Differentiability

Introduction:

In case of function of one variable, we know that if $y = f(x)$ is a function of one variable x then we say that the function f is differentiable at $x = x_0$ if the increment or change in f from x to $x_0 + \triangle x$.
$\triangle y = f(x_0 + \triangle x) - f(x_0)$ is expressed as
$\triangle y = f'(x_0)\triangle x + \epsilon_1 \triangle x$; where as $\triangle x \to 0, \epsilon_1 \to 0$.
Here, $f'(x_0)$ is called the differential (total) of function f. It is denoted by df.
Thus, df = differential of $f = f'(x_0)h$.
Now we shall extend this concept for the function of two variables.
Suppose $f(x, y)$ is a function of two variables x and y. Let (x_0, y_0) be a point in the domain $\mathbb{R}^2$ of $f(x, y)$ and $(x_0 + \triangle x, y_0 + \triangle y)$ be any point in a neighbourhood of point (x_0, y_0) and in the domain of f.
The increment (or change) in the function f is the difference
$f(x_0 + \triangle x, y_0 + \triangle y) - f(x_0, y_0)$ from point (x_0, y_0) to $(x_0 + \triangle x, y_0 + \triangle y)$.
This is denoted by $(\triangle)f(x_0, y_0)$ or $\triangle f$.
Thus $\triangle f(x_0, y_0) = f(x_0 + \triangle x, y_0 + \triangle y) - f(x_0, y_0)$

Example:

If $f(x,y) = x^2y$

$\triangle f(x_0, y_0) = f(x_0 + \triangle x, y_0 + \triangle y) - f(x_0, y_0)$

$= (x_0 + \triangle x)^2(y_0 + \triangle y) - x_0^2 y_0$

$\triangle f(x_0, y_0) = 2x_0y_0\triangle x + x_0^2\triangle y + y_0(\triangle x)^2 + 2x_0\triangle x\triangle y + (\triangle x)^2\triangle y ...(i)$

Now if we put $A = 2x_0y_0, B = x_0^2, \epsilon_1 = y_0\triangle x + x_0\triangle y$ and $\epsilon_2 = x_0\triangle x + (\triangle x)^2$ then expression (i) can be written as

$\triangle f(x_0, y_0) = A\triangle x + B\triangle y + \epsilon_1\triangle x + \epsilon_2\triangle y ...(ii)$

where A and B are independent of $\triangle x$ and $\triangle y$, and

$\lim_{(\triangle x, \triangle y)\to(0,0)} \epsilon_1 = 0, \lim_{(\triangle x, \triangle y)\to(0,0)} \epsilon_2 = 0.$

Here, the function $f(x,y)$ is said to have a differential at point (x_0, y_0). It is denoted by df.

Thus $df = A\triangle x + B\triangle y$.

Note that when $\triangle x$ and $\triangle y$ are sufficiently small df gives a good approximation of $\triangle f(x_0, y_0)$.

3.1: Definition (Differentiability)

A function $f(x,y)$ is said to be differentiable at a point (x_0,y_0) if there exists a neighbourhood $(x_0+\triangle x, y_0+\triangle y)$ of (x_0,y_0) in which the increment $\triangle f(x_0,y_0)$ can be expressed in the form

$\triangle f(x_0,y_0) = f(x_0+\triangle x, y_0+\triangle y) - f(x_0,y_0) = A\triangle x + B\triangle y + \epsilon_1\triangle x + \epsilon_2\triangle y$ where A and B are independent of $\triangle x$ and $\triangle y$, and

$\lim\limits_{(\triangle x,\triangle y)\to(0,0)} \epsilon_1 = 0, \lim\limits_{(\triangle x,\triangle y)\to(0,0)} \epsilon_2 = 0.$

Theorem 1: (Necessary conditions for differentiability :-)

Suppose $f(x,y)$ is a real valued function defined on a neighbourhood of (x_0,y_0). If $f(x,y)$ is differentiable at (x_0,y_0) then

$(i) f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ both exists

$(ii) f(x,y)$ is continuous at (x_0,y_0).

Proof :

Assume that $f(x,y)$ is differentiable at point (x_0,y_0).

(i) $\therefore$ By the definition of differentiability at (x_0,y_0)

$\triangle f(x_0,y_0) = f(x_0+\triangle x, y_0+\triangle y) - f(x_0,y_0)$

$= A\triangle x + B\triangle y + \epsilon_1\triangle x + \epsilon_2\triangle y ...(1)$

where A and B are independent of $\triangle x$ and $\triangle y$, and

$\lim\limits_{(\triangle x,\triangle y)\to(0,0)} \epsilon_1 = 0, \lim\limits_{(\triangle x,\triangle y)\to(0,0)} \epsilon_2 = 0.$

Equation (1) is true for small values of $\triangle x$ and $\triangle y$.

Put $\triangle y = 0$ in equation (1), we get

$f(x_0+\triangle x, y_0+\triangle y) - f(x_0,y_0) = A\triangle x + \epsilon_1\triangle x$

$(A+\epsilon_1)\triangle x = f(x_0+\triangle x, y_0+\triangle y) - f(x_0,y_0)$

$A+\epsilon_1 = \frac{f(x_0+\triangle x, y_0+\triangle y) - f(x_0,y_0)}{\triangle x}$

$\lim\limits_{\triangle x\to 0}[A+\epsilon_1] = \lim\limits_{\triangle x\to 0}[\frac{f(x_0+\triangle x, y_0+\triangle y) - f(x_0,y_0)}{\triangle x}]$

$\therefore A = \lim\limits_{\triangle x\to 0}(\frac{f(x_0+\triangle x, y_0+\triangle y) - f(x_0,y_0)}{\triangle x})$

$A = f_x(x_0,y_0)$.

i.e. $A = f_x(x_0,y_0)$ exist.

Similarly by putting $\triangle x = 0$ in equation (1) we get $B = f_y(x_0,y_0)$.

This proves condition (i).

(ii) Taking limit as $(\triangle x,\triangle y)\to(0,0)$ of Equation (1) we get

$\lim\limits_{(\triangle x,\triangle y)\to(0,0)}[f(x_0+\triangle x, y_0+\triangle y) - f(x_0,y_0)] = 0$

$\because$ the limit of each term on R.H.S. is 0.

$\lim\limits_{(\triangle x,\triangle y)\to(0,0)}[f(x_0+\triangle x, y_0+\triangle y)] = f(x_0,y_0)$

This shows that $f(x,y)$ is continuous at (x_0,y_0).

