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A MULTIVARIATE MODIFIED GROUP RUNS CONTROL CHART FOR PROCESS DISPERSION

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Abstract : In this article, we propose the 'Modified Group Runs' (MGR) charts useful to detect a shift in the process variability in case of bivariate and multivariate normal data. It is verified that the proposed charts significantly reduce the out of control 'Average Time to Signal' (ATS) as compared to the most of the well-known control charts. The control charts considered for comparison are the standard S chart, adaptive sample size chart, the synthetic S chart (Ghute and Shirke, 2008) and the GR S chart .

Key Words : Average Time to Signal (ATS), Determinant Ratio, Sample Generalized Variance, Steady State ATS, Synthetic Chart, Zero State ATS.

1. Introduction

Due to today's competitive environment, it is essential for the producer to manufacture the products of very high standard. Statistical process control is a very effective tool used to improve the quality. Quality of the product may depend on more than one characteristic; these may be correlated and need to be monitored simultaneously. In the recent past, various multivariate procedures have been developed for simultaneously monitoring these characteristics. Many of these have been developed to detect a shift in the process mean vector. Woodall and Montgomery (1999) mentioned the need to develop the methods for monitoring the process variability.

In the multivariate case, the process variability is measured by the $p \times p$ covariance matrix Σ, where p is the number of process variables. Ghute and Shirke (2008) proposed the bivariate and the multivariate synthetic S (BI-Syn S and MV-Syn S) control charts for detecting shifts in the covariance matrix of the respective normal processes. They have studied the performance of the proposed synthetic charts based on the ARL criterion in the zero state only and illustrated that, the proposed charts are superior to most of the well-known control charts. Ghute and Shrike (2008) have not studied the steady state behavior of the charts.

Once there is an evidence for the process being running smoothly, a subsequent indication of the process being in an out of control status may be examined by observing the process further. Based on this fact, Gadre and Rattihalli (2004) developed the 'Group Runs' (GR) control chart to detect shifts in the process mean (μ say) of the univariate normal process.

Recently, Gadre (2011) has applied this technique to develop the bivariate and the multivariate GR S (BI-GR S and MV - GR S) charts. Then by considering the 'Average Run Length' (ARL) model it has been illustrated that in the zero state as well as in the steady state, the GR S chart is superior to the matched synthetic S chart.

Including an additional design parameter, the GR chart can be modified. Using such a technique, Gadre and Rattihali (2006) have developed the control charts called the 'Modified Group Runs' (MGR) charts to detect a shift in the process mean and to identify increases in fraction non-conforming. It has been illustrated that the MGR charts perform better than the related GR charts.

In this article, we apply the MGR technique to develop the 'Bivariate Modified Group Runs' (BI-MGR S) chart and the 'Multivariate Modified Group Runs' (MV-MGR S) chart to detect a shift in the process variation. It is expected that the BI-MGR S chart and the MV-MGR S chart should perform better than the matched Synthetic S chart as well the matched GR S chart.

The description of various bivariate and multivariate charts for the process dispersion is given in Section 2. Section 3 includes the description and the design of the BI-MGR S chart and the MV-MGR S chart. In the subsequent section it is illustrated that the proposed charts outperform the compatible synthetic S chart and the GR S chart when the minimum required value of in control ATS is fixed. In Section 5, the steady state performance of the MV-MGR S chart is studied. Concluding remarks are included in the last section.

2. *Some bivariate and multivariate control charts for the process dispersion*

In this section we briefly describe the bivariate and the multivariate standard *S* chart, the synthetic *S* chart and the GR S chart.

Let the vectors $\underline{\mathbf{X}}_1, \underline{\mathbf{X}}_2, \ldots, \underline{\mathbf{X}}_n$ be a random sample from Np ($\underline{\mu}_0$, Σ_0) distribution. Here $\underline{\mu}_0$ is the in-control process mean vector and Σ_0 is the known in-control process covariance matrix. The problem is to detect shifts in the covariance matrix Σ. The problem is equivalent to test the null hypothesis

$$H_0: \Sigma = \Sigma_0 \text{ against } H_1: \Sigma \neq \Sigma_0 \qquad \ldots(1)$$

2.1 *Bivariate Standard* S *Control Chart*

Based on the determinant of the sample covariance matrix S (say), Alt and Smith (1988) introduced the procedures to monitor the process variability of a multivariate normal process. The sample covariance matrix is given by

$$S = \frac{1}{n-1}\sum_{i=1}^{n}\left(\underline{\mathbf{X}}_i - \underline{\overline{\mathbf{X}}}\right)\left(\underline{\mathbf{X}}_i - \underline{\overline{\mathbf{X}}}\right)' . \qquad \ldots(2)$$

By using the distributional properties of $|S|^{(1/2)}$, Alt and Smith (1988) introduced the 'Bivariate Standard S' (BI-S) chart. Ghute and Shirke (2008) discussed the construction of the BI-S chart.

For a given Type I error probablity α (say), the upper control limit (UCL) and the lower control limit (LCL) of this chart are computed in such a way that

$$P(LCL < |S| < UCL) = 1-\alpha \ .$$

As for small sample sizes LCL is zero, it is not used. Then to calculate UCL, we have,

$$P\ (|S| > UCL) = \alpha \ .$$

As mentioned in Ghute and Shirke (2008), we get,

$$UCL = \frac{\left(X^2_{2n-4}(1-\alpha)\right)^2 |\Sigma_0|}{4(n-1)^2}. \qquad ...(3)$$

Here $X^2_{2n-4}(1-\alpha)$ is the upper αth percentage point of X^2_{2n-4} variate. The implemention of the chart is as usual.

