Taylor's Theorem and Extreme Values

Definition 1: Local (Relative) Maximum Value:
Suppose $f(x,y)$ is defined on region R. (a,b) is a point in R and in domain of $f(x,y)$. $f(a,b)$ is called a local (relative) maximum value of function $f(x,y)$ if there exists some neighbourhood of (a,b) such that for every point $(a+h,b+k)$ of this neighbourhood $f(a,b) \geq f(a+h,b+k)$.
The point (a,b) is called Local (Relative) Maximum point.

Definition 2: Local (Relative) Minimum Value:
Suppose $f(x,y)$ is defined on region R. (a,b) is a point in R and in domain of $f(x,y)$. $f(a,b)$ is called a local (relative) minimum value of function $f(x,y)$ if there exists some neighbourhood of (a,b) such that for every point $(a+h,b+k)$ of this neighbourhood $f(a,b) \leq f(a+h,b+k)$.
The point (a,b) is called Local (Relative) Minimum point.

Definition 3:Local (Relative) extreme Value:
$f(a,b)$ is said to be a local (relative) extreme value of the function $f(x,y)$ if it is either a local (relative) maximum or local (relative) minimum value.

First Derivative Test:(Necessary condition for extremum):
If $f(x,y)$ has a local maximum or minimum value at an interior point (a,b) of its domain and if the first partial derivatives $f_x(x,y)$ and $f_y(x,y)$ exists in a neighbourhood of (a,b) then $f_x(a,b)=0$ and $f_y(a,b)=0$.

Remark:
The converse of the above theorem is not true.
For example: Consider $f(x,y)=x^2-y^2$.
Here $f_x(x,y)=2x, f_y(x,y)=-2y$.
Take the point $(a,b)=(0,0)$. Then $f_x(0,0)=0$ and $f_y(0,0)=0$.
This shows that both the first order partial derivatives at $(0,0)$ vanish but f has neither maxima nor minima at $(0,0)$.
Consider any neighbourhood of $(0,0)$ for small values of h both $(2h,h)$ and $(h,2h)$ points are in neighbourhood of $(0,0)$ and we have $f(2h,h)>0, f(h,2h)<0 \rightarrow f$ has neither maxima nor minima at $(0,0)$.
Thus, it is clear that vanishing of the first order partial derivatives is a necessary condition but not sufficient condition.

Definition 4: Critical Point or Stationary Point:
A point (a,b) is said to be a critical point or a stationary point of a function $f(x,y)$ if $f_x(a,b)=0=f_y(a,b)$.

Definition 5: Saddle Point:
A point (a,b) is said to be saddle point of a function f if in every neighbourhood of (a,b) there are points (x,y) for which $f(x,y)<f(a,b)$.

Second Derivative Test For Extrema:

Suppose $f(x,y)$ is a function of two variables x and y defined in region R such that its first and second order partial derivatives are continuous in some neighbourhood of (a,b) of the region R and $f_x(a,b) = 0 = f_y(a,b)$ then

(i) f has local maximum at (a,b) if
$f_{xx}(a,b) < 0, f_{xx}(a,b) \cdot f_{yy}(a,b) - f_{xy}^2(a,b) > 0.$

(ii) f has local minimum at (a,b) if
$f_{xx}(a,b) > 0, f_{xx}(a,b) \cdot f_{yy}(a,b) - f_{xy}^2(a,b) > 0.$

(iii) f has saddle point at (a,b) if
$f_{xx}(a,b) \cdot f_{yy}(a,b) - f_{xy}^2(a,b) < 0.$

(iv) Test is inconclusive at (a,b) if
$f_{xx}(a,b) \cdot f_{yy}(a,b) - f_{xy}^2(a,b) = 0.$