Remark 1: A function $f(x,y)$ is differentiable at (x_0,y_0) iff the partial derivatives $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exists and
$\triangle f = f(x_0+\triangle x, y_0+\triangle y) - f(x_0,y_0)$
$= f_x(x_0,y_0)\triangle x + f_y(x_0,y_0)\triangle y + \epsilon_1\triangle x + \epsilon y\triangle y;$
where $\epsilon_1 \to 0, \epsilon_2 \to 0$ as $(\triangle x, \triangle y) \to (0,0)$

Remark 2: The converse of the above theorem is not true i.e. above conditions are not sufficient.

Example 1: Show that the function $f(x,y) = \sqrt{|xy|}$ has first partial derivatives at the origin but it is not differentiable at the origin.
Solution : Given that $f(x,y) = \sqrt{|xy|}(x_0,y_0) = (0,0)$.
First let us find the first partial derivatives of $f(x,y)$ at the origin.
$\therefore f_x(0,0) = \lim_{\triangle x \to 0}(\frac{f(0+\triangle x,0)-f(0,0)}{\triangle x})$

$\therefore f_x(0,0) = \lim_{\triangle x \to 0}(\frac{\sqrt{|\triangle x,0|\cdot\sqrt{|0|}}}{\triangle x})$

$= \lim_{\triangle x \to 0}(\frac{0}{\triangle x}) = 0$
$f_x(0,0) = 0...(i)$
Similarly, $f_y(0,0) = 0...(ii)$
From (i) and (ii) both the first partial derivatives of $f(x,y)$ exists at $(0,0)$.
Now, suppose that f is differentiable at $(0,0)$ then by the definition of differentiability
$f(\triangle x, \triangle y) - f(0,0) = f_x(0,0)\triangle x + f_y(0,0)\triangle y + \epsilon_1\triangle x + \epsilon_2\triangle y$

$\therefore \sqrt{|\triangle x \cdot \triangle y|} - \sqrt{|0|} = 0\cdot\triangle x + 0\cdot\triangle y + \epsilon_1\triangle x + \epsilon_2\triangle y...(iii)$
$\epsilon_1 \to 0, \epsilon_2 \to 0$ as $(\triangle x, \triangle y) \to (0,0)$.
Since (iii) holds for all small values of $\triangle x$ and $\triangle y$, put $\triangle y = \triangle x$ in (iii), we get
$\sqrt{|(\triangle x)^2|} = \epsilon_1\triangle x + \epsilon_2\triangle x$
$\therefore |\triangle x| = \triangle x(\epsilon_1+\epsilon_2)$
$\therefore \frac{|\triangle x|}{\triangle x} = \epsilon_1 + \epsilon_2$
Taking limit as $\triangle x \to 0$ of both sides.
$\therefore \lim_{\triangle x\to 0}\frac{|\triangle x|}{\triangle x} = \lim_{\triangle x \to 0}\epsilon_1 + \epsilon_2$
$\therefore \pm 1 = 0$
which is absurd. Hence f is not differentiable at $(0,0)$.
Moreover,
For continuity of $f(x,y)$ at $(0,0)$. Consider
$|f(x,y) - f(0,0)| = |\sqrt{|xy|}| = \sqrt{x}\cdot\sqrt{y} \le x^2+y^2 < \epsilon$
$\because \sqrt{x} \le \sqrt{x^2+y^2}$
$\sqrt{y} \le \sqrt{x^2+y^2}$
$\Rightarrow \sqrt{x^2+y^2} < \sqrt{\epsilon}(=\delta)$ Thus, $|f(x,y)-f(0,0)| < \epsilon$ whenever $\sqrt{x^2+y^2} < \delta$.
$\Rightarrow f(x,y)$ is continuous at $(0,0)$.

Example 2: Show that the function $f(x,y) = |x|(1+y)$ is not differentiable at $(0,0)$ but is continuous at $(0,0)$.

Solution :

Given that $f(x,y) = |x|(1+y)$.

$(x_0, y_0) = (0,0) \therefore f(x_0,y_0) = f(0,0)$

$= 0$

$\lim\limits_{\triangle x \to 0} (\frac{f(x_0+\triangle x, y_0) - f(x_0,y_0)}{\triangle x}) = \lim\limits_{\triangle x \to 0} \frac{(\triangle x, 0) - f(0,0)}{\triangle x}$

$= \lim\limits_{\triangle x \to 0} \frac{|\triangle x|(1+0) - 0}{\triangle x}$

$= \lim\limits_{\triangle x \to 0} \frac{|\triangle x|}{\triangle x}$

Now

$= \lim\limits_{\triangle x \to 0^+} \frac{|\triangle x|}{\triangle x} = \lim\limits_{\triangle x \to 0^+} (\frac{\triangle x}{\triangle x}) = 1 ... (i)$

$= \lim\limits_{\triangle x \to 0^-} \frac{|\triangle x|}{\triangle x} = \lim\limits_{\triangle x \to 0^-} (\frac{-\triangle x}{\triangle x}) = -1 ... (ii)$

$\therefore \lim\limits_{\triangle x \to 0} \frac{|\triangle x|}{\triangle x}$ does not exist. ($\because$ by(i) and (ii))

i.e. $\lim\limits_{\triangle x \to 0} \frac{f(\triangle x, 0) - f(0,0)}{\triangle x}$ does not exist, which means that $f_x(0,0)$ does not exist.

Since existence of $f_x(0,0)$ and $f_y(0,0)$ is a necessary condition for differentiability, therefore f is not differentiable at $(0,0)$.

To show that $f(x,y)$ is continuous at $(0,0)$ we will use $\epsilon - \delta$ definition.

Let $\epsilon > 0$. Consider

$|f(x,y) - f(0,0)| = |f(x,y) - 0| = |x(1+y)| = |x|.|1+y| \leq 2|x|$, if $|y| < 1$

$\therefore |f(x,y) - f(0,0)| \leq 2|x| < \epsilon$

$\therefore |f(x,y) - f(0,0)| < \epsilon$, if $|x| < \frac{\epsilon}{2} = \delta$

take $\delta = min\{\frac{\epsilon}{2}, 1\}$ then $|f(x,y) - f(0,0)| < \epsilon$ when $|x| < \delta, |y| < \delta$

$\lim\limits_{\triangle x \to 0} f(x,y) = 0 = f(0,0) \Rightarrow f(x,y)$ is continuous at $(0,0)$

$\therefore \lim\limits_{\triangle x \to 0} f(x,y) = 0 = f(0,0) \Rightarrow f(x,y)$ is continuous at $(0,0)$.