'Average Run Length' (ARL) or the 'Average Time to Signal' (ATS) can be used as a measure of the efficiency of the chart in detecting an increase in the process dispersion. ARL is the expected number of samples required by the chart to declare the process as out of control, whereas ATS is the expected number of units required by the chart to declare the process as out of control. Thus for the fixed size case ATS = n(ARL). Let $|\Sigma_1|$ be the value of $|\Sigma|$ that impairs the quality of the process and has to be determined. Usally the in control and the out of control ARLs and the ATSs are respectively denoted by ARL_0, ARL_1, ATS_0 and ATS_1. Thus $ARL_0 = 1/\alpha$ and $ARL_1 = 1/(1-\beta)$. Here, β is the probablity of type-II error when $|\Sigma| = |\Sigma_1|$ $(>|\Sigma_0|)$. Hence

$$\beta = (|S| < UCL \mid |\Sigma| = |\Sigma_1|)$$

$$= F_{\chi^2_{2n-4}}\left(\frac{\chi^2_{2n-4}(1-\alpha)}{\sqrt{DR}}\right). \qquad ...(4)$$

where $F_{\chi^2_{2n-4}}(.)$ is the cumulative distribution function of χ^2_{2n-4} variate and $DR_1 = |\Sigma_1| / |\Sigma_0|$ and is known as the 'Determinant Ratio' (DR) between the out-of-control and the in-control covariance matrices. The process variability can be measured in terms of DR= $|\Sigma| / |\Sigma_0|$. DR = 1 indicates that there is no change in the process variability; DR > 1 indicates that the process variability has increased and DR < 1 indicates that the process variability has been decreased. Therefore, when the BI-S chart is used, the process shifts are measured through DR. Thus for the BI-S chart, the probablity of detecting a shift of size DR is given by,

$$P(DR) = P(|S| > UCL \mid |\Sigma| > |\Sigma_0|)$$

$$P(DR) = 1 - F_{\chi^2_{2n-4}} = \left(\frac{\chi^2_{2n-4}(1-\alpha)}{\sqrt{DR}}\right). \qquad ...(5)$$

For the notational convenience, we denote P(DR) by P. Therefore, for this chart,

$$ARL\ (DR) = 1/P \qquad ...(6)$$

To construct BI-S chart, the following ARL model can be used.

$$\left.\begin{array}{ll} & \text{Minimise } ARL_1 \\ \text{Subject to} & \\ & ARL_0 \geq \tau, \end{array}\right\} \qquad ...(7)$$

where, τ is the minimum required value of ARL_0.

Remark - 1 : In this criterion, the condition $ARL_0 \geq \tau$ restricts the probablity of false alarm to $\alpha = 1/\tau$ and minimization of ARL_1 is equivalent to maximization of the power.

As mentioned in Ghute and Shirke (2008), though the exact distribution of S for more than two variables is available, it is not one of the known distributions. Therefore for $p > 2$, to monitor the process variability, below we describe another approach mentioned by Alt and Smith (1988).

2.2 *Multivariate Standard* S *Control Chart*

This approach introduced by Alt and Smith (1988) is based on the mean and variance of |S| and making the use kσ control limits. It is well known that, under H_0,

$$E(|S|) = b_1 |\Sigma_0| \text{ and } Var(|S|) = b_2 |\Sigma_0|^2. \qquad \text{...(8)}$$

Here

$$b_1 = \frac{1}{(n-1)^p} \prod_{i=1}^{p} (n-i) \qquad \text{...(9)}$$

and

$$b_2 = \frac{1}{(n-1)^{2p}} \prod_{i=1}^{p} (n-i) \left\{ \prod_{i=1}^{p} (n-j+2) - \prod_{i=1}^{p} (n-j) \right\}. \qquad \text{...(10)}$$

Here the control limits for the 'Multivariate Standard S' (MV-S) chart are obtain by using $UCL = |\Sigma_0| \left(b_1 + k\sqrt{b_2}\right)$ and $LCL = |\Sigma_0| \left(b_1 - k\sqrt{b_2}\right)$ where $k > 0$ is the control limit coefficient of the S chart. The value k depends on the desired in-control ARL of the Chart. The operation of the chart is as usual. In this case,

$$P = 1 - P\left(LCL < |S| < UCL \mid |\Sigma| \neq |\Sigma_0|\right)$$

$$= 1 - \Phi\left(\frac{k}{DR} - \left(1 - \frac{1}{DR}\right)\frac{b_1}{\sqrt{b_2}}\right) + \Phi\left(\frac{-k}{DR} - \left(1 - \frac{1}{DR}\right)\frac{b_1}{\sqrt{b_2}}\right) \quad \text{...(11)}$$

where $\Phi(.)$ is the cumulative distribution function of the standard normal distribution. In this case also, to obtain the design parameters, ARL model is used.

2.3 *MEWMA - V Chart*

Let there be k sample each of size n from N_p (μ_0, Σ_0) distribution.

Compute $\bar{\bar{X}} = \frac{1}{k}\sum_{i=1}^{k}\bar{X}i$ and $\bar{S} = \frac{1}{k}\sum_{i=1}^{k} S_i$

Bivariate and Multivariate Synthetic Control Charts for the Process Variation

In order to develop the BI-Syn S and the MV-Syn S control charts for the process variation, Ghute and Shirke (2008) applied the technique of synthetic control chart due to Wu and Spedding (2000).

It is well known that the Shewhart's $\bar{X}$ chart is inefficient to detect small shifts in the process mean, and the Conforming Run Lerngth' (CRL) chart proposed by Bourke (1991) detects small shifts effectively, but inefficient to detect large sifts. Wu and Spedding (2000) combined these two charts to introduced the synthetic control chart to have a more effective technique. In synthetic control chart procedure, a sample of size n is observed. If the sample mean $\bar{X} \notin \left[\mu_0 - k\sigma/\sqrt{n}, \mu_0 + k\sigma/\sqrt{n}\right]$, it is declared as a non-conforming sample. The chart declares the process as out of control when the sample based CRL is not exceeding L (the control limit). Wu and Spedding (2000) have illustrated that, to detect small to moderate shifts in the process mean of the univariate process data, the synthetic chart performs better than the $\bar{x}$ chart and the CRL chart.