The expression $f_{xx} \cdot f_{yy} - f_{xy}^2$ is called discriminant of f and

$$f_{xx} \cdot f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

Example 1 Find extreme values of the function
$f(x,y) = 2(x^2 - y^2) - x^4 + y^4$.
Solution:
$f_x = 4x - 4x^3, f_{xx} = 4 - 12x^2,$
$f_y = -4y + 4y^3, f_{yy} = -4 + 12y^2, f_{xy} = 0$
For extremum we have
$f_x = 0, f_y = 0.$
$\therefore 4x - 4x^3 = 0$ and $-4y + 4y^3 = 0$
$\therefore 4x(1 - x^2) = 0$ and $4y(-1 + y^2) = 0$
$\Rightarrow x = 0, 1 - x^2 = 0$ and $y = 0, -1 + y^2 = 0$
$\Rightarrow x = 0, x = \pm 1$ and $y = 0, y = \pm 1$.
So we have 9 possibilities that is $(0,0), (0,1), (0,-1), (1,0), (-1,0), (-1,1), (-1,-1)$.
Now at point (0,0)
$f_{xx}(0,0) = 4, f_{yy}(0,0) = -4, f_{x,y}(0,0) = 0$
$f_{xx} \cdot f_{yy} - f_{xy}^2 = (4)(-4) - 0 = -16 < 0$
So $(0,0)$ is a saddle point for the function f.
Now at point (0,1)
$f_{xx}(0,1) = 4 > 0, f_{yy}(0,1) = 8, f_{x,y}(0,1) = 0$
$f_{xx} \cdot f_{yy} - f_{xy}^2 = (4)(8) - 0 = 32 > 0$
So $(0,1)$ is a local minimum for the function f.
Now at point (0,-1)
$f_{xx}(0,-1) = 4 > 0, f_{yy}(0,-1) = 8, f_{x,y}(0,-1) = 0$
$f_{xx} \cdot f_{yy} - f_{xy}^2 = (4)(8) - 0 = 32 > 0$
So $(0,-1)$ is a local minimum for the function f.
Now at point (1,0)
$f_{xx}(1,0) = -8 < 0, f_{yy}(1,0) = -4, f_{x,y}(1,0) = 0$
$f_{xx} \cdot f_{yy} - f_{xy}^2 = (-8)(-4) - 0 = 32 > 0$
So $(1,0)$ is a local maximum for the function f.
Now at point (-1,0)
$f_{xx}(-1,0) = -8 < 0, f_{yy}(-1,0) = -4, f_{x,y}(-1,0) = 0$
$f_{xx} \cdot f_{yy} - f_{xy}^2 = (-8)(-4) - 0 = 32 > 0$
So $(-1,0)$ is a local maximum for the function f.
Now at point (1,1)
$f_{xx}(1,1) = -8 < 0, f_{yy}(1,1) = 8, f_{x,y}(1,1) = 0$
$f_{xx} \cdot f_{yy} - f_{xy}^2 = (-8)(8) - 0 = -64 < 0$
So $(1,1)$ is a saddle point for the function f.
Now at point (-1,-1)
$f_{xx}(-1,-1) = -8 < 0, f_{yy}(-1,-1) = 8, f_{x,y}(-1,-1) = 0$
$f_{xx} \cdot f_{yy} - f_{xy}^2 = (-8)(8) - 0 = -64 < 0$
So $(-1,-1)$ is a saddle point for the function f.
Now at point (1,-1)
$f_{xx}(1,-1) = -8 < 0, f_{yy}(1,-1) = 8, f_{x,y}(1,-1) = 0$
$f_{xx} \cdot f_{yy} - f_{xy}^2 = (-8)(8) - 0 = -64 < 0$
So $(1,-1)$ is a saddle point for the function f.
Now at point (-1,1)

$f_{xx}(-1,1) = -8 < 0, f_{yy}(-1,1) = 8, f_{x,y}(-1,1) = 0$
$f_{xx} \cdot f_{yy} - f_{xy}^2 = (-8)(8) - 0 = -64 < 0$
So $(-1,1)$ is a saddle point for the function f.

Example 2 Investigate the maximum and minimum values of
$f(x,y) = (x+y-1)(x^2+y^2)$.
Solution:
$f_x = x^2 + y^2 + 2x(x+y-1)$,
$f_y = x^2 + y^2 + 2y(x+y-1)$,
$f_{xx} = 6x + 2y - 2$,
$f_{yy} = 2x + 6y - 2$,
$f_{xy} = 2y + 2x$.
For extremum, we have $f_x = 0 = f_y$
$\therefore x^2 + y^2 + 2x(x+y-1) = 0...(i)$ and
$x^2 + y^2 + 2y(x+y-1) = 0...(ii)$
Subtracting (i) and (ii), we get $(x+y-1)(x-y) = 0$.
$\Rightarrow x = y$ or $x = 1 - y$.

Case (1): With $x = y$ (i) becomes
$x^2 + x^2 + 2x(x + x - 1 = 0)$
$\therefore 6x^2 - 2x = 0$
i.e. $2x(3x-1) = 0$
$\Rightarrow x = 0$ or $x = \frac{1}{3}$
As $x = y$, we get the points as $(0,0)$ and $(\frac{1}{3}, \frac{1}{3})$.

Case (2): With $x = 1 - y$ in (i), we get $1 - 2y + 2y^2 = 0$
This has imaginary roots.
$\therefore$ The stationary points are $(0,0)$ and $(\frac{1}{3}, \frac{1}{3})$.
At point $(0,0)$,
$f_{xx} = -2 < 0, f_{yy} = -2, f_{xy} = 0$
$\therefore f_{xx} \cdot f_{yy} - f_{xy}^2 = (-2)(-2) = 4 > 0$
$\Rightarrow f$ has maximum at $(0,0)$ and $f(0,0) = 0$.
At point $(\frac{1}{3}, \frac{1}{3})$,
$f_{xx} = \frac{2}{3} > 0, f_{yy} = \frac{2}{3}, f_{xy} = \frac{4}{3}$
$\therefore f_{xx} \cdot f_{yy} - f_{xy}^2 = (\frac{2}{3})(\frac{2}{3}) - (\frac{16}{9}) = \frac{-4}{3} < 0$
$\Rightarrow f$ has saddle point at $(\frac{1}{3}, \frac{1}{3})$.

