.

$\therefore$ The game has a mixed strategies solution.

Here we apply rule of dominance to reduce the matrix to 2×2 size.

From player A's point of view, row I is dominated by row III as each element of row I is less than or equal to corresponding elements of row III.

$\therefore$ Delete row I.

The reduced matrix is,

	I	II	III	IV
II	3	4	2	4
III	4	2	4	0
IV	0	4	0	8

From player B's point of view, column I is dominated by column III as each element of column I is greater than or equal to corresponding elements of column III.

$\therefore$ Delete column I.

The reduced matrix is,

	II	III	IV
II	4	2	4
III	2	4	0
IV	4	0	8

In the above matrix, no single row (column) dominates another row(column). However row II dominates the average of row III and IV, which is

$$\left(\frac{2+4}{2}, \frac{4+0}{2}, \frac{0+8}{2}\right) = (3,2,4)$$

Hence delete one of the row involved in average combination.

The reduced matrix is,

	II	III	IV
II	4	2	4
III	2	4	0

In the above matrix, column II is dominated by column IV.

$\therefore$ Delete column II.

	III	IV
II	2	4
III	4	0

The resultant payoff matrix is of size 2×2.

	III	IV	Row minima
II	2	4	2
III	4	0	0
Col maxima	4	4	

Here maximin value = 2 ≠ minimax value = 4

$\Rightarrow$ Saddle point does not exist.

$\therefore$ The game has a mixed strategies solution.

The game can be solved by either algebraic or arithmetic method.

Algebraic Method: Let x_1 and x_2 be the probabilities with which player A plays his II and III strategies respectively. Similarly player B plays his strategies III and IV with the probabilities y_1 and y_2 respectively.

	III	IV	Prob.
II	2	4	x_1
III	4	0	x_2
Prob.	y_1	y_2	

$$x_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{0-4}{(2+0)-(4+4)} = 2/3 \qquad x_2 = 1 - x_1 = 1/3$$

$$y_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{0-4}{(2+0)-(4+4)} = 2/3 \qquad y_2 = 1 - y_1 = 1/3,$$

$$v = \frac{a_{11}a_{22}-a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{0-16}{(2+0)-(4+4)} = 8/3$$

Optimal strategy for player A is $\left(0, \frac{2}{3}, \frac{1}{3}, 0\right)$

Optimal strategy for player B is $\left(0,0,\frac{2}{3},\frac{1}{3}\right)$

Value of the game for A is $^{8}/_{3}$.

4) Graphical Solution of games

The graphical solution is suitable for games in which at least one player has exactly two pure strategies (i.e. for the game with the payoff matrix is of the size $2 \times n$ or $m \times 2$).

Consider the case of $(2 \times n)$ games in which player A has two strategies.

		y_1	y_2	$\cdots$	y_n
		B_1	B_1	$\cdots$	B_n
x_1	A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
$1-x_1$	A_2	a_{21}	a_{22}	$\cdots$	a_{2n}

The game assumes that player A mixes strategies A_1 and A_2 with the respective probabilities x_1 and $1-x_1$, $0 \le x_1 \le 1$. Player B mixes strategies B_1 to B_n with the probabilities $y_1, y_2, \dots, y_n$, where $y_j \ge 0$ for $j = 1,2,\dots,n$ and $y_1 + y_2 + \cdots + y_n = 1$.

In this case, A's expected payoff corresponding to B's j^{th} pure strategy is computed as $\left(a_{1j} - a_{2j}\right)x_1 + a_{2j}, j = 1,2,\dots,n$

Player A thus seek to determine the value of x_1 that maximizes the minimum expected payoffs— that is,

$$\max_{x_1} \min_{j} \{\left(a_{1j} - a_{2j}\right)x_1 + a_{2j}\}$$

Example 1: Use the graphical method for solving the following game and find the value of the game.