In the very similar way, Ghute and Shirke (2008) have combined the standard S chart and the CRL chart proposed by Bourke (1991) to introduce the BI-Syn S and the MV -Syn S control charts. To construct the synthetic chart, they considered the ARL model given in (7).

If n is the fixed sample size, for given values of (n, DR_1, τ), the optimum values of the design parameters (L, UCL) in case of BI-Syn S chart and the optimum values of (k , L) in case of MV-Syn S chart are computed. As shown in Ghute and shirke (2008), for the synthetic S chart, we have

$$ARL(DR) = \frac{1}{P}\frac{1}{\left\{1-Q^{L}\right\}}, \quad ...(12)$$

Where $Q = 1 - P$. Ghute and Shirke (2008) have illustrated that in zero state, the proposed synthetic control charts outperform the standard |S| chart, adaptive sample size |S| chart, MEWMA - V chart and some other competitive charts to detect changes in the process variability. It is to be noted that in case of Bivariate charts as only the UCL considered, these charts are dedicated to the detection an increase in the process variation instead of any type of shift.

2.4 *Bivariate and Multivariate GR* S *Charts for the Process Variation:*

Gadre (2011) have developed the GR S charts for the process dispersion.GR S charts for detecting a change in the process variation consist of two components. The first is to identify quality of the group using the standard S chart-based procedure and the second to decide status of the process depending on the GR procedure.

Standard S chart - based procedure : If the value |S| based on the group of *n* units falls outside the control limit (s), declare a group as non-conformed; otherwise, it is treated as a conformed group.

GR procedure : Let Y_r denotes the r^{th} (r = 1,2,...) group based CRL. In other words it is the number of groups inspected between $(r-1)^{th}$ (if one such exists) and the r^{th} non-conformed group, including the r^{th} non-conformed group. L indicates the lower limit of the GR chart. GR procedure declares the process as out of control if $Y_1 < L$ or for some r (>1) Y_r and $Y_{r+1} \leq L$ for the first time.

The expression for ARL :

To obtain the expression for ARL (DR) of the GR chart, we discuss the runs rule representation of the chart, using the group run lengths (Y_r, r = 1,2,...). If it is assumed that at time zero a non-conformed group is observed with $Y_0 \leq L$, the GR S chart is identical to the following runs rule.

'Declare the process as out of control as soon as two successive Y_r $(r \geq 0)$ are not exceeding L'.

The formula for ARL (DR) can easily be obtained by using the 'Transition Probablity Matrix' (t.p.m.)of an absorbing Markov chain based on CRL values used to model the GR S chart . Let m and l respectively denote the states {Y > L} and {Y ≤ L}.Note that Y_r, r = 1,2,...are idependently and identically distributed (i.i.d.) geometric random variables with mean 1/P(DR). If we define, A = P (Y ≤ L), the t.p.m. related to the GR S chart is given by

$$
\begin{array}{c c}
 & \begin{array}{ccc} m & l & \text{Signal} \end{array} \\
\begin{array}{c} m \\ l \\ \text{Signal} \end{array} &
\begin{bmatrix} (1-A) & A & 0 \\ (1-A) & 0 & A \\ 0 & 0 & 1 \end{bmatrix}
\end{array}
$$

with $A = 1-Q^L$.

For the runs rule representation of GR S chart one has to take *l* as the initial state. Let **R** be the matrix obtained by deleting the last row and column of the above matrix. Let N be the number of non conformed groups observed by the time GR S chart declare the process has gone out of control. Then, the average number of nonconformed groups observed before declaring the process as out of control is identical to the average time for the Markov chain to enter the absorbing state 'Signal'. The vector of average times (in terms of number of non conformed groups) corresponding to the various states is given by

$$E\,(\underline{N}) = (\mathbf{I} - \mathbf{R})^{-1}\underline{1}, \qquad \text{...(13)}$$

where $\underline{1}$ is a column vector of appropriate order having all elerments unity. As 'l' is the initial state for the GR S chart, the second element of $E\,(\underline{N})$ is E (N). Thus $E(N) = 1/A^2$. Therefore ARL (DR) for GR S chart is given by

$$ARL\,(DR) = \frac{1}{P}\frac{1}{P\{1-Q^L\}^2} \qquad \text{...(14)}$$

3. *Modified Group Runs Control Chart*

3.1 The Procedure of Implementation

The MGR chart for detecting a change in the process variation consists of two components. The first is to identify quality of the group using the standard S chart-based procedure as discussed in GR S chart and the second to decide status of the process depending on the MGR procedure.

MGR procedure : Let Y_r denotes the r^{th} (r = 1,2,...) group based CRL. L_2 indicates the lower limit of the MGR S chart. MGR procedure declares the process as out of control if $Y_1 \le L_2$ or for some r (>1) $Y_r \le L_1$ and $Y_{r+1} \le L_2$ for the first time. Here the role of L_1 is like a warning limit.

Remark : In the ARL criterion, charts are developed when the sample size n is fixed. Instead of this, in case of 100% inspection or uniform sampling inspection, one can consider the sample size n as the design parameter to be obtained, and use the ATS model below instead of the ARL model given in (7). As we are choosing the optimal value of n, the use of ATS model given below will definitely increase the efficiency of the chart.

$$\left.\begin{array}{l} \text{Minimise } ATS_1 \\ \text{Subject to} \quad ATS_0 \geq \tau, \end{array}\right\} \quad \ldots(15)$$

Therefore to develop the MGR S charts, we use the ATS model and compare the performance of the various charts under the ATS model.