Example 3 Find extreme value of a function.
$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$.
Solution:
$f_x = y - 2x - 2, f_y = x - 2y - 2, f_{xx} = -2 < 0, f_{yy} = -2, f_{xy} = 1$
For extremum, we have $f_x = 0 = f_y$
$\therefore y - 2x - 2 = 0$ and $x - 2y - 2 = 0$ solving these for x and y we get $x = y = -2$.
$\therefore$ The point $(-2, -2)$ is the only point where f may have extreme values.
Now $f_{xx} \cdot f_{yy} - f_{xy}^2$ at $(-2, -2) = (-2)(-2) - 1^2 = 3 > 0$.
$\Rightarrow f$ has local maximum at $(-2, -2)$, and $f(-2, -2) = 8$.

Example 4 Find and classify the extreme points of the function
$f(x, y) = x^4 - 3x^2y + y^3$.
Solution:
$f_x = 4x^3 - 6xy$,
$f_y = -3x^2 + 3y^2$,
$f_{xx} = 12x^2 - 6y$,
For extremum, we have $f_x = 0 = f_y \therefore 4x^3 - 6xy = 0, 3x^2 + 3y^2 = 02x(2x^2 - 3y) = 0$,
$\therefore y = \pm x \Rightarrow x = 0$ or $2x^2 = 3y$
$\therefore x = \frac{3}{2}$ or $x = \frac{-3}{2}$. $\therefore$ The critical points are $(\frac{3}{2}, \frac{3}{2}), (\frac{-3}{2}, \frac{3}{2})$ and $(0, 0)$.

At point $(\frac{3}{2}, \frac{3}{2})$,
$f_{xx} = 12(\frac{3}{2})^2 - 6(\frac{3}{2}) = 18 > 0$,
$f_{yy} = 6(\frac{3}{2}) = 9$,
$f_{xy} = -6(\frac{3}{2}) = -9$
$\therefore f_{xx} \cdot f_{yy} - f_{xy}^2 = (18)(9) - (-9)^2 = 81 > 0$.
$\Rightarrow f$ has local minimum at $(\frac{3}{2}, \frac{3}{2})$.
Similarly f has also local minimum at $(\frac{-3}{2}, \frac{3}{2})$

At point $(\frac{-3}{2}, \frac{3}{2})$,
$f_{xx} = 12(\frac{-3}{2})^2 - 6(\frac{3}{2}) = 18 > 0$,
$f_{yy} = 6(\frac{3}{2}) = 9$,
$f_{xy} = -6(\frac{-3}{2}) = 9$
$\therefore f_{xx} \cdot f_{yy} - f_{xy}^2 = (18)(9) - (9)^2 = 81 > 0$. $\Rightarrow f$ has local minimum at $(\frac{-3}{2}, \frac{3}{2})$.

At point $(0, 0)$
$f_{xx} = 0, f_{yy} = 0, f_{xy} = 0$.
$\therefore f_{xx} \cdot f_{yy} - f_{xy}^2 = 0$
$\therefore$test fails.
But $f(x, x) = x^3(1 - 2x)$
$\therefore$ For $0 < x < \frac{1}{2}, f(x, x) > 0$ and for $\frac{-1}{2} < x < 0, f(x, x) < 0$.
$\Rightarrow f$ has saddle point at $(0, 0)$.

Example 5

Find extreme values of the function

$f(x,y) = xy + \frac{50}{x} + \frac{20}{y}$

Solution:

$f_x = y - \frac{50}{x^2}$,

$f_y = x - \frac{20}{y^2}$,

$f_{xx} = \frac{100}{x^3}$,

$f_{yy} = \frac{40}{y^3}$,

$f_{xy} = 1$.

For extremum, we have $f_x = 0 = f_y$

$\therefore y - \frac{50}{x^2} = 0$ and $x - \frac{20}{y^2} = 0$.

$\Rightarrow y = \frac{50}{x^2}$ with this $x - \frac{20}{y^2} = 0$ becomes $x - \frac{200x^4}{2500} = 0$.

$\Rightarrow x(1 - \frac{x^3}{2500}) = 0$

$\Rightarrow x = 0$ or $(1 - \frac{x^3}{2500}) = 0$.

$\therefore x = 5$.

Putting this in $y - \frac{50}{x^2} = 0$ gives $y = 2$.

$\therefore (5,2)$ is the only point where f take extreme value.

$f_{xx}(5,2) = \frac{4}{5} > 0$,

$f_{yy}(5,2) = 5$,

$f_{xy}(5,2) = 1$.

$\therefore f_{xx} \cdot f_{yy} - f_{xy}^2 = (\frac{4}{5})(5) - (1)^2 = 3 > 0$

$\Rightarrow f$ has minimum at $(5,2)$ and $f(5,2) = 30$

Example 6

Find extreme values of the function

$f(x,y) = 3x^2(y-1) + y^2(y-3) + 1$.