Example 3: Let

$f(x,y) = \frac{2xy}{x^2+y^2}$, if $f(x,y) \neq (0,0)$

$= 0$ if $f(x,y) = (0,0)$

Show that $f(x,y)$ is not differentiable at $(0,0)$ even though $f_x(0,0)$ and $f_y(0,0)$ exists

Solution:

First let us show that $f_x(0,0) \& f_y(0,0)$ exist

$f_x(0,0) = \lim\limits_{\triangle x \to 0} \frac{f(\triangle x, 0) - f(0,0)}{\triangle x}$

$f_x(0,0) = \lim\limits_{\triangle x \to 0} \frac{0-0}{\triangle x} = 0$

Similarly $f_y(0,0) = 0$ i.e. both $f_x(0,0) \& f_y(0,0)$ exist.

Now, we will find the limit of $f(x,y)$ along a path $y = mx, m \neq 0$.

$\therefore \lim\limits_{(x,y) \to (0,0)} f(x,y) = \lim\limits_{(x,mx) \to (0,0)} f(x,mx)$

$= \lim_{x\to 0}(\frac{2x\cdot mx}{x^2+m^2x^2})$
$= \frac{2m}{1+m^2}$
which depends upon the path. i.e. $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Hence, f is not continuous at $(0,0)$.
Therefore f is not differentiable at $(0,0)$.

Example 4:
$f(x,y) = 2xy\frac{x^2-y^2}{x^2+y^2}, (x,y) \neq (0,0)$
$= 0, (x,y) = (0,0)$
Show that $f(x,y)$ is differentiable at $(0,0)$.
Solution :
$f_x(0,0) = \lim_{\triangle x\to 0} \frac{f(\triangle x,0)-f(0,0)}{\triangle x}$
$f_x(0,0) = \lim_{\triangle x\to 0} \frac{0-0}{\triangle x} = 0$
Similarly $f_y(0,0) = 0$ i.e. both $f_x(0,0)\& f_y(0,0)$ exist.
Now $\triangle f = f(x_0+\triangle x, y_0+\triangle y) - f(x_0,y_0)$
$\triangle f = f(\triangle x,\triangle y) - f(0,0)$
$\therefore f(\triangle x,\triangle y) - f(0,0) = 0\cdot\triangle x + 0.\triangle y + \epsilon_1\triangle x + \epsilon_2\triangle y$; where

$\epsilon_1 = \frac{2(\triangle x)^2\triangle y}{(\triangle x)^2+(\triangle y)^2}$, if $(\triangle x,\triangle y) \neq (0,0)$
$= 0$ if $(\triangle x,\triangle y) = (0,0)$

$\epsilon_2 = \frac{-2(\triangle x)(\triangle y)^2}{(\triangle x)^2+(\triangle y)^2}$, if $(\triangle x,\triangle y) \neq (0,0)$
$= 0$ if $(\triangle x,\triangle y) = (0,0)$

Here as $(\triangle x,\triangle y) \to (0,0), \epsilon_1 \to 0, \epsilon_2 \to 0$.
$\therefore$ $(\triangle x,\triangle y) - f(0,0) = f_x(0,0)\triangle x + f_y(0,0)\triangle y + \epsilon_1\triangle x + \epsilon_2\triangle y; \epsilon_1 \to 0, \epsilon_2 \to 0$ as $(\triangle x,\triangle y) \to (0,0)$

Hence by the definition, $f(x,y)$ is differentiable at $(0,0)$.

Theorem 3: (Sufficient Conditions for Differentiability) :
If $f(x,y)$ is a function of two variables x and y such that
$(i) f_x(a,b)$ and $f_y(a,b)$ exist
(ii) One of the first partial derivatives f_x, f_y is continuous at (a,b).
Then $f(x,y)$ is differentiable at (a,b).

Proof :
Suppose f_y is continuous at $(a,b) \Rightarrow f_y$ exist in the neighbourhood of (a,b), (say square δ neighbourhood of (a,b))
i.e.$\exists\delta > 0$ so that the point $(a+h, b+k)$ lies in the δ-neighbourhood of (a,b) where $|h| < \delta, |k| < \delta$.
Now $\triangle f = f(a+h), b+k) - f(a,b)$
$= f(a+h,b+k) - f(a+h,b) + f(a+h,b) - f(a,b)...*$
Define the function $g(y)$ as $g(y) = f(a+h,y)$
Here g is derivable in $(b, b+k)$ and we have $g'(y) = f_y(a+h,y)$.
Also g is continuous in $[b, b+k]$.
Hence by LMVT (IInd form)
$g(b+k) - g(b) = kg'(b+k\theta); 0 < \theta < 1.$
i.e. $f(a+h,b+k) - f(a+h,b) = kf_y(a+h,b+k\theta)...(1)$

Since f_y is continuous at (a,b)
$\lim\limits_{(h,k)\to(0,0)} f_y(a+h,b+k\theta) = f_y(a,b)$
$\lim\limits_{(h,k)\to(0,0)} f_y(a+h,b+k\theta) - f_y(a,b) = 0$
If we put $f_y(a+h,b+k\theta) - f_y(a,b) = \psi(h,k)$
$\lim\limits_{(h,k)\to(0,0)} \psi(h,k) = 0.$
With this equation (1) becomes,
$f(a+h,b+k) - f(a+h,b) = k(f_y(a,b) + \psi(h,k))$
$f(a+h,b+k) - f(a+h,b) = kf_y(a,b) + k\psi(h,k)...(2)$

Now, we have, $f_x(a,b) = \lim\limits_{h\to0} \frac{f(a+h,b)-f(a,b)}{h}$
$\therefore \lim\limits_{h\to0}[\frac{f(a+h,b)-f(a,b)}{h} - f_x(a,b)] = 0$
Put $\phi(h) = \frac{f(a+h,b)-f(a,b)}{h} - f_x(a,b)$ then $\lim\limits_{h\to0}\phi(h) = 0$ i.e. $\phi(h) \to 0$ as $(h,k) \to (0,0)$.
$\therefore f(a+h,b) - f(a,b) = hf_x(a,b) + h\phi(h,k)...(3)$
Putting $(2), (3)$ and (1) in $*$ we get

$\triangle f = f(a+h,b+k) - f(a,b) = hf_x(a,b) + kf_y(a,b) + h\phi(h,k) + k\psi(h,k)$; where $\phi(h,k) \to 0$ and $\psi(h,k) \to 0$ as $(h,k) \to (0,0)$.
Hence, by the definition of differentiability, $f(x,y)$ is differentiable at (a,b).