3.2 *The expression for ATS :*

The expression for ATS (DR) of the MGR S chart is similar to that discussed in Gadre and Rattihalli (2006). Below we discuss the same in brief.

If it is assumed that at time zero a non-conformed group is observed with $Y_0 \leq L_1$, the MGR S chart is identical to the following runs rule :

'Declare the process as out of control if for some r (≥ 0), $Y_r \leq L_1$, $Y_{r+1} \leq L_2$ for the first time'.

Using the above runs representation an expression for the ATS of the MGR S chart can be obtained. For the purpose, in the following we discuss the CRL based Markov chain representation of the MGR S chart.

CRL Based Markov Chain Representation of the MGR S Chart

A CRL based Markov chain representation of the MGR S chart can be given with the help of the states $M = \{Y > L_1\}$, $L = \{Y \leq L_1\}$ under the first level of group inspection and $m = \{Y > L_2\}$, $l = \{Y \leq L_2\}$ under the second level of group inspection. The related transition probablity matrix (t.p.m.) is

$$\begin{array}{c|cccc} & L & M & m & Signal \\ \hline L & 0 & 0 & 1-A_2 & A_2 \\ M & A_1 & 1-A_1 & 0 & 0 \\ m & A_1 & 1-A_1 & 0 & 0 \\ Signal & 0 & 0 & 0 & 1 \end{array}$$

where $A_i = 1 - Q^{L_i}$ for i = 1,2. It is worth to note that L corresponds to the initial state and *l* corresponds to the absorbing state. The expression for ATS of the MGR S chart can easily be obtained by computing ATS when the initial state is L. Let R_1 be a matrix obtained by deleting the last row and column of the above t.p.m. and $\underline{N}$ be a column vector of order 3 indicating the number of non-conformed groups observed before declaring the process has gone out of control when the process is in the states L, M and m respectively. Then $E(\underline{N}) = (I-R_1)^{-1}\underline{1}$. As L is the initial state of MGR, the first element $(1-A_2+A_1)/(A_1A_2)$ of $E(\underline{N})$ is E(N) for MGR chart and hence

$$ATS(P) = (n/P)\,(1-A_2+A_1)/(A_1A_2)$$

$$= \frac{n}{P}\left\{\frac{Q^{L_2}+1-Q^{L_1}}{\left(1-Q^{L_1}\right)\left(1-Q^{L_2}\right)}\right\}. \qquad ...(15)$$

The procedure of finding the value of P_i (i = 0, 1) is the procedure similar to that discussed in Ghute and Shirke (2008). Then the procedure used to find the design parameters is similar to that discussed in Gadre and Rattihalli (2006). For the sake of completeness below we brief the procedure of obtaining the design parameters.

3.3 *A Procedure of obtaining the Design Parameters of the MGR S charts .*

The optimal design procedure for the BI-MGR S chart

The procedure has following steps.

Step-1 : Specify Σ_0, DR_1 and τ.

Step - 2 : Suppose m1 is the minimum ever attained value of ATS_1. As ATS_1 is always less than ATS_0, initialise m_1 to τ. Initialise n to 1.

Step - 3 : If $m_1 < n$, terminate the execution; else initialise L_1 and L_2 to 1.

Step - 4 : Obtain P_1 by solving (15) numerically. For this take $\in > 0$ as a very small real constant. We say that ATS_0 and τ are very close to each other if the difference between them is not more that ε and the condition $ATS_0 \geq \tau$ holds.

Step - 5 : Obtain the upper P_1 percentage point $X^2_{2n-4}(P_1)$ of Chi-square distribution with $(2n-4)$ degrees of freedom by using Equation (5).

Step - 6 : From the current values of $X^2_{2n-4}(P_1)$, L_1 and L_2, compute UCL and ATS_1 using Equations (3) and (15).

Step - 7 : Test whether the conditions $ATS_1 < m_1$ and $ATS_0 \geq \tau$ hold. If so, change m_1 to ATS_1 and move to the next step ; else move to the next step without changing m_1.

Step- 8 : If $L_2 < 100$, increase the value of L_2 by unity and go back to Step - 4; else if $L_1 < 100$, increase the value of L_1 by unity, initialise L_2 to 1 and then go back to step - 4; else increase the value of n by unity and go back to Step - 3.

Step - 9 : The value of m_1 is the required value of ATS_1. The corresponding values of UCL, L_1 and L_2 as the design parameters of the proposed bivariate chart.

4. *Numerical Examples and Comparison in Zero State*

In this section, we consider various examples and illustrate that under ATS criterion, in zero state as well as in steady state, the MGR S charts perform better than the synthetic S chart and the GR S chart. Under ATS criterion, the sample size n is also one of the design parameters.

4.1 *Examples Related to BI-MGR S chart and its performance in Zero state*

***Example* - 1 :** For the input parameters $DR_1 = 3$ and $\tau = 1200$, values of the design parameters of the four charts along with respective zero state ATS_1 are given below.

BI-S chart : $n = 18$, $UCL = 2.3179$, $ATS_1 = 31.3624$.

BI-Syn S chart : $n_s = 11$, $L_s = 4$, $UCL_s = 2.0878$, $ATS_{1s} = 21.0928$.

BI-GR S chart : $n_g = 9$, $L_g = 4$, $UCL_g = 1.8431$, $ATS_{1g} = 17.222$. .

BI - MGR S Chart : $n_{mg} = 7$, $L_{1mg} = 1$, $L_{2mg} = 6$, $UCL_{mg} = 1.8345$, $ATS_{1mg} =$ 14.8179.