Solution:

$f_x = 6x(y-1)$,

$f_y = 3(x^2 - 2y + y^2)$

$f_{xx} = 6(y-1)$

$f_{yy} = 6(y-1)$

$f_{xy} = 6x$

For extremum, we have $f_x = 0 = f_y$

$\therefore 6x(y-1) = 0$ and $3(x^2 - 2y + y^2) = 0$

$\Rightarrow x = 0$ or $y = 1$.

When $x = 0, x^2 - 2y + y^2 = 0$

$\Rightarrow y = 0$ or $y = 2$.

When $y = 1, x^2 - 2y + y^2 = 0$

$\Rightarrow x = \pm 1$.

$\therefore$ The stationary points are $(0,0), (0,2), (1,1), (-1,1)$.

At point $(0,0)$,

$f_{xx} = -6 < 0$,

$f_{yy} = -6$

$f_{xy} = 0$
$\therefore f_{xx} \cdot f_{yy} - f_{xy}^2 = (-6)(-6) - 0 = 36 > 0.$
$\Rightarrow f$ has maximum at $(0, 0)$ and $f(0, 0) = 1$.

At point $(0, 2)$,
$f_{xx} = 6 > 0,$
$f_{yy} = 6,$
$f_{xy} = 0$
$\therefore f_{xx} \cdot f_{yy} - f_{xy}^2 = 36 - 0 = 36 > 0.$ $\Rightarrow f$ has minimum at $(0, 2)$ and $f(0, 2) = -3$.

At point $(1, 1)$
$f_{xx} = 0, f_{yy} = 0, f_{xy} = 6.$
$\therefore f_{xx} \cdot f_{yy} - f_{xy}^2 = 0 - 36 = -36 < 0$
$\Rightarrow f$ has saddle point at $(1, 1)$.

At point $(-1, 1)$
$f_{xx} = 0, f_{yy} = 0, f_{xy} = -6.$
$\therefore f_{xx} \cdot f_{yy} - f_{xy}^2 = 0 - 36 = -36 < 0$
$\Rightarrow f$ has saddle point at $(-1, 1)$.

Example 7
A rectangular box open at the top is to have a volume of $32m^3$. What must be the dimensions so that the total surface area is minimum?
Solution
Let the length, breadth and height of the rectangular box be x, y, z respectively, with surface S and volume V.
Here, $V = 32m^3 \Rightarrow xyz = 32...(i)$
We want to minimize the surface area of the rectangular box.
But surface area $= S$ is given by $S = 2z(x + y) + xy$.
But from (i), $z = \frac{32}{xy}$
$\therefore S = xy + 64(\frac{1}{x} + \frac{1}{y}) = f(x, y) say,$
Now $S_x = y - \frac{64}{x^2}, S_y = x - \frac{64}{y^2}, S_{xx} = \frac{128}{x^3}, S_{yy} = \frac{128}{y^3}, S_{xy} = 1.$
For extremum, $s_x = 0 = S_y$
$y - \frac{64}{x^2} = 0$ and $x - \frac{64}{y^2} = 0$
$\therefore x^2 y = 64$ and $y^2 x = 64$
$\Rightarrow x^2 y = y^2 x \Rightarrow x = y \therefore y - \frac{64}{x^2} = 0$
$\Rightarrow x - \frac{64}{x^2} = 0 \Rightarrow x^3 = 64 \Rightarrow x = 4$ and hence $y = 4$.
$\therefore$ The point $(4, 4)$ is only point at which S may take extreme value.
At point $(4, 4)$,
$\frac{128}{64} = 2 > 0, S_{yy}(4, 4) = 2, S_{xy} = 1.$
$S_{xx} \cdot S_{yy} - S_{xy}^2 = 2(2) - 1^2 = 3 > 0$
$\Rightarrow S$ has minimum at $(4, 4)$.
We have $V = xyz = 32 \therefore (4)(4)z = 32 \Rightarrow z = 2.$
$\therefore$ At $(4, 4, 2)$, S has minimum value.

$\therefore (S)_{min} = 2z(x+y) + xy$
$= 2(2)(4+4) + (4)(4)$
$(S)_{min} = 32 + 16 = 48$
Hence, Length $= 4m$, breadth $= 4m$, Height $= 2m$.

In example (4), we have obtained the minimum of the function $x^4 - 3x^2y + y^3$ and in example (7), we have found the minimum of the function $2z(x+y) + xy$ subject to the condition $xyz = 32$. Here we observe that these two problems are of different types. example (4) is a problem of free extrema where as example (7) we have an additional condition called constraint or side condition i.e. problem is of constrained extrema.
To solve example (7) we have obtained the function S in terms of two variables x and y by replacing the value of z from the side condition. Another method to solve the problems of constrained extrema is given by 'Lagrange'. The method is known as 'Lagrange's method of multipliers.