Differentials : Let $z = f(x, y)$ be a differentiable function of two variables x and y. The differential or total differential of z; denoted by dz; is defined as
$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$
where dx and dy (are called the differentials of x and y) are two new independent variables.
Suppose $z = f(x, y)$ is differentiable at (x_0, y_0). Then
$\triangle z = f(x_0 + \triangle x, y_0 + \triangle y) - f(x_0, y_0)$
$\triangle z = \frac{\partial z}{\partial x}\triangle x + \frac{\partial z}{\partial y}\triangle y + \epsilon_1 \triangle x + \epsilon_2 \triangle y;$
$\epsilon_1, \epsilon_2 \to 0$ as $(\triangle x, \triangle y) \to (0, 0)$.
For small values of $\triangle x \& \triangle y$
$\triangle z = dz + \epsilon_1 \triangle x + \epsilon_2 \triangle y$; where
$\triangle x, \triangle y$ are increments in x and y respectively.
Hence, the increment $\triangle z$ is approximately equal to the differential dz.
i.e. we can compute the approximate value of the given function by using differential.
Formula is
$f(x_0 + \triangle x, y_0 + \triangle y) \approx f(x_0, y_0) + df$; where
$df = \frac{\partial f}{\partial x}(x_0, y_0)\triangle x + \frac{\partial f}{\partial y}(x_0, y_0)\triangle y$

Working Rule : Given any function $f(x, y)$
(i) Decide x_0, y_0 and $\triangle x, \triangle y$.
(ii) Find $f(x_0, y_0)$.
$(iii)(\frac{\partial f}{\partial x})_(x_0, y_0), (\frac{\partial f}{\partial y})_(x_0, y_0)$ obtain these values. (iv) Use the formula.

Example 1: Using differentials find the approximate value of $(2.01)(3.02)^2$.

Solution :

Let $f(x,y) = xy^2$

$f(x_0 + \triangle x, y_0 + \triangle y) = (2.01)(3.02)^2$

Here, $x_0 = 2, y_0 = 3$ and $\triangle x = 0.01, \triangle y = 0.02$.

$f(x_0, y_0) = f(2,3) = 2(3)^2 = 18$

$f_x(x_0, y_0) = (\frac{\partial f}{\partial x})_(x_0, y_0) = y_0^2$

$\therefore f_x(2,3) = 3^2 = 9$

$f_y(x_0, y_0) = (\frac{\partial f}{\partial y})_(x_0, y_0) = 2x_0y_0$

$\therefore f_y(2,3) = 2(2)(3)12.$

$\therefore df = f_x(x_0, y_0)\triangle x + f_y(x_0, y_0)\triangle y$

$\therefore df = y_0^2 \triangle x + 2x_0y_0\triangle y$

$= 9(0.01) + 12(0.02)$

$df = 0.33.$

Hence

$f(x_0 + \triangle x, y_0 + \triangle y) \approx f(x_0, y_0) + df$

$\therefore (2.01)(3.02)^2 \approx 18 + 0.33$

$= 18.33.$

Example 2: Find approximate value of $\sqrt{\frac{4.1}{25.01}}$ by using differentials.

Solution :

Let $f(x,y) = \sqrt{\frac{x}{y}}$.

Here, $x_0 = 4, y_0 = 25$ and $\triangle x = 0.1, \triangle = 0.01$

$\therefore f(x_0, y_0) = f(4.25) = \sqrt{\frac{4}{25}} = \frac{2}{5}.$

$f_x(x_0, y_0) = \frac{1}{2\sqrt{x_0, y_0}}$

$\therefore f_x(4.25) = \frac{1}{2\sqrt{4.25}} = \frac{1}{20}$

$f_y(x_0, y_0) = \frac{-1}{2}\sqrt{\frac{x_0}{y_0^3}}$

$\therefore f_y(4.25) = \frac{-1}{2}\sqrt{\frac{4}{25^3}} = \frac{-1}{25}$

$\therefore df = f_x(x_0, y_0)\triangle x + f_y(x_0, y_0)\triangle y$

$\therefore \frac{1}{20}(0.1) - \frac{1}{125}(0.01)$

$= 0.005 - 0.00008$

$\therefore df = 0.00492.$

Hence,

$f(x_0 + \triangle x, y_0 + \triangle y) \approx f(x_0, y_0) + df$

$\therefore \sqrt{\frac{4.1}{25.01}} \approx \frac{2}{3} + 0.00492$

$= 0.4 + 0.00492 = 0.40492.$

Composite Function:Chain Rule

For a function of one variable $y = f(x)$ and $= \phi(t)$ then $y = f(\phi(t))$ is called composite function of t
its derivative w.r.t. t is given by $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$
which is known as chain rule.
For a function of two variables also we have composite function and chain rule.
1. Suppose $u = f(x, y)$ is a function of two independent variables x, y and x, y are themselves function of single variable t
that is $x = \phi(t)$ and $y = \psi(t)$ then $u = f(\phi(t), \psi(t)) = F(t)$
is called a composite function of a single variable t
For eg. 1.$u = f(x, y) = x + y$ and $x = at, y = bt^2$
then $u = f(at, bt^2) = at + bt^2$ is a composite function of a single variable t
2. $u = sin(x + y^2)$ and $x = cost, y = t^2$
then $u = sin(cost + t^4)$ is a composite function of t
3. Suppose $W = f(u, v)$ is a function of two variables u, v and u, v are functions of two variables x, y
that is $u = \phi(x, y)$ and $v = \psi(x, y)$
$W = f[\phi(x, y), \psi(x, y)] = F(x, y)$ is called a composite function of two variables x, y
for eg. $W = f(u, v)$ and $u = x + y, v = x - y$ then
$W = f(x + y, x - y)$ is a composite function of two variables x and y.
4. Suppose $Z = f(x)$ is a function in one variable x and x itself a function of two variables u and v i.e. $x = \phi(u, v)$
then $Z = f(\phi(u, v))$ is a composite function of two variables u and v.
for eg. $Z = f(u) : u = ax + by$ then $Z = f(ax + by)$ is a composite function of x and y.

Theorem : Chain Rule (I):-

If $u = f(x, y)$ is a differentiable function of x and y, $x = \phi(t)$ and $y = \psi(t)$ are themselves a functions of single variable t then composite function $u = f[\phi(t), \psi(t)]$ is a differentiable function of a single variable t and its total derivative is given by
$\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt}$.