		Player B				
		B_1	B_2	B_3	B_4	B_5
	A_1	-5	5	0	-1	8
Player A	A_2	8	-4	-1	6	-5

Solution 1: The given payoff matrix is

	B_1	B_2	B_3	B_4	B_5	Row minima
A_1	-5	5	0	-1	8	-5
A_2	8	-4	-1	6	-5	-5 ← Maximin value
Column maxima	8	5	0	6	8	

Minimax value (↑ at column maximum 0 of B_3)

Here maximin value = -5 ≠ minimax value = 0

⇒ Saddle point does not exist.

That is, game does not have a pure strategy solution.

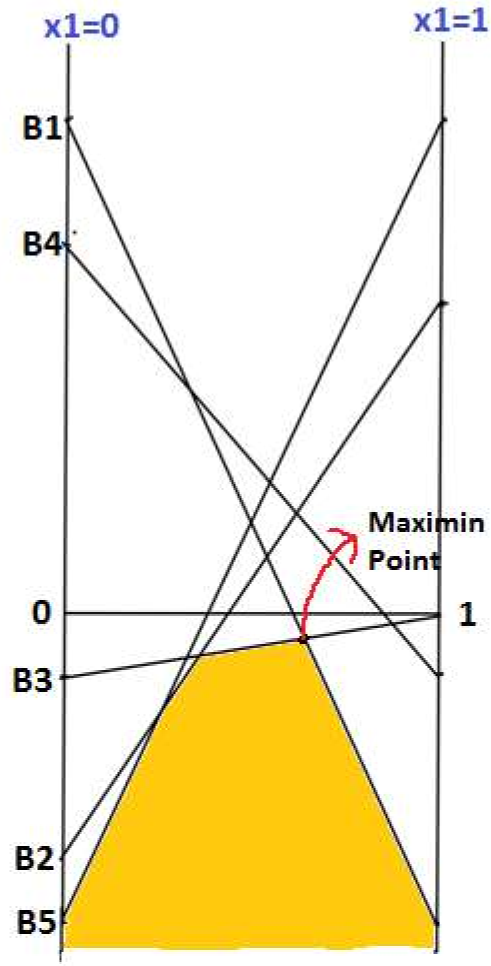
Since the given matrix is of the size 2×5, this matrix can be solved by graphical method.

Graphical Method

Let x_1 and x_2 be the probabilities with which player A plays his A_1 and A_2 strategies respectively and $x_1 + x_2 = 1$.

Then A's expected payoffs corresponding to B's pure strategies are

B's Pure strategies	A's expected payoff
1	$-13x_1+8$
2	$9x_1-4$
3	x_1-1
4	$-7x_1+6$
5	$13x_1-5$



Since player A wishes to maximize his minimum expected payoffs, the two lines which intersect at the highest point of the lower envelope of the five lines gives the maximin expected payoff to player A regardless of what B does. This highest point is called as maximin point.

The two optimum strategies B_1 and B_3 for player B are given by the two lines which pass through this maximin point reduce the given matrix to 2×2 size.

	B_1	B_3
A_1	-5	0
A_2	8	-1

Solution of game:

	B_1	B_3	Row minima
A_1	-5	0	-5
A_2	8	-1	-1
Col maxima	8	0	

Here maximin value = -1 and minimax value = 0

$\Rightarrow$ Saddle point does not exist.

$\therefore$ The game has a mixed strategies solution.

Algebraic method:

Let x_1 and x_2 be the probabilities with which player A plays his A_1 and A_2 strategies respectively. Similarly player B plays his strategies B_3 and B_4 with the probabilities y_1 and y_2 respectively.

$$x_1 + x_2 = 1 \text{ and } y_1 + y_2 = 1$$

	B_1	B_3	Prob.
A_1	-5	0	x_1
A_2	8	-1	x_2
Prob.	y_1	y_2	

$$x_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{-1-8}{(-5-1)-(0+8)} = 9/14 \qquad x_2 = 1 - x_1 = 5/14$$

$$y_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{-1-0}{(-5-1)-(0+8)} = 1/14 \qquad y_2 = 1 - y_1 = 13/14,$$

$$v = \frac{a_{11}a_{22}-a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{5+0}{(-5-1)-(0+8)} = -5/14$$

Optimal strategy for player A is $\left(\frac{9}{14}, \frac{5}{14}\right)$

Optimal strategy for player B is $\left(\frac{1}{14}, 0, \frac{13}{14}, 0,0\right)$

And value of the game for A is $-5/14$.