4.2: Lagrange's Method of undetermined multiplier(s) :

M-(1): Let $f(x, y, z)$ be a function of three variables x, y, z which is to be examplained for extremum and let the variables x, y, z are connected by the relation $\phi(x, y, z) = 0 ...(1)$
Since $f(x, y, z)$ is to have extremum
$\therefore \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0,$
so that $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 ...(2)$
Differentiating the relation (1) we have
$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0 ...(3)$
Multiply equation (3) by a parameter λ and adding in equation (2) we get
$(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x})dx + (\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y})dy + (\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z})dz = 0.$
This equation will be satisfied identically if coefficients of dx, dy, dz are 0.
i.e. if
$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 ...(4)$
$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 ...(5)$
$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 ...(6)$
The equation (1),(4),(5) and (6) will determine the values of x, y, z and λ for which $f(x, y, z)$ stationary.

Example 1. Divide the number 36 into three parts so that continued product of the first, square of second and cube of third may be maximum.
Solution :
Let the numbers be x, y, z respectively.
and $f(x, y, z) = xy^2z^3$ and $g(x, y, z) = x + y + z = 36$ Construct the auxiliary function F as
$F = f(x, y, z) + \lambda g(x, y, z)$
$F = xy^2z^3 + \lambda(x + y + z - 36)$
Differentiating F partially w.r.t. x, y, z and λ, and then equating to 0, we get
$F_x = y^2z^3 + \lambda = 0 ...(1)$
$F_y = 2xyz^3 + \lambda = 0 ...(2)$
$F_z = 3xy^2z^2 + \lambda = 0 ...(3)$
$F_\lambda = x + y + z - 36 = 0 ...(4)$

Now multiply equation (1) by x, (2) by y, (3) by z and adding,we get
$6xy^2z^3 + \lambda(x + y + z) = 0$
$\therefore 6xy^2z^3 + 36\lambda = 0$
$\Rightarrow \lambda = \frac{-xy^2z^3}{6}$
Putting this value in
$xy^2z^3 + \lambda x = 0$
$\therefore xy^2z^3 - \frac{xy^2z^3}{6} \cdot x = 0$
$\therefore xy^2z^3(1 - \frac{x}{6}) = 0$
$\Rightarrow 1 - \frac{x}{6} = 0 \because xy^2z^3 \neq 0$
$\Rightarrow x = 6.$
Similarly putting the values of λ in $2xy^2z^3 + \lambda y = 0$ and $3xy^2z^3 + \lambda z = 0$ respectively we get $y = 12$ and $z = 18$.

The three numbers are $6, 12, \&18$.
and $f(6, 12, 18) = 6(12)^2(18)^3 = 5038848$.

Example 2: Obtain the shortest distance of the point $(1, 2, -3)$ from the plane $2x - 3y + 6z = 20$.
Solution :
Suppose $A(1, 2, -3)$ and let $p(x, y, z)$ be any point on the plane (say) $\phi(x, y, z) = 2x - 3y + 6z - 20 = 0$.
The distance $= d^2 = Ap = (x-1)^2 + (y-2)^2 + (z+3)^2 \equiv f(x, y, z)$. Which is to be minimize.
Construct the auxiliary function
$F = f(x, y, z) + \lambda\phi(x, y, z)$
$\therefore F = (x-1)^2 + (y-2)^2 + (z+3)^2 + \lambda(2x - 3y + 6z - 20)$.
Differentiating F w.r.t. x, y, z and λ, equate to zero
$F_x = 2(x-1) - 2\lambda = 0...(1)$
$F_y = 2(y-2) - 3\lambda = 0...(2)$
$F_z = 2(z+3) + 6\lambda = 0...(3)$
$F_\lambda = 2x - 3y + 6z - 20 = 0$

Multiply equation (1) by $(x-1)$, (2) by $(y-2)$, (3) by $(z+3)$ we get
$2(x-1)^2 + 2\lambda(x-1) = 0...(5)$
$2(y-2)^2 - 3\lambda(y-2) = 0...(6)$
$2(z+3)^2 + 6\lambda(z+3) = 0...(7)$
Adding (5), (6) and (7) we get $2[(x-1)^2 + (y-2)^2 + (z+3)^2] + \lambda(2x - 3y + 6z) + 22\lambda = 0$
$\therefore 2d^2 + 42\lambda = 0 \Rightarrow \lambda = \frac{-d^2}{14}, z + 3 = \frac{d^2}{7}$
Taking value of λ in equation (1), (2) and (3) we get
$x - 1 = \frac{d^2}{21}, y - 2 = \frac{-d^2}{14}$
$\therefore d^2 = (\frac{d^2}{21})^2 + (\frac{-d^2}{14})^2 + (\frac{d^2}{7})^2$
$\Rightarrow d = 6$.
$\therefore$ The shortest distance is 6 unit.

