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T.Y.B.Sc. (Mathematics)

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TRANSPORTATION PROBLEM (T.P)

The structure of T.P involves a large number of shipping routes from several supply origins to several demand destinations. The objective is to determine the number of units of an item (commodity or product) that should be shipped from an origin to a destination in order to satisfy the required quantity of goods or services at each destination centre. This should be done within the limited quantity of goods or services available at each supply centre, at the minimum transportation cost and/or time.

Note: There are the methods to solve the T.P are only for minimization.

General Mathematical Model of Transportation Problem

Let there be m sources of supply $S_1, S_2, \ldots \ldots \ldots \ldots \ldots \ldots \ldots .. S_m$ having a_i $(i = 1,2, \ldots \ldots \ldots \ldots .. m)$ units of supply (or capacity) respectively, to be transported among *n* destinations, $D_1, D_2, \ldots \ldots \ldots \ldots . \ldots D_n$ with b_j $(j = 1,2, \ldots \ldots \ldots \ldots n)$ units of demand (or requirement) respectively. Let c_{ij} be the cost of shipping one unit of the commodity from source *I* to destination *j* for each route. If x_{ij} represents number of units shipped per route from source *i* to destination *j*, the problem is to determine the transportation schedule so as to minimize the total transportation cost while satisfying the supply and demand conditions.

LPP formulation of transportation problem:

Mathematically, the problem, in general, may be stated as follows:

Minimize (total cost) Z= $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

Subject to the constraints

$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1,2, \ldots \ldots \ldots \ldots \ldots . m$ (Supply constraints)

$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1,2, \ldots \ldots \ldots \ldots \ldots .. n$ (Demand constraints)

and $x_{ij} \geq 0 \; for \; all \; i \; and \; j.$

Existence of feasible Solution:

A necessary and sufficient condition for existence of a feasible solution to T.P is

Total supply = Total demand

$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ (Also called ***Rim conditions***)

The tabular form of T.P is as follows:

From \ To	D_1	D_2	………	D_n	Supply a_i
S_1	c_{11} x_{11}	c_{12} x_{12}	………	c_{1n} x_{1n}	a_1
S_2	c_{21} x_{21}	c_{22} x_{22}	………	c_{2n} x_{2n}	a_2
. .	. .	. .	………	. .	. .
S_m	c_{m1} x_{m1}	c_{m2} x_{m2}	………	c_{mn} x_{mn}	a_m
Demand b_j	b_1	b_2	………	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

Some Definitions:

1) **Balanced T.P:**
A T.P is said to be balanced if it satisfies Rim Condition. i.e Total supply = Total demand i.e

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

2) **Degenerate feasible solution:**
The feasible solution is said to be degenerate when the number of positive allocations at any stage of the feasible solution **is less than** $m+n-1$.
3) **Non-degenerate feasible solution:**
The feasible solution is said to be non-degenerate when the number of positive allocations at any stage of the feasible solution **is equal to** $m+n-1$.
4) Cells that have positive allocations are called **occupied cells**, otherwise they are known as **non-occupied cells (or unoccupied cell).**
5) **Unbalanced Transportation Problem:**
The Transportation problem is said to be unbalanced when Total supply $\neq$ Total demand.

T.P can be made balanced by adding a dummy source or destination when total demand is not equal to total supply by setting the unit transportation cost equal to zero for the cells in that corresponding row or column.

THE TRANSPORTATION ALGORITHM:

Step 1: Formulate the problem and arrange the data in the matrix form.

Step 2: Check whether the T.P is balanced or not. If T.P is not balanced, then first balance it by adding dummy row or column with suitable amount of supply or demand.

Step 3: Obtain The Initial Basic Feasible Solution by using any one of the following method.

i) North West Corner Method
ii) Least Cost Entry Method
iii) Vogel's Approximation Method (VAM)

Step 4: The IBFS obtained by any of the three methods must satisfy non-degeneracy condition.

Step 5: Test the IBFS for optimality by using Modified Distribution Method (MODI).

Step 6: Repeat Step 5 until an optimal solution is reached.

METHODS OF FINDING INITIAL BASIC FEASIBLE SOLUTION

[A] North West Corner Method (NWCM):

1) Start with the upper left (north-west) corner of the transportation matrix and allocate commodity equal to the min (a_1, b_1)
2) (a) If allocation made in step 1 is equal to the supply available at first source (a_1, in first row), then move vertically down to the cell (2,1)in the second row and first column. Apply Step 1 again, for next allocation.