The above computation clearly indicates that ATS_1 of the BI-MGR S chart is significantly less as compared to the remaining three charts. Further, to study the behaviour of BI- MGR S and the other three charts corresponding to the changes in DR value, we have computed normalised ATS (DR) values (normalised with respect to the BI-Syn S chart). The graphs of these values against DR values for various charts are given in Fig. 1.

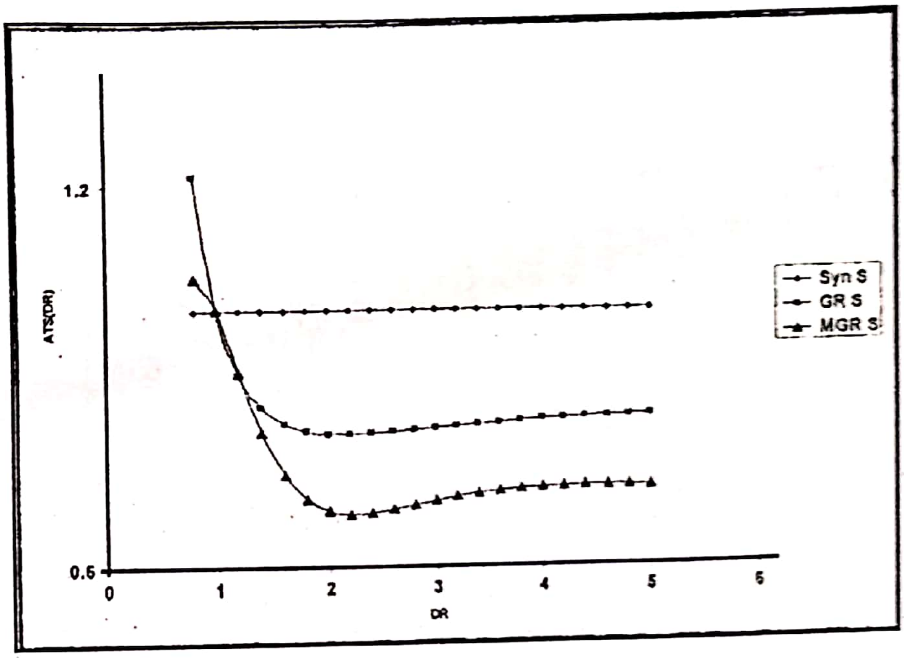

Fig.1 : Graphs of ATS vs DR values for the three charts

Figure-1 clearly indicates the superiority of the BI-MGR S chart over the BI-Syn S chart and the BI-GR S chart in detecting an increase in the value of DR.

Example - 2: Here we consider six values of DR_1 as {1.2, 1.7, 2.2, 2.5, 3.0, 3.5} and three values of τ as {1200, 5000, 6000} respectively. For all these 18 combinations of the input parameters (DR_1, τ), values of the design parameters along with the respective values of ATS_1 are computed for each of the three control charts and are given in the following table.

τ	DR_1	Bivariate Synthetic S hart				Bivariate GR S Chart				Bivariate MGR S Chart				
		n_s	L_s	UCL_s	ATS_{1s}	n_g	L_g	UCL_{mg}	ATS_{1g}	n_{mg}	L_{1mg}	L_{2mg}	UCL_{mg}	ATS_{1mg}
1200	1.2	85	3	1.1930	224.7	65	4	1.2027	188.95	37	1	9	1.3384	154.53
	1.7	29	3	1.4783	59.0	21	4	1.4784	47.56	13	1	8	1.6474	38.89
	2.2	17	3	1.7248	33.78	13	4	1.6648	27.24	9	1	7	1.7715	22.8
	2.5	15	3	1.7982	27.2780	12	4	1.7014	22.1655	8	1	7	1.8230	18.7013
	3.0	11	4	2.0878	21.0928	9	4	1.8431	17.2220	7	1	6	1.8345	14.8179
	3.5	9	4	2.6663	17.5914	9	3	1.7286	14.5559	6	1	6	1.8937	12.6111
5000	1.2	180	3	1.1754	394.08	139	3	1.1577	317.59	75	1	8	1.2756	255.93
	1.7	41	4	1.5647	83.51	39	4	1.4383	66.54	24	1	7	1.5677	53.78
	2.2	18	4	2.0376	46.59	32	3	1.4536	40.23	19	1	3	1.5366	31.75
	2.5	19	4	1.9964	35.4401	17	4	1.7717	28.2220	9	1	9	2.1322	23.7719
	3.0	15	4	2.1912	26.6943	16	2	1.6157	22.6252	9	1	9	2.1322	18.5607
	3.5	13	3	2.2582	26.6943	16	2	1.6157	19.6962	7	1	7	2.2419	15.2107
6000	1.2	208	3	1.1652	419.27	152	3	1.1554	335.76	83	1	9	1.2742	270.75
	1.7	49	3	1.4877	86.64	33	3	1.4617	67.9	15	1	12	1.8938	5651
	2.2	24	3	1.8230	46.489	18	2	1.5947	39.41	15	1	12	1.8938	32.36
	2.5	21	3	1.9088	36.5062	27	2	1.4519	33.0423	11	1	7	1.9753	24.2506
	3.0	15	4	2.2332	27.4236	18	1	1.4183	27.7105	9	1	7	2.1070	18.6628
	3.5	13	4	2.3761	22.3596	15	1	1.4721	22.7865	8	1	6	2.1490	15.5163

From Table 1 we observe the following

1. $n_{mg} \leq n_g \leq n_s$;
2. $ATS_{1mg} < ATS_{1g} < ATS_{1s}$.