Example 3: Show that the greatest value of $8xyz$ under the condition
$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{4} = 1$ is $\frac{64}{\sqrt{3}}$

Solution :
Let $f(x, y, z) = 8xyz, g(x, y, z) = \frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{4} - 1 = 0$.
We construct the auxiliary function F as $F = 8xyz + \lambda(\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{4} - 1)$
Differentiate F partially w.r.t. x, y, z and λ, then equating to zero, we get
$F_x = 8yz + \frac{2\lambda x}{9} = 0...(1)$
$F_y = 8xz + \frac{2\lambda y}{16} = 0...(2)$
$F_z = 8xy + \frac{2\lambda z}{4} = 0...(3)$

Multiply equation (1) by x, (2) by y, (3) by z we get
$F_\lambda = \frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{4} - 1 = 0...(4)$

$8xyz + \frac{2\lambda x^2}{9} = 0...(5)$
$8xyz + \frac{2\lambda y^2}{16} = 0...(6)$
$8xyz + \frac{2\lambda z^2}{4} = 0...(7)$

Adding these equations we get
$24xyz + 2\lambda(\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{4}) = 0$
$\Rightarrow 24xyz + 2\lambda = 0 \Rightarrow \lambda - 12xyz.$
Putting the value of λ in equation (5), we get $8xyz + \frac{2(-12xyz)\cdot x^2}{9} = 0$
$\Rightarrow 8xyz(1 - \frac{3x^2}{9}) = 0$
$\Rightarrow 9 - 3x^2 = 0 \because xyz \neq 0$
$\Rightarrow x = \sqrt{3}.$

Similarly putting the value of λ in equation (6) and (7) respectively, we get
$y = \frac{4}{\sqrt{3}}$ and $z = \frac{2}{\sqrt{3}}$.
$\therefore$ The point $(\sqrt{3}, \frac{4}{\sqrt{3}}, \frac{2}{\sqrt{3}})$ is the stationary point.
$\therefore$ Maximum value of xyz is $8(\sqrt{3})(\frac{4}{\sqrt{3}})(\frac{2}{\sqrt{3}}) = \frac{64}{3}$

Example 4:
Find the greatest and smallest values of the function $f(x, y) = xy$ takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$.
Solution :
We have $f(x, y) = xy$, Suppose $g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$.
$\nabla f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j}$
$\nabla g = \frac{\partial g}{\partial x}\bar{i} + \frac{\partial g}{\partial y}\bar{j}$
$\nabla f = y\bar{i} + x\bar{j}$ and $\nabla g = \frac{2x}{8}\bar{i} + \frac{2y}{2}\bar{j} = \frac{x}{4}\bar{i} + y\bar{j}$.
Now consider
$\nabla f = \lambda \nabla g$
$\therefore y\bar{i} + x\bar{j} = \lambda(\frac{x}{4}\bar{i} + y\bar{j})$
$\therefore y\bar{i} + x\bar{j} = \lambda\frac{x}{4}\bar{i} + \lambda y\bar{j}$
$\Rightarrow y = \frac{\lambda}{4}x$ and $x = \lambda y$
$\therefore y = \frac{\lambda}{4}\lambda y \Rightarrow y(\frac{\lambda^2}{4} - 1) = 0$
$\therefore y = 0$ or $(\frac{\lambda^2}{4} - 1) = 0$
$\Rightarrow \lambda = \pm 2$

Case 1: If $y = 0$ then $x = 0 \therefore$ we get the point $(0, 0)$. But $(0, 0)$ is not on the given ellipse. $\therefore y \neq 0$.

Case 2: If $y \neq 0$ then $\lambda = \pm 2 \therefore x = \pm 2y$ with this $g(x, y) = 0$
gives $\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1 \Rightarrow 4y^2 + 4y^2 = 8 \Rightarrow y = \pm 1$.
$\therefore$ The critical points are $(\pm 2, 1)$ and $(\pm 2, -1)$ The greatest value of function $f(x, y) = xy = 2$. and the smallest value of function $f(x, y) = xy = -2$.

Example 5: Find the extreme value of the function $f(x,y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

Solution:

We have $f(x,y) = 3x + 4y$. Suppose $g(x,y) = x^2 + y^2 - 1 = 0$

$\nabla f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j}$

$\nabla f = 3\bar{i} + 4\bar{j}$ and $\nabla g = 2x\bar{i} + 2y\bar{j}$

Consider

$\nabla f = \lambda \nabla g$

$3\bar{i} + 4\bar{j} = \lambda(2x\bar{i} + 2y\bar{j})$

$\Rightarrow 2x\lambda = 3$ and $2y\lambda = 4$

Since $\lambda \neq 0, x = \frac{3}{2\lambda}$ and $y = \frac{2}{\lambda}$

With this, $g(x,y) = 0$ becomes

$(\frac{3}{2\lambda})^2 + (\frac{2}{\lambda})^2 - 1 = 0 \Rightarrow 4\lambda^2 = 25\lambda = \pm\frac{5}{2}$.

$\therefore x = \pm\frac{3}{5}$ and $y = \pm\frac{4}{5}$

$\therefore$ The stationary points are $(\pm\frac{3}{5}, \pm\frac{4}{5})$.

The extreme values of $f(x,y) = 3x + 4y$ are 5 and -5.

Example 6: Find the extreme values of $f(x,y,z) = x - 2y + 5z$ on $x^2 + y^2 + z^2 = 30$.

Solution:

We have $f(x,y,z) = x - 2y + 5z$.