   (b) If allocation made in step 1 is equal to the demand available at first destination (b_1, in first column), then move horizontally to the cell (1, 2) in the first row and second column. Apply step 1 again, for next allocation.

   (c) If $a_1 = b_1$, allocate $x_{11} = a_1 \ or \ b_1$ and move diagonally to the cell (2, 2)

3) Continue the procedure step by step till an allocation is made in south-east corner cell of the Transportation table.

[B] Least Cost Entry method:

1) Select the cell with the lowest unit cost in the entire transportation table and allocate as much as possible to the cell. Then eliminate that row or column in which either the supply or demand is exhausted. **In case the smallest unit cost cell is not unique, then select the cell where the maximum allocation can be made.**

2) Repeat step 1 with the next lowest unit cost among the remaining rows and columns of the transportation table and allocate as much as possible to this cell. Then eliminate that row or column in which either the supply or demand is satisfied.
3) Repeat the procedure until the entire available supply at various sources and demand at various destinations is satisfied.

[C] Vogel's Approximation Method (VAM):

1) Calculate the **penalties** for each row and column by taking the difference between the smallest and the next smallest unit transportation cost in the same row and the same column. This difference indicates that the penalty or extra cost that has to be paid if one fails to allocate to the cell with the minimum unit transportation cost.
2) Select the row or column with **the largest penalty**, then select the least unit cost in that row or column and allocate as much as possible in the cell.
   If there is tie in the values of penalties then select the row or column with least unit transportation cost and if there is tie in the least unit transportation cost, then select the cell of row or column in which maximum allocation can be made.
3) Adjust the supply and demand and cross out the satisfied row or column.
4) Repeat steps 1 to 3 until the entire available supply at various sources and demand at various destinations are satisfied.

TEST FOR OPTIMALITY:

To check the optimality of the transportation problem we use the MODI method.

Modified Distribution Method: (MODI Method):

1) If the IBFS is non-degenerate (number of allocation$= m + n - 1$) then go to step 2.If IBFS is non degenerate (number of allocation$< m + n - 1$) then introduce ∈ in the unoccupied cell with minimum unit transportation cost (∈ is very small positive number) such that all allocations are at independent positions (i.e there should not form a loop)
2) Calculate $u_i's$ (for rows) and $v_j's$ (for columns) using the costs in occupied cells and the formula

$$\boldsymbol{u_i + v_j = c_{ij}},\ \textbf{for occupied cells (i, j).}$$

   It is better to assign zero to a particular u_i or v_j where there are maximum number of allocations in a corresponding row or column.

3) Calculate the opportunity cost $\boldsymbol{d_{ij}}$ for unoccupied cells using the relationship
   $\boldsymbol{d_{ij} = c_{ij} - (u_i + v_j)}$ for all *i* and *j*.
4) Examine sign of each d_{ij}
   (i) If $\boldsymbol{d_{ij}} > 0$**,** then the current basic feasible solution is optimal.
   (ii) If $\boldsymbol{d_{ij} = 0}$**,** then the current will remain unaffected but an alternative solution exists.
   (iii) In general, If $\boldsymbol{d_{ij}} \geq 0$, The transportation problem has optimal solution.
   (iv) If one or more $\boldsymbol{d_{ij}} < 0$, then an improved solution by entering unoccupied cell (i, *j*) in the basis. An unoccupied cell having largest negative value of $\boldsymbol{d_{ij}}$ is chosen for entering into the solution mix (new transportation schedule).
5) Construct the closed path (loop) for unoccupied cell with largest negative opportunity cost. Start the loop with selected unoccupied cell and mark plus sign (+) sign in this cell. Trace a path along the rows (or columns) to an occupied cell, mark the corner minus sign (-) and continue down the column (or row) to an occupied cell. Then mark the corner with plus sign (+) and minus sign (-) alternatively. Close the loop to the selected unoccupied cell.
6) Select the smallest quantity amongst the cells marked with minus sign on the corners of loop. Allocate this value to the selected unoccupied cell and add it to other occupied cells marked with plus signs. Now subtract this from the occupied cells marked with minus signs.
7) Obtain a new improved solution by allocating units to the unoccupied cell according to step 5 and calculate the new total transportation cost.
8) Further test the revised solution for optimality. The procedure terminates when all $\boldsymbol{d_{ij} \geq 0}$ for unoccupied cells.