Example 2: Use the graphical method for solving the following game and find the value of the game.

		Player B			
		B_1	B_2	B_3	B_4
	A_1	2	2	3	-1
Player A	A_2	4	3	2	6

Solution 2: The given payoff matrix is

	B_1	B_2	B_3	B_4	Row minima	
A_1	2	2	3	-1	-1	
A_2	4	3	2	6	2	← Maximin value
Column maxima	4	3 (↑ Minimax value)	3	6		

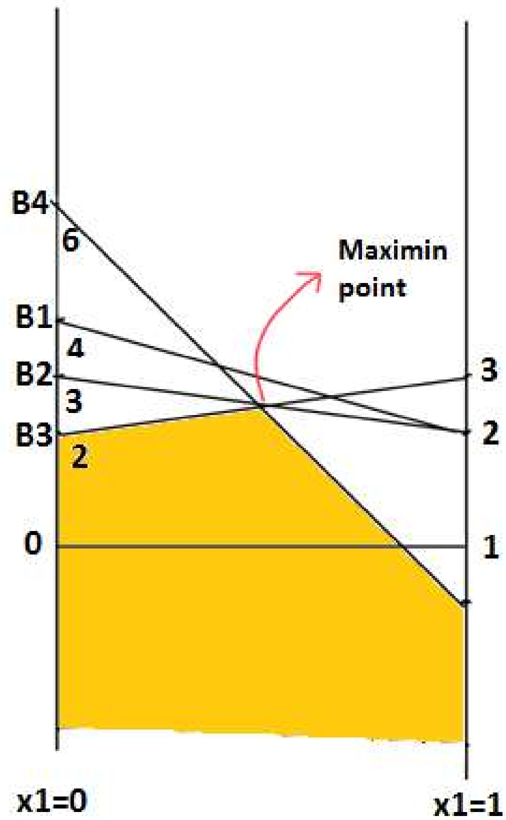
Here maximin value $= 2 \neq$ minimax value $= 3$

$\Rightarrow$ Saddle point does not exist.

That is, game does not have a pure strategy solution.

B's Pure strategies	A's expected payoff
$\boldsymbol{B_1}$	$-2x_1 + 4$
$\boldsymbol{B_2}$	$-x_1 + 3$
$\boldsymbol{B_3}$	$x_1 + 2$
$\boldsymbol{B_4}$	$-7x_1 + 6$

Graphical Solution:



Here, three lines passing through maximin point. Any two lines having opposite signs for their slopes will yield an alternative optimal solution. This means that combination of B_2 and B_4 must be excluded as both have the same sign for their slopes. So the game reduces to 2×2 size and which can be solved either by algebraic or arithmetic method.

	B_3	B_4	Row minima
A_1	3	-1	-1
A_2	2	6	2
Col maxima	3	6	

Here maximin value $=2$ and minimax value $= 3$

$\Rightarrow$ Saddle point does not exist.

$\therefore$ The game has a mixed strategies solution.

Algebraic method:

Let x_1 and x_2 be the probabilities with which player A plays his A_1 and A_2 strategies respectively. Similarly player B plays his strategies B_3 and B_4 with the probabilities y_1 and y_2 respectively.

$$x_1 + x_2 = 1 \text{ and } y_1 + y_2 = 1$$

	B_3	B_4	Prob.
A_1	3	-1	x_1
A_2	2	6	x_2
Prob.	y_1	y_2	

$$x_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{6-2}{(3+6)-(-1+2)} = 1/2 \qquad x_2 = 1 - x_1 = 1/2$$

$$y_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{6-(-1)}{(3+6)-(-1+2)} = 7/8 \qquad y_2 = 1 - y_1 = 1/8,$$

$$v = \frac{a_{11}a_{22}-a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{18+2}{(3+6)-(-1+2)} = 5/2$$

Optimal strategy for player A is $\left(\frac{1}{2}, \frac{1}{2}\right)$

Optimal strategy for player B is $\left(0, 0, \frac{7}{8}, \frac{1}{8}\right)$

And value of the game for A is 5/2.