Proof: Given: $u = f(x, y)$ and $x = \phi(t)$ and $y = \psi(t)$.
Let $\triangle x = \phi(t + \triangle t) - \phi(t)$ and $\triangle y = \psi(t + \triangle t) - \psi(t)$ be the increments in x and y respectively corresponds to an increment $\triangle t$ in t
Since $u = f(x, y)$ is differentiable, then by increment theorem
$\triangle u = \dfrac{\partial u}{\partial x}\triangle x + \dfrac{\partial u}{\partial y}\triangle y + \epsilon_1 \triangle x + \epsilon_2 \triangle y.....(1)$
where $\epsilon_1 \to 0, \epsilon_2 \to 0$ as $(\triangle x, \triangle y) \to (0, 0)$
$\triangle u = (\dfrac{\partial u}{\partial x} + \epsilon_1)\triangle x + (\dfrac{\partial u}{\partial y} + \epsilon_2)\triangle y$
$\dfrac{\triangle u}{\triangle t} = (\dfrac{\partial u}{\partial x} + \epsilon_1)\dfrac{\triangle x}{\triangle t} + (\dfrac{\partial u}{\partial y} + \epsilon_2)\dfrac{\triangle y}{\triangle t}.....(2)$
As $x = \phi(t), y = \psi(t)$ are differentiable functions in t
$\therefore$ they are continuous at t and hence $\triangle x, \triangle y \to 0$ as $\triangle \to 0$
$\therefore \epsilon_1 \to 0, \epsilon_2 \to 0$ as $\triangle \to 0$
Also $\lim\limits_{\triangle t \to 0} \dfrac{\triangle x}{\triangle t} = \dfrac{dx}{dt}$ and $\lim\limits_{\triangle t \to 0} \dfrac{\triangle y}{\triangle t} = \dfrac{dy}{dt}$
Taking limit as $\triangle t \to 0$ of equation (2)
$\lim\limits_{\triangle t \to 0} \dfrac{\triangle u}{\triangle t} = \lim\limits_{\triangle t \to 0} (\dfrac{\partial u}{\partial x} + \epsilon_1) \lim\limits_{\triangle t \to 0} \dfrac{\triangle x}{\triangle t} + \lim\limits_{\triangle t \to 0} (\dfrac{\partial u}{\partial y} + \epsilon_2) \lim\limits_{\triangle t \to 0} \dfrac{\triangle y}{\triangle t}$
$\therefore \dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt}$.

Theorem: Chain Rule(II):-

If $w = f(u,v)$ is a differentiable function of two variables u and v ,$u = \phi(x,y)$ and $v = \psi(x,y)$ are differentiable functions of x and y then the composite function $W = f[\phi(x,y),\psi(x,y)] = F(x,y)$ is also differentiable and

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$$

Proof: Since u, v, w are differentiable functions, by Chain rule(I)

$$\triangle u = \frac{\partial u}{\partial x}\triangle x + \frac{\partial u}{\partial y}\triangle y + \epsilon_1\triangle x + \epsilon_2\triangle y.....(1)$$

$$\triangle v = \frac{\partial v}{\partial x}\triangle x + \frac{\partial v}{\partial y}\triangle y + \epsilon_3\triangle x + \epsilon_4\triangle y.....(2)$$

$$\triangle w = \frac{\partial w}{\partial x}\triangle u + \frac{\partial w}{\partial y}\triangle v + \epsilon_5\triangle u + \epsilon_6\triangle v.....(3)$$

Where $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \to 0$ as $(\triangle x, \triangle y) \to (0,0)$
and $\epsilon_5, \epsilon_6 \to 0$ as $(\triangle u, \triangle v) \to (0,0)$

Now by (3) $\triangle w = (\frac{\partial w}{\partial u} + \epsilon_5)\triangle u + (\frac{\partial w}{\partial v} + \epsilon_6)\triangle v$

$$\triangle w = (\frac{\partial w}{\partial u}+\epsilon_5)(\frac{\partial u}{\partial x}\triangle x+\frac{\partial u}{\partial y}\triangle y+\epsilon_1\triangle x+\epsilon_2\triangle y)+(\frac{\partial w}{\partial v}+\epsilon_6)(\frac{\partial v}{\partial x}\triangle x+\frac{\partial v}{\partial y}\triangle y+\epsilon_3\triangle x+\epsilon_4\triangle y)$$

$$\triangle w = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x}\triangle x + \frac{\partial w}{\partial u}\frac{\partial u}{\partial y}\triangle y + \frac{\partial w}{\partial u}\epsilon_1\triangle x + \frac{\partial w}{\partial u}\epsilon_2\triangle y + \frac{\partial u}{\partial x}\triangle x\epsilon_5 + \frac{\partial u}{\partial y}\triangle y\epsilon_5 + \epsilon_1\epsilon_5\triangle x + \epsilon_2\epsilon_5\triangle y + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}\triangle x + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}\triangle y + \frac{\partial w}{\partial u}\epsilon_3\triangle x + \frac{\partial w}{\partial v}\epsilon_4\triangle y + \frac{\partial v}{\partial x}\triangle x\epsilon_6 + \frac{\partial v}{\partial y}\triangle y\epsilon_6 + \epsilon_3\epsilon_6\triangle x + \epsilon_4\epsilon_6\triangle y$$

$$\triangle w = (\frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x})\triangle x + (\frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y})\triangle y + \alpha_1\triangle x + \alpha_2\triangle y.....(4)$$

where α_1, α_2 are sum of terms containing the factors $\epsilon_1, \epsilon_2, ..., \epsilon_6$
$\therefore \alpha_1 \to 0, \alpha_2 \to 0$ as $(\triangle x, \triangle y) \to (0,0)$.
From (4) $w = F(x,y)$ is differentiable at (x,y)
Now put $\triangle y = 0$ and divide by $\triangle x$;
Equation (4) becomes $\frac{\triangle w}{\triangle x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} + \alpha_1$
Taking limit as $\triangle x \to 0$ we get

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

Similarly put $\triangle x = 0$ and divide by $\triangle y$ to equation (4); taking limit as $\triangle y \to 0$ we get

$$\frac{\triangle w}{\triangle y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$$

Theorem: Chain rule for the functions of three variables

If $W = f(x, y, z)$ is a differentiable function of three variables x, y, z and x, y, z are differentiable functions of single variable t then the composite function $w = f(t)$ is also differentiable function of t and its derivative is

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Theorem: Chain rule for the functions of many variables