Suppose $g(x,y,z) = x^2 + y^2 + z^2 - 30 = 0$

$\nabla f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k}$ and $\nabla g = \frac{\partial g}{\partial x}\bar{i} + \frac{\partial g}{\partial y}\bar{j} + \frac{\partial g}{\partial z}\bar{k}$

$\nabla f = \bar{i} - 2\bar{j} + 5\bar{k}$ and $\nabla g = 2x\bar{i} - 2y\bar{j} + 2z\bar{k}$.

Consider

$\nabla f = \lambda \nabla g$

$\therefore \bar{i} - 2\bar{j} + 5\bar{k} = \lambda(2x\bar{i} - 2y\bar{j} + 2z\bar{k})$

$\Rightarrow 2x\lambda = 1, 2y\lambda = -2, 2z\lambda = 5$

$x = \frac{1}{2\lambda}, y = \frac{-1}{\lambda}, z = \frac{5}{2\lambda}$

$\therefore g(x,y,z) = 0$ becomes $(\frac{1}{2\lambda})^2 + (\frac{-1}{\lambda})^2 + (\frac{5}{2\lambda})^2 - 30 = 0 \Rightarrow \lambda = \pm\frac{1}{2}$

Putting this value of λ in x, y, z we get $x = \pm 1, y = \pm 2, z = \pm 5$.

$\therefore$ The stationary point is $(x,y,z) = (\pm 1, \pm 2, \pm 5)$. So that the extreme values of function $f(x,y,z)$ are 22 and -20.

Taylors Formula For Functions of Two Variables

Theorem: If $f(x,y)$ and its partial derivatives of order $(n+1)$ are continuous in the neighbourhood of a point (a,b) and if $(a+h,b+k)$ is any point in this neighbourhood then there exists a positive number c, $0<c<1$ such that

$$f(a+h,b+k)=f(a,b)+(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})f(a,b)+\frac{1}{2!}(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^2f(a,b)+...+\frac{1}{n!}(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^nf(a,b)+\frac{1}{(n+1)!}(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^{n+1}f(a+ch,b+ck)$$

Proof: :Let $x=a+ht, y=b+kt$; where $0\leq t\leq 1$ is a parameter

$\therefore f(x,y)=f(a+ht,b+kt)=F(t)$.

Since $f(x,y)$ possesses continuous partial derivatives of order $n+1$ in any neighbourhood of point (a,b), $F(t)$ is continuous in $[0,1]$ and

$$F'(t)=\frac{\partial f}{\partial x}.\frac{dx}{dt}+\frac{\partial f}{\partial y}.\frac{dy}{dt}$$

$$F'(t)=h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}=(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})f$$

$$F''(t)=\frac{\partial f'}{\partial x}.\frac{dx}{dt}+\frac{\partial f'}{\partial y}.\frac{dy}{dt}$$

$$=\frac{\partial}{\partial x}(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}).h+\frac{\partial}{\partial y}(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}).k$$

$$=(h\frac{\partial^2 f}{\partial x^2}+k\frac{\partial^2 f}{\partial x}\partial y)h+(h\frac{\partial^2 f}{\partial y\partial x}+k\frac{\partial^2 f}{\partial y^2})k$$

$$=h^2\frac{\partial^2}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\partial y}+k^2\frac{\partial^2}{\partial y^2}$$

$$=(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^2f$$

Continuing in this way we have $F^{n+1}(t)=(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^{n+1}f$

By Maclaurin's theorem, we have

$F(1)=F(0)+F'(0)+\frac{1}{2!}F''(0)+...+\frac{1}{n!}F^n(0)+\frac{1}{(n+1)!}F^{n+1}(c).....(1)$

But $F(1)=f(a+h,b+k)$

$F(0)=f(a,b)$

$$F'(0)=(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})f(a,b)$$

$$F''(0)=(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^2f(a,b)...$$

$$F^n(0)=(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^nf(a,b)$$

$$F^{n+1}(0)=(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^{n+1}f(a+ch,b+ck)$$

Putting all these values in equation (1) we get

$$f(a+h,b+k)=f(a,b)+(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})f(a,b)+\frac{1}{2!}(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^2f(a,b)+...+\frac{1}{n!}(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^nf(a,b)+\frac{1}{(n+1)!}(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y})^{n+1}f(a+ch,b+ck)$$

Remark: 1.The last term is called the remainder and the theorem is called Taylor's expansion about the point (a,b)
2.Another form of Taylor's Formula is
$f(x,y) = f(a,b) + [(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}]f(a,b) + \frac{1}{2!}[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}]^2 f(a,b) + ... + \frac{1}{n!}[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}]^n f(a,b) + \frac{1}{(n+1)!}[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}]^{n+1} f(a+c(x-a), b+c(y-b))$
This is called Taylor's expansion of $f(x,y)$ about the point (a,b) in the powers of $(x-a),(y-b)$.
3.If $a=0, b=0$ and h,k are independent variables that is $h=x, k=y$ then we get
$f(x,y) = f(0,0)+(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y})f(0,0)+\frac{1}{2!}(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y})^2 f(0,0)+...+\frac{1}{n!}(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y})^n f(0,0)+ \frac{1}{(n+1)!}(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})^{n+1} f(cx,cy)$
This is called Maclaurin's expansion.