As the Table 1 covers almost all the situations, we conclude that under ATS criterion, the BI-MGR S chart is superior to the matched BI-GR S chart and the BI-Syn S chart in detecting an increase in the process dispersion. If n_s is the optimal sample size of ther BI-Syn S chart under ATS criterion, BI-Syn S chart is equivalent to the synthetic control chart of Ghute and Shirke (2008) with the sample size n_s and the constraint $ARL_0 \geq \tau_1$ $(=\tau / n_s)$. Further Ghute and Shirke (2008) illustrated that the BI-Syn S chart is superior to the matched BI-S chart, the adaptive sample size S chart and MEWMA-V chart proposed by Yeh et.al. (2003). Thus we conclude that the BI-MGR S chart is superior to all these charts.

A Procedure of Obtaining the Design Parameters of the MGR S charts.

The optimal design procedure for the MV-MGR S chart :

The procedure has the following steps.

Step-1 : Specify Σ_0, DR_1 and τ.

Step-2 : Suppose m_1 is the minimum ever attained value of ATS_1. As ATS_1 is always less than ATS_0, initialise m_1 to τ. Initialise n to 1.

Step-3 : Compute b_1 and b_2 using Equations (9) and (10)

Step-4 : If $m_1 < n$, terminate the execution; else initialise L_1 and L_2 to 1.

Step-5 : Obtain P_1 by solving (15) numerically. For this take $\varepsilon > 0$ as a very small real constant. We say that ATS_0 and τ are very close to each other if the difference between them is not more than ε and the condition $ATS_0 \geq \tau$ holds. From, this value of P_1, obtain the value of k such that $P_1 = 2\Phi(-k)$.

Step-6 : Compute UCL, LCL and ATS_1.

Step-7 : Test whether the conditions $ATS_1 < m_1$ and $ATS_0 \geq \tau$ hold. If so, change m_1 to ATS_1 and move to the next step; else move to the next step without changing m_1.

Step-8 : If $L_2 < 100$, increase the value of L_2 by unity and go back to Step-5; else if $L_1 < 100$, increase the value of L_1 by unity, initialise L_2 to 1 and then go back to Step-5; else increase the value of n by unity and go back to Step-3.

Step-9 : The value of m_1 is the required value of ATS_1. The corresponding values of UCL, L_1 and L_2 are the design parameters of the proposed bi-variate chart.

4.2 *Examples Related to the MV-MGR S chart and its performance in Zero state*

Example -3: When $p = 3$; for the input parameters $DR_1 = 3$ and $\tau = 1200$, values of the design parameters of the four charts along with respective zero state ATS_1 are given below.

MV-S Chart : $n = 4$, $k = 2.1796$, $ATS_1 = 11.7790$.
MV-Syn S chart : $n_s = 4$, $k_{(s)} = 1.7239$, $L_s = 6$, $ARL_{1(s)} = 8.8344$.
MV-GR S chart : $n_g = 4$, $k_g = 1.5612$, $L_g = 7$, $ATS_{1g} = 7.8891$.
MV-MGR S Chart : $n_{mg} = 4$, $k_{1mg} = 1.5636$, $L_{1mg} = 4$, $L_{2mg} = 10$, $ATS_{1mg} = 7.8113$.

The above computation clearly indicates that ATS_1 of the MV-MGR S chart is less as compared to the remaining three charts. Further, to study the behaviour of the MV-MGR S and the other three charts corresponding to the changes in DR value, one can compute the normalised ATS(DR) values (normalised with respect to the MV-Syn S chart) and draw the graphs of these values against DR values for various charts similar to Figure 1.

Example 4 : The optimal values of (n, L,UCL) for dimensions p = 2,3 and 4, DR_1 = 1.2, 1.7, 2.2, 2.5, 3.0 and 3.5 with τ = 1200 for the MV-Syn S chart, MV - GR S chart and the MV-MGR S chart along with the respective ATS_1 values are presented in Table - II.

Table II : Optimal design parameters along with ATS_1 values of the MV-Syn S chart, the MV- GR S chart and the MV- MGR S chart

p	DR_1	MV - Syn S chart				MV - GR S chart				MV - MGR S chart				
		n_s	L_s	UCL_s	ATS_{1s}	n_g	L_g	UCL_g	ATS_{1g}	n_{mg}	L_{1mg}	L_{2mg}	UCL_{mg}	ATS_{1mg}
2	1.2	3	46	3.4629	149.78	3	40	3.2537	112.4	3	1	73	3.0323	58.2345
	1.7	3	14	3.2482	21.37	3	13	2.9675	16.5972	3	1	15	2.6981	12.4426
	2.2	3	9	3.1633	10.71	3	10	2.8959	8.9765	3	1	9	2.5985	7.5934
	2.5	3	8	3.1392	8.5160	3	7	2.7942	7.2915	3	1	9	2.5985	6.4670
	3.0	3	6	3.0815	6.6641	3	6	2.7495	5.8989	3	1	6	2.5203	5.4164
	3.5	3	5	3.0441	5.7133	3	6	2.7493	5..1829	3	1	6	2.5203	4.8463
3	1.2	4	42	1.9436	170.33	4	39	1.8329	130.7810	4	1	66	1.6936	70.9162
	1.7	4	14	1.8243	27.0898	4	12	1.6521	21.3241	4	1	16	1.5156	16.1274
	2.2	4	9	1.7730	13.99	4	8	1.5842	11.7235	4	1	9	1.4469	10.007
	2.5	4	7	1.7427	11.2132	4	7	1.5612	9.9506	4	1	8	1.4329	8.5436
	3.0	4	6	1.7242	8.8365	4	6	1.5342	7.8466	4	1	6	1.3989	7.1994
	3.5	4	5	1.7014	7.5992	4	7	1.5612	6.9668	4	1	6	1.3989	6.4578
4	1.2	5	36	0.9760	187.3937	4	32	0.9143	145.9844	5	1	61	0.8542	82.2494
	1.7	5	13	0.9171	32.2856	5	12	0.8342	25.6796	5	1	15	0.7599	19.59
	2.2	5	8	0.8869	17.0567	5	8	0.7983	14.3874	5	1	9	0.7273	12.3223
	2.5	5	7	0.8782	13.7570	5	8	0.7983	11.9653	5	1	7	0.7111	10.5618
	3.0	5	6	0.8683	10.9108	5	5	0.7546	9.7318	5	1	6	0.7013	8.9281
	3.5	5	5	0.8562	9.4155	5	8	0.7983	8.7442	5	1	5	0.6897	8.0268