Loop: An ordered set of at least four cells in a transportation table forms a loop.

Characteristics: (i) A loop contains an even numbers of cells
(ii) Any two adjacent cells of the ordered set lie either in the same row or column.

Alternate solution: If at least one $\boldsymbol{d_{ij} = 0}$ in the optimal solution of T.P then T.P has alternate solution.
To find alternate solution construct the loop starting from the cell in which $\boldsymbol{d_{ij} = 0}$.

Maximization Transportation Problem:

In maximization T.P instead of unit cost $\boldsymbol{c_{ij}}$**,** the unit profit or payoff $\boldsymbol{p_{ij}}$ is associated. Convert maximization T.P into minimization by

a) selecting the maximum payoff and subtracting all payoffs from it. **Or**
b) multiplying each $\boldsymbol{p_{ij}}$ by -1.

To find optimal solution use the profit per unit values in the original T.P.

Assignment Problem

The assignment model deals with matching workers (with varying skills) to jobs. The goal is to determine the minimum cost assignment of workers to jobs.

The general assignment model with *n* workers and *n* jobs is represented in the following table:

Jobs

	1	2		n	Supply
1	c_{11}	c_{12}		c_{1n}	1
2	c_{21}	c_{22}		c_{2n}	1
n	c_{n1}	c_{n2}		c_{nn}	1
Demand	1	1		1	

Let x_{ij} denote the assignment of facility *i* to job *j* such that

$$x_{ij} = \begin{cases} 1 \text{ if facility } i \text{ is assigned to job } j \\ 0 \text{ otherwise} \end{cases}$$

LPP Formulation of Assignment Problem:

Mathematically, the problem, in general, may be stated as follows:

Minimize (total cost) Z= $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$

subject to the constraints

$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1,2, \dots \dots \dots \dots .n$ (Supply constraints)

$\sum_{i=1}^{m} x_{ij} = 1, \quad j = 1,2, \dots \dots \dots \dots ..n$ (Demand constraints)

and $x_{ij} \geq 0 \; for\ all\ i\ and\ j.$

Hungarian Method to Find optimal solution of A.P.:

A. Check whether the number of rows is equals to the number of columns. If not, then as required a dummy row or dummy column. The cost element in dummy cells is always zero.

B. (a) Identify the smallest element in each row and subtract it from all the elements in that row.

(b) In the reduced matrix obtained from 2(a), identify the smallest element in each column and subtract it from all the elements in that column. Each row and each column now have at least one zero element.

C. The procedure of making assignments is as follows:
   a) First round of making assignments
      - Identify rows successively from top to bottom until a row with exactly one zero element is found. Make an assignment to this single zero by making □ around it. Then cross (X) all other zeros in that corresponding column. Repeat this till no row with exactly one zero is left.
      - Identify columns successively from left to right until a column with exactly one zero element is found. Make an assignment to this single zero by making □ around it. Then cross (X) all other zeros in that corresponding row. Repeat this till no column with exactly one zero is left.
      - **Note that in each row and each column there is exactly one assignment can be done.**
   b) Second round for making assignments
      - If a row and/or column has two or more unmarked zeros and one cannot be chosen by inspection, then choose the zero cells arbitrarily for assignment. In this case the Assignment problem has **alternate solutions**.
      - Repeat steps (a) and (b) successively until one of the following situation arise..

D. **Optimality Criterion**
   a. **I**f all zero elements in the matrix are either marked with □ or are crossed off (X) and if number of assignments is equal to number of row/or columns, then the solution is optimal.
   b. If number of assignments is less than the number of rows/or columns, then proceed to step 5.

E. Draw a set of horizontal and vertical lines to cover all the zeros in the revised cost matrix obtained from step 3, by using the following procedure:
   a. For each row in which no assignment was made, mark a tick (√)
   b. Examine the marked rows. If any crossed zero element occurs in those rows, mark a tick (√) to the respective columns containing those zeros.
   c. Examine crossed columns. If any assigned zero element occurs in those columns, tick (√) the respective rows containing those assigned zeros.
   d. Repeat this process until no more rows or columns can be marked.
   e. **Draw the lines through marked columns and unmarked rows.**

F. Develop the new revised opportunity cost matrix
   a. From uncovered elements (which are not on the lines) choose the smallest element say k, and subtract it from all uncovered elements.
   b. Add the smallest element k to the elements which lies at the intersection of two lines.
   c. Remaining elements on those lines remain unchanged.