If $W = f(x_1, x_2, ...x_n)$ is a differentiable function of finite set of variables $x_1, x_2, ...x_n$ and each $x_1, x_2, ...x_n$ is a differentiable function of finite set of variables $p_1, p_2, ...p_r$. Then $w = f[p_1, p_2, ...p_r]$ is differentiable function of finite set of variables $p_1, p_2, ...p_r$ and we have

$$\frac{\partial w}{\partial p_1} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial p_1} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial p_1} + \frac{\partial w}{\partial x_3}\frac{\partial x_3}{\partial p_1} + ... + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial p_1}$$

$$\frac{\partial w}{\partial p_2} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial p_2} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial p_2} + \frac{\partial w}{\partial x_3}\frac{\partial x_3}{\partial p_2} + ... + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial p_2}$$

and so on

$$\frac{\partial w}{\partial p_r} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial p_r} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial p_r} + \frac{\partial w}{\partial x_3}\frac{\partial x_3}{\partial p_r} + ... + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial p_r}$$

Examples:

1. If $w = f(ax+by)$ then show that $b\dfrac{\partial w}{\partial x} - a\dfrac{\partial w}{\partial y} = 0$

Solution: We have given that $w = f(ax+by)$ and put $u = ax+by$ then $w = f(u)$.
Then by chain rule

$\dfrac{\partial w}{\partial x} = \dfrac{dw}{du}.\dfrac{\partial u}{\partial x} = a\dfrac{dw}{du}$

$\therefore b\dfrac{\partial w}{\partial x} = ab\dfrac{dw}{du}$.....(1)

$\dfrac{\partial w}{\partial y} = \dfrac{dw}{du}.\dfrac{\partial u}{\partial y} = b\dfrac{dw}{du}$

$\therefore a\dfrac{\partial w}{\partial y} = ab\dfrac{dw}{du}$.....(2)

From (1) and (2)

$b\dfrac{\partial w}{\partial x} - a\dfrac{\partial w}{\partial y} = 0$

2. If $z = f(y+ax) + g(y-ax)$ prove that $z_{xx} = a^2 z_{yy}$, assuming that second order partial derivatives of f, g exist and a is constant.

Solution: Put $u = y+ax, v = y-ax$ hence $z = f(u) + g(v)$
Where $u = \phi(y,x) = y+ax, v = \psi(y,x) = y-ax$
$\therefore$ by chain rule

$z_x = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x} = f'(u)a + g'(v)(-a)$

$z_x = a(f'(u) - g'(v))$.....(1)

Again differentiating w.r.t. x

$z_{xx} = \dfrac{\partial^z}{\partial x^2} = \dfrac{\partial}{\partial u}[af'(u) - g'(v)].\dfrac{\partial u}{\partial x} + \dfrac{\partial}{\partial v}[af'(u) - g'(v)].\dfrac{\partial v}{\partial x}$

$z_{xx} = a^2 f"(u) + a^2 g"(v)$.....(2)

Now $z_y = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y} = f'(u) + g'(v)$.....(3)

Differentiating again w.r.t. y

$z_{yy} = \dfrac{\partial^z}{\partial y^2} = f"(u) + g"(v)$....(4)

from (2) and (4)

$z_{xx} = a^2 z_{yy}$

3. If $u = xy^2\log(\frac{y}{x})$ then find du.

Solution: We know that $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$.....(1)

Now $\frac{\partial u}{\partial x} = y^2\log(\frac{y}{x}) + xy^2\frac{1}{y/x}(\frac{-1}{x^2})y = y^2\log(\frac{y}{x}) - y^2$....(2)

$\frac{\partial u}{\partial y} = 2xy\log(\frac{y}{x}) + xy^2\frac{1}{y/x}(\frac{1}{x}) = 2xy\log(\frac{y}{x}) + xy$....(3)

from (2) and (3)

$du = [y^2\log(\frac{y}{x}) - y^2]dx + [2xy\log(\frac{y}{x}) + xy]dy$

4.if $u = u(\frac{y-x}{xy}, \frac{z-x}{xz})$, Show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

Solution: Let $u = u(\frac{y-x}{xy}, \frac{z-x}{xz})$

Put $r = \frac{y-x}{xy} = \frac{1}{x} - \frac{1}{y}$

and $s = \frac{z-x}{xz} = \frac{1}{x} - \frac{1}{z}$

$\therefore u = u(r,s)$ is a composite function of x and y

$\therefore$ by chain rule $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x}$.....(1)

Since $\frac{\partial r}{\partial x} = -\frac{1}{x^2}, \frac{\partial r}{\partial y} = \frac{1}{y^2}, \frac{\partial r}{\partial z} = 0$

And $\frac{\partial s}{\partial x} = -\frac{1}{x^2}, \frac{\partial s}{\partial y} = 0, \frac{\partial s}{\partial z} = \frac{1}{z^2}$

Equation (1) becomes

$x^2\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial r} - \frac{\partial u}{\partial s}$.....(2)

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y}$

$\therefore y^2\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}$....(3)

Also $\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z}$

$\therefore z^2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}$.....(4)

Adding (2),(3),(4) we get

$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

5. If $u = f(r)$ and $x = rcos\theta, y = rsin\theta$ then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} = f"(r) + \frac{1}{r}f'(r)$.

Solution: Given that $u = f(r)$ and $x = rcos\theta, y = rsin\theta$ then $r^2 =^2 + y^2$ and $\frac{\partial r}{\partial x} = \frac{x}{r}$

Now $\frac{\partial u}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x}$ and $\frac{\partial u}{\partial y} = \frac{df}{dr}\frac{\partial r}{\partial y}$

$\therefore \frac{\partial u}{\partial x} = \frac{df}{dr}\frac{x}{r}$

$\frac{\partial^2 u}{\partial x^2} = (\frac{d^2 f}{dr^2}\frac{\partial r}{\partial x})\frac{x}{r} + \frac{df}{dr}(\frac{r - x\frac{\partial r}{\partial x}}{r^2})$

$\frac{\partial^2 u}{\partial x^2} = (\frac{d^2 f}{dr^2}\frac{x}{r})\frac{x}{r} + \frac{df}{dr}(\frac{r - x\frac{\partial r}{\partial x}}{r^2})$

$\frac{\partial^2 u}{\partial x^2} = (\frac{d^2 f}{dr^2}\frac{x^2 r^2}{+}\frac{df}{dr}\frac{r^2 - x^2}{r^3}$

$\frac{\partial^2 u}{\partial x^2} = (\frac{d^2 f}{dr^2}\frac{x^2 r^2}{+}\frac{df}{dr}\frac{y^2}{r^3}$.....(1)

Similarly $\frac{\partial^2 u}{\partial x^2} = (\frac{d^2 f}{dr^2}\frac{y^2 r^2}{+}\frac{df}{dr}\frac{x^2}{r^3}$.....(2)

Adding (1) and (2) we get

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (\frac{d^2 f}{dr^2}\frac{x^2 + y^2 r^2}{+}\frac{df}{dr}\frac{x^2 + y^2}{r^3}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{d^2 f}{dr^2}.1 + \frac{df}{dr}\frac{r^2}{r^3}$$
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{d^2 f}{dr^2} + \frac{df}{dr}\frac{1}{r}$$
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f"(r) + \frac{1}{r}f'(r)$$

Directional derivatives:

If $f(x, y)$ is differentiable function and $x = \phi(t), y = \psi(t)$ then $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$ gives the rate of change of f with respect to t. This depends on the direction of motion along the curve. If curve is a straight line and parameter t is the arc length measured from point $p_0(x_0, y_0)$ in the direction of a given unit vector u then $\frac{df}{dt}$ is the rate of change of f with respect to distance in the direction of $\bar{u}$. These values of $\frac{df}{dt}$ through p_0 are called directional derivatives.