4.3 Performance Comparison of the MV-MGR S Chart with the MV - Syn S chart and the MV-GR S chart

Here we make the comparison of the MV-Syn S chart, the MV-GR S chart and the MV-MGR S chart. The performances of the charts are compared by taking $ATS_0 = 1200$. Further the three different values of p as

Table III : ATS (DR) comparison of the three charts for $p = 2,3,4$; $DR_1 = 3$ and $\tau = 1200$

DR	p = 2			p = 3			p = 4		
	MV Syn-S	MV GR-S	MV MGR-S	MV Syn-S	MV GR-S	MV MGR-S	MV Syn-S	MV GR-S	MV MGR-S
1.0	1201.1	1201.1	1202	1203.5	1200.6	1201.1	1201.6	1200.8	1200.5
1.1	422.8653	377.1884	395.2231	446.759	399.6518	413.492	464.5111	431.8355	427.0979
1.2	189.49	157.5494	163.7155	209.4617	175.3775	178.7376	225.1527	200.1351	189.8321
1.3	101.4403	81.0288	81.2828	116.3618	93.8528	92.1814	128.486	111.1797	100.3307
1.5	41.886	32.5826	30.1189	50.6795	39.8867	36.2941	58.1769	49.2068	41.1464
2.0	13.8024	11.1821	9.7086	17.7738	14.5261	12.5807	21.394	18.2389	15.0936
3.0	6.6641	5.8989	5.4164	8.8365	7.8466	7.1994	10.9108	9.7318	8.8679
5.0	4.519	4.2562	4.0874	6.0361	5.6892	5.4591	7.5179	7.0436	6.7789

2,3,4 and various shifts in the covariance matrix are considered. Here we assume that underlying process data follow a p-variate normal distribution.

The probablities of detecting these shifts, P(DR) and the Related ATS vaLues of different charts are calculated using a Macro developed in MatLab.From the above table, we observe that for $DR_1 > 1.3$, the MV-MGR S chart consistently produces smaller out-of-control ATS than the MV-Syn S chart and the MV-GR S chart. Therefore in this range MV-MGR S chart is superior to the other two charts. Ghute and Shirke (2008) have already illustrated that, for various sample sizes, the MV-Syn S chart is superior to the standard S chart, MLRT, SSVPC, $|S|^{(1/2)}$ and the decomposition scheme introduced by Tang and Barnett (1996). Thus by using a similar explanation given below Table 1, we conclude that the MV-MGR S cahrt is preferred to the remaining charts mentioned above with sample size n_s.

5. Steady State Behaviorsw of the Various Charts

5.1 The Markov Chain Representation of the Synthetic S and GR S chart

As the synthetic S chart, the GR S chart as well as the MV-MGR S can be represented as a Markov chain, it is desirable to study its performance in steady state. The steady state ATS measures average time to signal (in terms of samples), when the effect of head start has been faded away.To compare the steady state performance of the MGR S chart with the other compatible charts, we consider the Markov chain representation of the MGR S chart depending on standard S chart - based procedure.

To distinguish between the levels of group inspection in the MGR S chart, let the groups in the first (second) level of inspection be classified as 0 $(\tilde{0})$ or $(\tilde{1})$ according as it being conformed or non--conformed. For the illustration purpose, let $L_1 = 3$ and $L_2 = 2$. Thus the MGR S chart will produce a signal if $Y_1 < 3$ or for some $r(>1)$ $Y_r < 4$ and $Y_{r+1} < 3$ for the first time. As in Davis and Woodall (2002), a Markov chain representation in this situation can be described by using the 21 states listed in Table IV.

Table IV : States of the MGR S chart aNd their Labels

State Label	State	State Label	State	StateLabel	State
1.	Õ Õ	8.	Õ Õ Ĩ 01 Õ	15.	00011
2.	Õ Õ Ĩ	9.	Õ Õ Ĩ 001	16.	00011 Õ
3.	Õ Õ Ĩ 0	10.	Õ Õ Ĩ 001 Õ	17.	000101
4.	Õ Õ Ĩ 00	11.	000	18.	000101 Õ
5.	Õ Õ Ĩ 1	12.	0001	19.	0001001 Õ
6.	Õ Õ Ĩ 1 Õ	13.	00010	20.	0001001 Õ
7.	Õ Õ Ĩ 01	14.	000100	21.	Signal

For the general values of L_1 and L_2, the matrix **W** (of non absorbing states) has the following states.

1. A Sequence of at least L2 Õ′ S.
2. A sequence in (1) followed by Ĩ and further appended by at most (L_1-1) 0′ s. There are L_1 such sequences.
3. Each of the sequences in (2) followed by 1 and is further appended by a sequence of at most (L_2-1) Õ's. The toatal number of such sequences in L_1L_2.
4. A sequence of at least L_1 0's.
5. A sequence in (4) followed by 1 and further appended by at most (L_1-1) zeros. The number of such sequences in L_1.
6. Each sequence in (5) appended by 1 and further followed by at most (L_2-1) Õ's. There are L_1L_2 such sequences and the last state is Signal.