Example 3: Use the graphical method for solving the following game and find the value of the game.

		Player B	
		I	II
Player A	I	2	5
	II	2	3
	III	3	2
	IV	-2	8

Solution 3: The given payoff matrix is

	I	II	Row minima
I	2	5	2
II	2	3	2
III	3	2	2
IV	-2	8	-2
Col maxima	3	8	

Here maximin value $= 2 \neq$ minimax value $= 3$

$\Rightarrow$ Saddle point does not exist.

$\therefore$ The game has a mixed strategies solution.

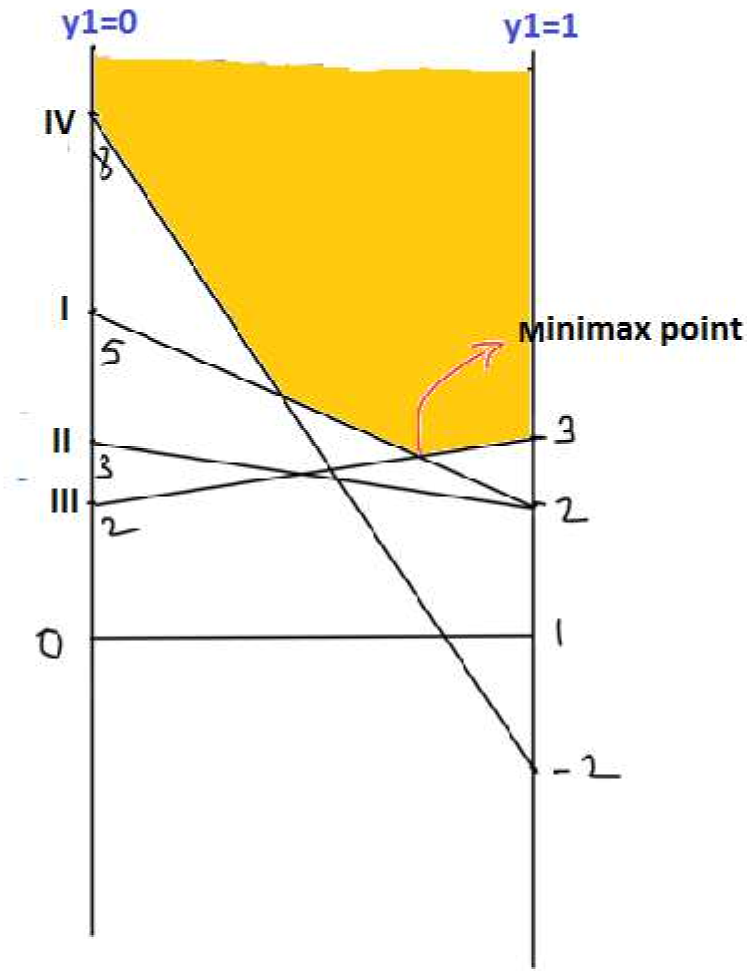
Since the given matrix is of the size 4×2, this matrix can be solved by graphical method.

Graphical Method

Let y_1 and y_2 be the probabilities with which player B plays his I and II strategies respectively and $y_1 + y_2 = 1$.

Then B's expected payoffs corresponding to A's pure strategies are

A's Pure strategies	B's expected payoff
I	$-3y_1 + 5$
II	$-y_1 + 3$
III	$y_1 + 2$
IV	$-10y_1 + 8$



Since player B wishes to minimize his maximum expected losses, the two lines which intersect at the lowest point of the upper envelope of the four lines gives the minimax expected payoff to player B regardless of what A does. This lowest point is called minimax point.

The two optimum strategies I and III for player A are given by the two lines which pass through this minimax point reduce the given matrix to 2 × 2 size.

	I	II
I	2	5
III	3	2

Solution of game:

	I	II	Row minima
I	2	5	2
III	3	2	2
Col maxima	3	5	

Here maximin value =2 and minimax value = 3

$\Rightarrow$ Saddle point does not exist.

$\therefore$ The game has a mixed strategies solution.