Definition: Directional derivatives in the planes

Suppose the function $f(x, y)$ is defined on a region R in the xy plane. $p_0(x_0, y_0)$ is a point in R and $u = u_1\bar{i} + u_2\bar{j}$ is a unit vector. $x = x_0 + su_1, y = y_0 + su_2$ are the parametric equations of a line passing through p_0 parallel to $\bar{u}$; where s is the arc length measured from point p_0 in the direction of $\bar{u}$.

The derivative of f at point $p_0(x_0, y_0)$ in the direction of $\bar{u}$ is

$(\frac{df}{ds})_{u,p_0} = \lim_{s\to 0}(\frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s})$ if R.H.S. exist is called the directional derivative of f at point p_0. It is denoted by $(D_u f)_{p_0}$.

Note: If $\bar{u} = \bar{i}$ then $(D_u f)_{p_0}$ gives $\frac{\partial f}{\partial x}$ at p_0, and If $\bar{u} = \bar{u}$ then $(D_u f)_{p_0}$ gives $\frac{\partial f}{\partial y}$ at p_0

Examples:

1. Find the directional derivative of $f(x,y) = x^2 + xy$ at point $(1,2)$ in th direction of a unit vector $\bar{u} = \frac{1}{\sqrt{2}}\bar{i} + \frac{1}{\sqrt{2}\bar{j}}$

Solution: Let $f(x,y) = x^2 + xy$, $p_0 = (1,2)$ and $\bar{u} = u_1\bar{i} + u_2\bar{j} = \frac{1}{\sqrt{2}}\bar{i} + \frac{1}{\sqrt{2}}\bar{j}$

Since $(\dfrac{df}{ds})_{u,p_0} = \lim\limits_{s\to 0}(\dfrac{f(x_0+su_1, y_0+su_2) - f(x_0,y_0)}{s})$

$= \lim\limits_{s\to 0}(\dfrac{f(1+\frac{s}{\sqrt{2}}, 2+\frac{2}{\sqrt{2}}) - f(1,2)}{s})$

$= \lim\limits_{s\to 0}[\dfrac{((1+\frac{s}{\sqrt{2}})^2 + (1+\frac{s}{\sqrt{2}})(2+\frac{s}{\sqrt{2}})) - (1^2+1.2)}{s}] = \lim\limits_{s\to 0}[\dfrac{\frac{5}{\sqrt{2}} + s^2}{s}]$

$= lim_{s\to 0}(\dfrac{5}{\sqrt{2}} + s) = \dfrac{5}{\sqrt{2}}$

$\therefore (\frac{df}{ds})_{\bar{u},p_0} = (D_u f)_{p_0} = \frac{5}{\sqrt{2}}$

2. Find the directional derivative of $f(,y,z) = x^2 + 2y^2 + 3z^2$ at the point $(1,1,0)$ in the direction of $\bar{u} = \bar{i} - \bar{j} + 2\bar{k}$

Solution: Let $f(,y,z) = x^2 + 2y^2 + 3z^2$, $p_0 = (1,1,0)$ and $\bar{u} = \bar{i} - \bar{j} + 2\bar{k}$ Since $\bar{u}$ is not a unit vector so $\hat{u} = \frac{1}{\sqrt{6}}(\bar{i} - \bar{j} + 2\bar{k})$

$(\dfrac{df}{ds})_{\hat{u},p_0} = \lim\limits_{s\to 0}[\dfrac{f(x_0+su_1, y_0+su_2, z_0+su_3) - f(x_0,y_0,z_0)}{s}]$

$= \lim\limits_{s\to 0}[\dfrac{f(1+\frac{s}{\sqrt{6}}, 1-\frac{s}{\sqrt{6}}, \frac{2s}{\sqrt{6}}) - f(1,1,0)}{s}]$

$= \lim\limits_{s\to 0}[\dfrac{((1+\frac{s}{\sqrt{6}})^2 + 2(1-\frac{s}{\sqrt{6}})^2 + 3(\frac{2s}{\sqrt{6}})^2) - 3}{s}]$

$= \lim\limits_{s\to 0}[\dfrac{(\frac{-2s}{\sqrt{6}} + \frac{15s^2}{\sqrt{6}})}{s}]$

$= \lim\limits_{s\to 0}(\dfrac{-2}{\sqrt{6}} + \dfrac{15s}{\sqrt{6}}) = \dfrac{-2}{\sqrt{6}}$

$\therefore (\dfrac{df}{ds})_{\hat{u},p_0} = (D_u f)_{p_0} = \dfrac{-2}{\sqrt{6}}$

The Gradient Vector Definition: The gradient vector of $f(x, y)$ at a point $p_0(x_0, y_0)$ is the vector $\nabla f = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j}$

Note: We can find the directional derivative of f in the direction of $\bar{u}$ at point p_0 using the dot product of $\bar{u}$ with gradient of f at p_0 :

Since by chain rule we can write $(\frac{df}{ds})_{u,p_0} = (\frac{\partial f}{\partial x})_{p_0} . \frac{dx}{ds} + (\frac{\partial f}{\partial y})_{p_0} . \frac{dy}{ds}$

$(\frac{df}{ds})_{u,p_0} = (\frac{\partial f}{\partial x})_{p_0} . u_1 + (\frac{\partial f}{\partial y})_{p_0} . u_2$

$(\frac{df}{ds})_{u,p_0} = ((\frac{\partial f}{\partial x})_{p_0}\bar{i} + (\frac{\partial f}{\partial y})_{p_0} . \bar{j}) . (u_1\bar{i} + u_2\bar{j})$

Examples:

1. Find the directional derivative of $f(x,y) = xe^y + cos(xy)$ at the point $(2,0)$ in the direction of $3\bar{i} - 4\bar{j}$.