Thus W is a square matrix of order 2 $(L_1(L_2+1)+1)$. Note that the $(i,j)^{th}$ element of W is

$$w(i,j) = \begin{cases} Q & \text{if the } i^{th} \text{ state leads to } j^{th} \text{ state, and } j^{th} \text{ state corre-} \\ & \text{sponds to the sequence ending with 0 or } \tilde{0} \\ P & \text{if the } i^{th} \text{ state leads to } j^{th} \text{ state, and } j^{th} \text{ state} \\ & \text{corresponds to the sequence ending with 1 or } \tilde{1} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\underline{\pi}$ be a 1 x $2(L_1(L_2 + 1)+1)$ row vector corresponding to the stationary probablity distribution that the Markov chain will be in each of the non-absorbing sates, which is conditioned on no signal. The steady state ATS of the MGR chart is given by $\{n_{mg}\,(I-W)^{-1}\,\underline{1}\}$. For further details on Markov chain representation of the chart, one may refer to Brooke and Evans (1972).

5.2. *The Steady State ATS performance of the bivariate charts*

Example-1 (Cont.)

It is to be noted that for any run length based control chart, the steady state ATS is not smaller than zero state ATS. If the signal depends on one point only, both ATS's are same. Hence by making the $(S.S.\ ATS)_0$ of the two charts same, their performance can be compared. Hence we compute adjusted sate ATS of the chart II with respect to the chart I as $[Adj.\ S.S.\ ATS\ (DR)]_{II} = \{[S.S.\ ATS\ (DR)]_{II} / [S.S.\ ATS(1)]_{II}\}\ \{[S.S.\ ATS(1)]_{I}\}$.

Table-V gives adjusted steady state ATS values corresponding to the different values of DR for various charts.

Table V : Adj. SSATS for the BI-Syn S chart, the BI-GRS chart and the BI-MGR S chart

DR	Synthetic	GR	MGR	DR	Synthetic	GR	MGR
0.6	122500.0	200988.1	98376.62	3	27.7786	24.21825	22.51579
0.8	7214.90	8481.649	6889.911	3.2	25.5284	22.24596	20.44003
1	1310.90	1310.9	1310.9	3.4	23.7636	20.69295	18.81011
1.2	435.4628	403.1048	431.2453	3.6	22.3513	19.44451	17.50182
1.4	205.784	183.81	197.8932	3.8	21.2016	18.42326	16.43205
1.6	120.8915	106.2946	112.543	4	20.2521	17.57549	15.54351
1.8	81.7476	71.42005	73.88912	4.2	19.4581	16.8626	14.79543
2	60.769	52.97864	53.55669	4.4	18.7869	16.25649	14.15811
2.2	48.2601	42.05797	41.63042	4.6	18.2141	15.73614	13.60963
2.4	40.1934	35.03601	34.03918	4.8	17.7212	15.28573	13.13328
2.6	34.6734	30.2326	28.89372	5	17.2939	14.8928	12.71628
2.8	30.7181	26.78607	25.22974				

From Table-V we observe the following

1. For FR in the left neighborhood of 3 (= DR_1),
   $(\text{Adj. SSATS})_{Syn} > (\text{Adj. SSATS})_{GR} > (\text{Adj. SSATS})_{MGR}$

and

2. For DR > 3,
   $(\text{Adj. SSATS})_{Syn} > (\text{Adj. SSATS})_{GR} > (\text{Adj. SSATS})_{MGR}$

The computations indicate that the bivariate MGR S chart is superior in detecting the significant shift as compared to the other two compatible charts in the steady state.

5.3 *The Steady State ATS performance of the Multivariate charts*
Example-3 (Cont.)

Table-VI gives adjusted steady state ATS values corresponding to the different values of DR for various charts.

Table VI : Adj. SSATS for the MV-Syn S chart, the MV-GR S chart and the MV-MGR S chart

DR	Synthetic	GR	MGR	DR	Synthetic	GR	MGR
0.6	11471000	62271747	74957725	3	11.3109	10.06653	8.83565418
0.8	27220	44704.18	50122.0365	3.2	10.488	9.376702	8.25252282
1	1281.10	1281.1	1281.1	3.4	9.8363	8.827424	7.78746597
1.2	240.221	204.3316	185.968047	3.6	9.3088	8.380556	7.40862226
1.4	89.74	73.80825	64.2280226	3.8	8.8738	8.010428	7.0944384
1.6	48.1352	39.81073	34.1651088	4	8.5095	7.699252	6.82992081
1.8	31.6114	26.52995	22.7469792	4.2	8.2003	7.434194	6.60442063
2	23.4322	19.95859	17.1729419	4.4	7.9348	7.205993	6.41001512
2.2	18.7582	16.18204	13.9877412	4.6	7.7045	7.007405	6.2407409
2.4	15.8088	13.78047	11.9658216	4.8	7.503	6.833297	6.09208286
2.6	13.809	12.13914	10.5836852	5	7.3252	6.67936	5.96063338
2.8	12.3782	10.95559	9.58610038				

From Table-VI we observe the following

1. For DR< 1, $(\text{Adj. SSATS})_{Syn} < (\text{Adj. SSATS})_{GR} < (\text{Adj. SSATS})_{MGR}$

and

2. For DR > 1, $(\text{Adj. SSATS})_{Syn} > (\text{Adj. SSATS})_{GR} > (\text{Adj. SSATS})_{MGR}$.

The computations indicate superiority of the MV-MGR S chart over other two charts in the steady state.

6. CONCLUSION

In this article, we have proposed BI–MGR S chart and MV-MGR S chart for monitoring the covariance matrix of a normally distributed process. The ATS comparison between the other procedures and the MGR S control charts are carried out. The comparisons indicate that in zero state as well in the steady state, the MGR S chart outperform the SYn S chart and hence the standard S chart, the adaptive sample size S chart and EWMA - V chart for all the shifts considered. It is also superior to the corresponding GR S charts proposed by Gadre(2011).

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