Algebraic method:

Let x_1 and x_2 be the probabilities with which player A plays his A_1 and A_2 strategies respectively. Similarly player B plays his strategies B_3 and B_4 with the probabilities y_1 and y_2 respectively.

$$x_1 + x_2 = 1 \text{ and } y_1 + y_2 = 1$$

	I	II	Prob.
I	2	5	x_1
III	3	2	x_2
Prob.	y_1	y_2	

$$x_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{2-3}{(2+2)-(5+3)} = 1/4 \qquad x_2 = 1 - x_1 = 3/4$$

$$y_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{2-5}{(2+2)-(5+3)} = 3/4 \qquad y_2 = 1 - y_1 = 1/4,$$

$$v = \frac{a_{11}a_{22}-a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{4-15}{(2+2)-(5+3)} = 11/4$$

Optimal strategy for player A is $\left(\frac{1}{2}, 0, \frac{1}{2}, 0\right)$

Optimal strategy for player B is $\left(\frac{7}{8}, \frac{1}{8}\right)$

And value of the game for A is 5/2.

Example 4: Use the graphical method for solving the following game and find the value of the game.

Player B

		B_1	B_2	B_3	B_4
Player A	A_1	19	6	7	5
	A_2	7	3	14	6
	A_3	12	8	18	4
	A_4	8	7	13	-1

Solution 4: The given payoff matrix is

	B_1	B_2	B_3	B_4	Row minima
A_1	19	6	7	5	5 (boxed)
A_2	7	3	14	6	3
A_3	12	8	18	4	4
A_4	8	7	13	-1	-1
Col maxima	19	8	18	6 (circled)	

Here maximin value $= 5 \neq$ minimax value $= 6$

$\Rightarrow$ Saddle point does not exist.

$\therefore$ The game has a mixed strategies solution.

To solve the given game by graphical method, the game must be reduced to $2 \times n$ or $m \times 2$ size using rule of dominance.

Rule of Dominance:

	B_1	B_2	B_3	B_4
A_1	19	6	7	5
A_2	7	3	14	6
A_3	12	8	18	4
A_4	8	7	13	-1

From player B's point of view, column B_1 is dominated by column B_2 as each element of column B_1 is greater than or equal to corresponding elements of column B_2.

$\therefore$ Delete columnB_1 .

The reduced matrix is,

	B_2	B_3	B_4
A_1	6	7	5
A_2	3	14	6
A_3	8	18	4
A_4	7	13	-1

From player A's point of view, row A_4 is dominated by row A_3 as each element of row A_4 is less than or equal to corresponding elements of row A_3.

$\therefore$ Delete row A_4.

The reduced matrix is,

	B_2	B_3	B_4
A_1	6	7	5
A_2	3	14	6
A_3	8	18	4

In the above matrix, column B_3 is dominated by column B_4.

$\therefore$ Delete column B_3.

The reduced matrix is,

	B_2	B_4
A_1	6	5
A_2	3	6
A_3	8	4

Solution of game:

	B_2	B_4	Row minima
A_1	6	5	5
A_2	3	6	3
A_3	8	4	4
Col maxima	8	6	

Here maximin value $= 5 \neq$ minimax value $= 6$

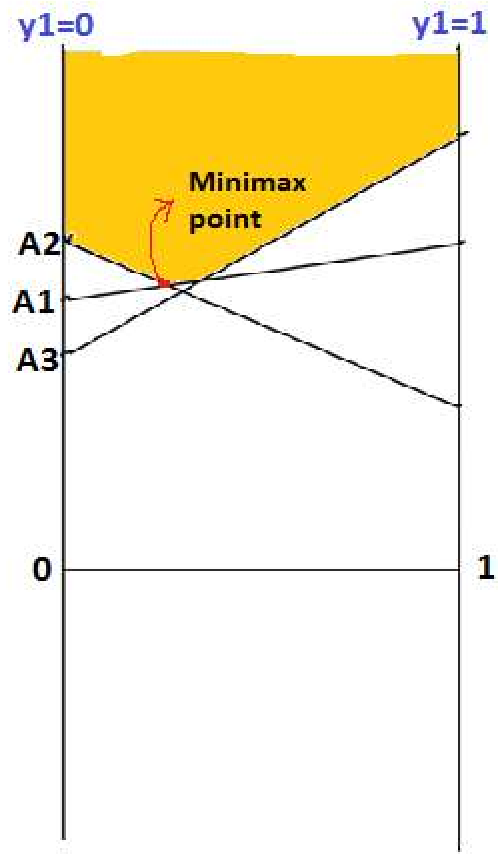
$\Rightarrow$ Saddle point does not exist.