Examples

1. Expand $f(x,y) = x^3 + xy^2$ in the powers of $(x-2)$ and $(y-1)$.
Solution: We have $f(x,y) = x^3 + xy^2, a = 2, b = 1$
$\therefore f(2,1) = 10.$
$f_x = 3x^2 + y^2 \therefore f_x(2,1) = 13, f_y = 2xy, f_y(2,1) = 4$
$f_{xx} = 6x, f_{xx}(2,1) = 12, f_{yy} = 2x, f_{yy}(2,1) = 4$
$f_{xy} = 2y, f_{xy}(2,1) = 2, f_{xxx} = 6, f_{xxx}(2,1) = 6, f_{yyy} = 0 = f_{yyy}(2,1)$
$f_{xyy} = 2, f_{xyy}(2,1) = 2, f_{xxy} = 0, f_{xxy}(2,1) = 0$
By Taylor's formula
$f(a+h, b+k) = f(a,b) + (h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y})f(a,b) + \dfrac{1}{2!}(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y})^2 f(a,b) + ... + \dfrac{1}{n!}(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y})^n f(a,b) + \dfrac{1}{(n+1)!}(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y})^{n+1} f(a+ch, b+ck)$
putting all th values in this , we have
$x^3 + xy^2 = 10 + 13(x-2) + 4(y-1) + \frac{1}{2}[12(x-2)^2 + 4(x-2)(y-1) + 4(y-1)^2] + \frac{1}{6}[6(x-2)^3 + 2(x-2)(y-1)^2]$

2. Expand $f(x,y) = sinxy$ in the powers of $(x-1)$ and $(y - \frac{\pi}{2})$ upto second degree terms.
Solution: Here $f(x,y) = sinxy, a = 1, b = \frac{pi}{2}$
$\therefore f(1.\frac{\pi}{2}) = 1$
$f_x = ycosxy \qquad \therefore f_x(1, \frac{\pi}{2}) = 0$
$f_y = xcosxy \qquad \therefore f_y(1, \frac{\pi}{2}) = 0$
$f_{xx} = -y^2 sinxy \qquad \therefore f_{xx}(1, \frac{\pi}{2}) = \frac{-\pi^2}{4}$
$f_{yy} = -x^2 sinxy \qquad \therefore f_{xx}(1, \frac{\pi}{2}) = -1$
$f_{xy} = -xysinxy + cosxy, \qquad \therefore f_{xy}(1, \frac{\pi}{2}) = \frac{-\pi}{2}$
Now by Taylor's formula
$f(a+h, b+k) = f(a,b) + (h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y})f(a,b) + \dfrac{1}{2!}(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y})^2 f(a,b)$
$\therefore sinxy = 1 + \frac{1}{2!}[(x-1)^2(\pi^2/4) + 2(x-1)(y-\pi/2)(-\pi/2) + (y-\pi/2)^2(-1)]$
$\therefore sinxy \approx 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)(y-\pi/2) - \frac{1}{2}(y-\pi/2)^2.$

3. Find the cubic approximation for $f(x,y) = x^y$ near the point $(1,1)$
Solution: Here $f(x,y) = x^y, a = 1, b = 1, \therefore f(1,1) = 1$
$f_x = yx^{y-1} \qquad \therefore f_x(1,1) = 1$
$f_y = x^y \log x \qquad \therefore f_y(1,1) = 0$
$f_{xx} = y(y-1)x^{y-2} \qquad \therefore f_{xx}(1,1) = 0$
$f_{yy} = \log x.x^y.logx \qquad \therefore f_{yy}(1,1) = 0$
$f_{xy} = x^y \frac{1}{x} + \log x.y.x^{y-1} \qquad \therefore f_{xy}(1,1) = 1$
$f_{xxx} = y(y-1)(y-2)x^{y-3} \qquad \therefore f_{xxx}(1,1) = 0$
$f_{yyy} = (\log x)^2 x^y \log x \qquad \therefore f_{yyy}(1,1) = 0$
$f_{xxy} = (y^2 - y)x^{y-2}\log x + x^{y-2}(2y-1) \qquad \therefore f_{xxy}(1,1) = 1$
$f_{xyy} = x^{y-1}\log x + \log x(-y.x^{y-1}logx + x^{y-1}) \qquad \therefore f_{xyy} = 0$

Putting all this values in Taylor's formula, we have

$f(x,y) = x^y = 1 + (x-1).1 + (y-1).0 + \frac{1}{2}[(x-1)^2.0 + 2(x-1)(y-1).1 + (y-1)^2.0] +$
$\frac{1}{6}[(x-1)^3.0 + 3(x-1)^2(y-1).1 + 3(x-1)(y-1)^2.0 + (y-1)^3.0]$
$x^y \approx 1 + (x-1) + (x-1)(y-1) + \frac{1}{2}(x-1)^2(y-1)$