Solution: Let $f(x,y) = xe^y + cos(xy)$, $p_0 = (2,0)$ and $\bar{u} = 3\bar{i} - 4\bar{j}$ Since u is not a unit vector so

$\hat{u} = \frac{3}{5}\bar{i} - \frac{4}{5}\bar{j}$

Now $f_x = e^y - sin(xy).y$ and $f_y = xe^y - sin(xy).x$

$\therefore f_x(2,0) = 1, f_y(2,0) = 2$

The gradient of f at $(2,0) = (\nabla f)_{(2,0)} = f_x(2,0)\bar{i} + f_y(2,0)\bar{j} = \bar{i} + 2\bar{j}$

The directional derivative of f at $(2,0)$ in the direction of $3\bar{i} - 4\bar{j}$ is

$(\dfrac{df}{ds})_{\hat{u},p_0} = (D_u f)_{p_0} = (\nabla f)_{p_0}.\hat{u} = (i + 2j).(\frac{3}{5}\bar{i} - \frac{4}{5}\bar{j}) = -1$

2. Find the derivative of $f(x,y) = 2xy - 2y^2$ at the point $(5,5)$ in the direction of $4\bar{i} + 3\bar{j}$.

Solution: Let $f(x,y) = 2xy - 2y^2$, $p_0 = (5,5)$ and $\bar{u} = 4\bar{i} + 3\bar{j}$ Since u is not a unit vector so $\hat{u} = \frac{4}{5}\bar{i} + \frac{3}{5}\bar{j}$

Now $f_x = 2y, f_x(5,5) = 10, f_y = 2x - 6y, f_y(5,5) = -20$

$\therefore$ the gradient of f at $(5,5) = (\nabla f)_{(5,5)} = 10\bar{i} - 20\bar{j}$

$\therefore (\frac{df}{dx})_{\hat{u},p_0} = (D_u f)_{p_0} = (\nabla f)_{p_0}.\hat{u} = (10\bar{i} - 20\bar{j})(\frac{4}{5}\bar{i} + \frac{3}{5}\bar{j}) = -4.$

3. Find the derivative of $f(x,y,z) = x^2 + 2y^2 - 3z^2$ at the point $(1,1,1)$ in the direction of $\bar{i} + \bar{j} + \bar{k}$.

Solution: Let $f(x,y,z) = x^2 + 2y^2 - 3z^2$, $p_0 = (1,1,1)$ and $\bar{u} = \bar{i} + \bar{j} + \bar{k}$

Since $\bar{u}$ is not a unit vector so $\hat{u} = \frac{1}{\sqrt{3}}(\bar{i} + \bar{j} + \bar{k})$

Now $f_x = 2x, f_x(1,1,1) = 2, f_y = 4y, f_y(1,1,1) = 4, f_z = -6z, f_z(1,1,1) = -6$

The gradient of f at $(1,1,1) = (\nabla f)_{(1,1,1)} = 2\bar{i} + 4\bar{j} - 6\bar{k}$

The derivative of f at point p_0 is

$(\frac{df}{ds})_{\hat{u},p_0} = (D_u f)_{p_0} = (\nabla f)_{p_0}.\hat{u} = (2\bar{i} + 4\bar{j} - 6\bar{k}).\frac{1}{\sqrt{3}}(\bar{i} + \bar{j} + \bar{k}) = 0$

Properties of directional derivatives:

The directional derivative definition revels that

$D_u f = \nabla f.u = |\nabla f||u|cos\theta = |\nabla f|cos\theta$ As u is unit vector.

It has following properties: 1.The function f increase most rapidly when $cos\theta = 1$ or when $\bar{u}$ is in the direction of ∇f.

that is $D_u f = |\nabla f|cos(0) = |\nabla f|$.

2.The function f decreases most rapidly when $cos\theta = -1$ or when $\bar{u}$ is in the direction of $-\nabla f$.

that is $D_u f = |\nabla f|cos(\pi) = -|\nabla f|$.

3.Any direction $\bar{u}$ orthogonal to the gradient is a direction of zero change in f when $\theta = \frac{\pi}{2}$ that is $D_u f = |\nabla f|cos(\frac{\pi}{2}) = |\nabla f|.0 = 0$.

Examples:

1. Find the direction in which $f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$

a)increase most rapidly at point (1,1)

b)decrease most rapidly at point (1,1)

c)What are the directions of zero change in f at (1,1)?

Solution: We have $f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$

a)$(\nabla f)_{(1,1)} = f_x(1, 1)\bar{i} + f_y(1, 1)\bar{j} = \bar{i} + \bar{j}$

Its direction is $|(\nabla f)_{(1,1)}| = \frac{1}{\sqrt{2}}\bar{i} + \frac{1}{\sqrt{2}}\bar{j} = \bar{u}$

b) f decreases most rapidly in the direction of $-(\nabla f)_{(1,1)}$

$-\bar{u} = -\frac{1}{\sqrt{2}}\bar{i} - \frac{1}{\sqrt{2}}\bar{j}$

c) The directions of zero change at $(1, 1)$ are the directions orthogonal to ∇f

$\therefore \bar{n} = -\frac{1}{\sqrt{2}}\bar{i} + \frac{1}{\sqrt{2}}\bar{j}$ and $-\bar{n} = \frac{1}{\sqrt{2}}\bar{i} - \frac{1}{\sqrt{2}}\bar{j}$

2. a) Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at point $(1, 1, 0)$ in the direction of $2\bar{i} - 3\bar{j} + 6\bar{k}$

b) In what direction f change most rapidly at point $(1, 1, 0)$ and what are the rate of change in these directions?

Solution: a) Suppose $\bar{u} = 2\bar{i} - 3\bar{j} + 6\bar{k}$ and $\hat{u} = \frac{2}{7}\bar{i} - \frac{3}{7}\bar{j} + \frac{6}{7}\bar{k}$

$f_x(1, 1, 0) = 2, f_y(1, 1, 0) = -2, f_z(1, 1, 0) = -1$

$\therefore (\nabla f)_{(1,1,0)} = 2\bar{i} - 2\bar{j} - \bar{k}$

Hence the derivative of f at given point is

$(D_u f)_{(1,1,0)} = (\nabla f)_{(1,1,0)}\hat{u} = (2\bar{i} - 2\bar{j} - \bar{k}).(\frac{2}{7}\bar{i} - \frac{3}{7}\bar{j} + \frac{6}{7}\bar{k}) = \frac{4}{7}$

b)The function f increase most rapidly in the direction of $\nabla f = 2\bar{i} - 2\bar{j} - \bar{k}$ and decreases most rapidly in the direction of $-\nabla f$. The rate of change in the directions are $|\nabla f| = 3$ and $-|\nabla f| = -3$