∴ The game has a mixed strategies solution.

Since the reduced matrix is of the size 3×2, this matrix can be solved by graphical method.

Graphical Method: Let y_1 and y_2 be the probabilities with which player B plays his B_2 and B_4 strategies respectively and $y_1 + y_2 = 1$.

A's Pure strategies	B's expected payoff
A_1	$y_1 + 5$
A_2	$-3y_1 + 6$
A_3	$4y_1 + 4$



The two optimum strategies A_1 and A_2 for player A are given by the two lines which pass through this minimax point reduce the given matrix to 2×2 size.

	B_2	B_4	Row minima
A_1	6	5	5
A_2	3	6	3
Col maxima	6	6	

Here maximin value = 5 ≠ minimax value = 6

$\Rightarrow$ Saddle point does not exist.

$\therefore$ The game has a mixed strategies solution.

Algebraic method: Let x_1 and x_2 be the probabilities with which player A plays his A_1 and A_2 strategies respectively. Similarly player B plays his strategies B_3 and B_4 with the probabilities y_1 and y_2 respectively.

$$x_1 + x_2 = 1 \text{ and } y_1 + y_2 = 1$$

	B_2	B_4	Prob.
A_1	6	5	x_1
A_2	3	6	x_2
Prob.	y_1	y_2	

$$x_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{6-3}{(6+6)-(5+3)} = 3/4 \qquad x_2 = 1 - x_1 = 1/4$$

$$y_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{6-5}{(6+6)-(5+3)} = 1/4 \qquad y_2 = 1 - y_1 = 3/4,$$

$$v = \frac{a_{11}a_{22}-a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{36-15}{(6+6)-(5+3)} = 21/4$$

Optimal strategy for player A is $\left(\frac{3}{4}, \frac{1}{4}, 0, 0\right)$

Optimal strategy for player B is $\left(0, \frac{1}{4}, 0, \frac{3}{4}\right)$

And value of the game for A is 21/4.

Linear Programming Solution of Games

		y_1	y_2	$\cdots$	y_n
		B_1	B_2	$\cdots$	B_n
x_1	A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
x_2	A_2	a_{21}	a_{22}	$\cdots$	a_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_m	A_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}

Player A’s optimal probabilities, x_1, x_2, ..., and x_m can be determined by solving the following maximin problem:

$$\max_{x_i}\left\{\min\left(\sum_{i=1}^{m} a_{i1}x_i\,, \sum_{i=1}^{m} a_{i2}x_i\,, \dots, \sum_{i=1}^{m} a_{in}x_i\right)\right\}$$

$$x_1 + x_2 + \cdots + x_m = 1$$

$$x_i \geq 0, i = 1,2, \dots, m$$

Let $v = \min(\sum_{i=1}^{m} a_{i1}x_i\,, \sum_{i=1}^{m} a_{i2}x_i\,, \dots, \sum_{i=1}^{m} a_{in}x_i)$

The equation implies that

$$\sum_{i=1}^{m} a_{ij}x_i \geq v, j = 1,2, \dots, n$$

Thus A’s problem thus can be written as

Maximize $z = v$

Subject to $v - \sum_{i=1}^{m} a_{ij}x_i \leq 0, j = 1,2, \dots, n$

$$x_1 + x_2 + \cdots + x_m = 1$$

$$x_i \geq 0, i = 1,2, \dots, m$$

v unrestricted

Player B’s optimal probabilities, y_1, y_2, ..., and y_n can be determined by solving the following minimax problem:

$$\min_{y_j}\left\{\max\left(\sum_{j=1}^{n} a_{1j}y_j\,, \sum_{j=1}^{n} a_{2j}y_j\,, \dots, \sum_{j=1}^{n} a_{mj}y_j\right)\right\}$$

$$y_1 + y_2 + \cdots + y_n = 1$$

$$y_j \geq 0, j = 1,2, \dots, n$$

Using a procedure similar to that of player A, B's problem reduces to

$$\text{Minimize } w = v$$

$$\text{Subject to } v - \sum_{j=1}^{n} a_{ij} x_i \geq 0, i = 1,2, \dots, m$$

$$y_1 + y_2 + \cdots + y_n = 1$$

$$y_j \geq 0, j = 1,2, \dots, n$$

$$v \text{ unrestricted}$$

The two problems optimize the same unrestricted variable v, the value of the game. The reason is that B's problem. This means that the optimal solution of one problem automatically yields the optimal solution of the other.

Example: Solve the following problem by linear programming

Player B

		B_1	B_2	B_3
	A_1	4	1	−3
Player A	A_2	3	1	6
	A_3	−3	4	−3

Solution: Given payoff matrix is

	B_1	B_2	B_3	Row minima
A_1	4	1	−3	−3
A_2	3	1	6	[1]
A_3	−3	4	−3	−3
Col maxima	(4)	4	6	

Here maximin value = 4 ≠minimax value = 1

$\Rightarrow$ Saddle point does not exist.

$\therefore$ The game has a mixed strategies solution.

Linear Programming Solution:

Player A's Linear Program

$$\text{Maximize } z = v$$

Subject to

$$\begin{aligned} v - 4x_1 - 3x_2 + 3x_3 &\le 0 \\ v - x_1 - x_2 - 4x_3 &\le 0 \\ v + 3x_1 - 6x_2 + 3x_3 &\le 0 \\ x_1 + x_2 + x_3 &= 1 \\ x_1, x_2, x_3, v &\ge 0 \end{aligned}$$

Player B's Linear Program

$$\text{Minimize } z = v$$

Subject to

$$\begin{aligned} v - 4y_1 - y_2 + 3y_3 &\ge 0 \\ v - 3y_1 - y_2 - 6y_3 &\ge 0 \\ v + 3y_1 - 6y_2 + 3y_3 &\ge 0 \\ y_1 + y_2 + y_3 &= 1 \\ y_1, y_2, y_3, v &\ge 0 \end{aligned}$$

Note that B's problem is dual of A's problem. This means that optimal solution of one problem automatically yields solution of the other.

Solution of A's Problem:

Equation form of A's linear program is,

$$\text{Maximize } z = v - MR_4$$

Subject to

$$\begin{aligned} v - 4x_1 - 3x_2 + 3x_3 + s_1 &= 0 \\ v - x_1 - x_2 - 4x_3 + s_2 &= 0 \\ v + 3x_1 - 4x_2 + 3x_3 + s_3 &= 0 \end{aligned}$$

$$x_1 + x_2 + x_3 + R_4 = 1$$
$$x_1, x_2, x_3, v, s_1, s_2, s_3, R_4 \geq 0$$

We write the objective equation as

$$z - v + MR_4 = 0$$

Take M=100

$$z - v + 100\, R_4 = 0$$

Initial Basic Feasible Solution

The system has $m = 4$ equations and $n = 8$ variables.

Set $n - m = 8 - 4 = 4$ variables equal to zero the solving for remaining 4 variables.

Simplex iteration starts at origin. So the associated set of nonbasic and basic variable is defined as

Nonbasic (zero) variables:(x_1, x_2, x_3, v)

Basic variables: (s_1, s_2, s_3, R_4)

Initial Simplex Table:

Basic	v	x_1	x_2	x_3	s_1	s_2	s_3	R_4	Solution
z	-1	0	0	0	0	0	0	100	0
s_1	1	-4	-3	3	1	0	0	0	0
s_2	1	-1	-1	-4	0	1	0	0	0
s_3	1	3	-6	3	0	0	1	0	0
R_4	0	1	1	1	0	0	0	1	1