G. Repeat steps 3 to 6 until an optimal solution is obtained.

H. **Alternate solution in A. P exists when any two or more rows or columns contains two or more than two zeros at the same positions. In this case the solutions can be obtained by assigning any zero randomly.**

Maximization Case in Assignment Problem:

Instead of costs, if c_{ij}, s are profits, then the Assignment Problem is of Maximization. Such problems may be solved by converting the given maximization problem into a minimization problem in either of the following two ways:

i) Change the sign of c_{ij}, s (i.e + to – and – to +) and solve by usual methods. But to find IBFs or Optimal solution use original cost.
ii) Locate the largest payoff element in the assignment table and then subtract all the elements of the table from the largest element.

The transformed assignment problem, so obtained, can be solved by using the Hungarian method.

Restrictions on Assignments

Sometimes it may so happen that a particular resource cannot be assigned a particular activity. In such cases, the cost of performing that particular activity by a particular resource is considered to be very large (written as M or ∞) so as to prohibit the entry of this pair of resource-activity into the final solution.

EXAMPLES

1) Determine the Initial basic feasible solution to the transportation problem by using NWCR, LCEM, VAM

	D1	D2	D3	D4	Supply
S1	21	16	15	3	11
S2	17	18	14	23	13
S3	32	27	18	41	19
Demand	6	10	12	15	

	D1	D2	D3	D4	Supply
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Demand	200	225	275	250	

	D1	D2	D3	D4	Supply
S1	1	2	1	4	30
S2	3	3	2	1	50
S3	4	2	5	9	20
Demand	20	40	30	10	

	D1	D2	D3	D4	Supply
O1	6	4	1	5	14
O2	8	9	2	7	16
O3	4	3	6	2	5
Demand	6	10	15	4	

2) Obtain IBFS by VAM and obtain optimal solution by MODI method.

	D1	D2	D3	D4	Supply
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8	70	20	18
Demand	5	8	7	14	

	D1	D2	D3	D4	Supply
A	6	3	5	4	22
B	5	9	2	7	15
C	5	7	8	6	8
Demand	7	12	17	9	

3)Obtain IBFS by LCM and find optimal solution by MODI

	A	b	c	d	Supply
A	8	9	6	3	18
B	6	11	5	10	20
C	3	8	7	9	18
Demand	15	16	12	13	

(Unbal)

	A	B	C	Supply
W	4	8	8	76
X	16	24	16	82
Y	8	16	24	77
Demand	72	102	41	

4) Obtain optimal solution.

	E	F	G	H	I	Availability
A	8	10	12	17	15	100
B	15	13	18	11	9	150
C	14	20	6	10	13	180
D	13	19	7	5	12	280
Requirement	90	170	50	210	190	

If in the above problem, the transportation cost from A to G is reduced to 10, what will be the new optimum solution?

5) The IBFS of T.P is given as below:

	I	II	III	IV
A	5	10	4 10	5
B	6 20	8	7	2 5
C	4 5	2 10	5 5	7

Answer the following questions:

i) Is the solution is feasible?
ii) Is the solution is degenerate?
iii) Is the solution optimum?
iv) Does the problem have more than one solution? If so, show at least one of them.
v) If the cost for the route B-III is reduced from Rs.7 to Rs. 6 per unit, what will be the new optimum solution?

Degeneracy, Unbalanced, mixed Transportation problems (Find optimal solution)

	D1	D2	D3	D4	D5	Supply
S1	5	8	6	6	3	8
S2	4	7	7	6	5	5
S3	8	4	6	6	4	9
Demand	4	4	5	4	8	

	M1	M2	M3	M4	M5	Supply
F1	4	2	3	2	6	8
F2	5	4	5	2	1	12
F3	6	5	4	7	7	14
Demand	4	4	6	8	8	

Maximization

	1	2	3	4	Supply
A	6	6	11	15	80
B	4	6	10	12	120
C	6	4	7	6	150
D	4	10	14	14	70
E	8	8	9	9	90
Demand	100	200	80	80	