From the table, the starting solution is given by

$z = 0, s_1 = 0, s_2 = 0, s_3 = 0, R_4 = 1$

But this solution is not consistent. To make the z −row consistent, we use the following operation

New z-row = Old z-row $-100R_4$-row

Basic	v	x_1	x_2	x_3	s_1	s_2	s_3	R_4	Solution	Ratio
z	-1	-100	-100	-100	0	0	0	0	-100	
s_1	1	-4	-3	3	1	0	0	0	0	-
s_2	1	-1	-1	-4	0	1	0	0	0	-
s_3	1	3	-6	3	0	0	1	0	0	0 →
R_4	0	1	1	1	0	0	0	1	1	1

From the table, the starting solution is given by

$z = -100, s_1 = 0, s_2 = 0, s_3 = 0, R_4 = 1$ which is consistent.

But here all $z -$ row coefficients are not nonnegative. Hence solution can be improved.

First Iteration Table:

Basic	v	x_1	x_2	x_3	s_1	s_2	s_3	R_4	Solution	Ratio
z	97/3	0	-300	0	0	0	100/3	0	-100	
s_1	7/3	0	-11	7	1	0	4/3	0	0	-
s_2	4/3	0	-3	-3	0	1	1/3	0	0	-
x_1	1/3	1	-2	1	0	0	1/3	0	0	-
R_4	-1/3	0	3	0	0	0	-1/3	1	1	1/3 →

Second Iteration Table:

Basic	v	x_1	x_2	x_3	s_1	s_2	s_3	R_4	Solution	Ratio
z	-1	0	0	0	0	0	0	100	0	
s_1	10/9	0	0	7	1	0	1/9	11/3	11/3	3.3
s_2	1	0	0	-3	0	1	0	1	1	1 →
x_1	1/9	1	0	1	0	0	1/9	2/3	2/3	6
x_2	-1/9	0	1	0	0	0	-1/9	1/3	1/3	-

Third Iteration Table:

Basic	v	x_1	x_2	x_3 ↓	s_1	s_2	s_3	R_4	Solution	Ratio
z	0	0	0	-3	0	1	0	101	1	
s_1	0	0	0	[31/3]	1	-10/9	1/9	23/9	23/9	23/93 →
v	1	0	0	-3	0	1	0	1	1	-
x_1	0	1	0	4/3	0	-1/9	1/9	5/9	5/9	5/12
x_2	0	0	1	-1/3	0	1/9	-1/9	4/9	4/9	-

Fourth Iteration Table:

Basic	v	x_1	x_2	x_3	s_1	s_2	s_3	R_4	Solution
z	0	0	0	0	9/31	21/31	1/31	3154/31	54/31
x_3	0	0	0	1	3/31	-10/93	1/93	23/93	23/93
v	1	0	0	0	9/31	21/31	1/31	54/31	54/31
x_1	0	1	0	0	-4/31	1/31	3/31	7/31	7/31
x_2	0	0	1	0	1/31	7/93	-10/93	49/93	49/93

Since all z-row coefficients are nonnegative (≥ 0)

$\Rightarrow$ Solution is optimal.

$$Z_{max} = 1.74, x_1 = 0.22, x_2 = 0.53, x_3 = 0.25$$

Solution of B's Problem:

$$\begin{pmatrix} \text{optimal value of} \\ \text{dual variable } y_i \end{pmatrix} = \begin{pmatrix} \text{optimal primal z} - \text{coefficient of starting varible } x_i \\ + \\ \text{original objective coefficient of } x_i \end{pmatrix}$$

Here starting variables are s_1, s_2, s_3

optimal primal z − coefficient of starting variables are

$s_1 = 9/31$, $s_2 = 21/31$ and $s_3 = 1/31$

And original objective coefficient of s_1, s_2, s_3 are 0

$$\Rightarrow \quad y_1 = \frac{9}{31} + 0 = 0.29, y_2 = \frac{21}{31} + 0 = 0.68, y_3 = \frac{1}{31} + 0 = 0.03,$$

So, the optimal strategies for player A are (0.22,0.53,0.25)

And the optimal strategies for player B are (0.29,0.68,0.03)

Value of the game= 1.74