Progressive Education Society's
Modern College of Art's, Science and Commerce (Autonomous), Shivajinagar, Pune-5

Department of Mathematics

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Subject: FOURIER SERIES AND BOUNDARY VALUE PROBLEMS.

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Chapter 5. Strum-Liouville Problems and Applications.

REGULAR STRUM-LIOUVILLE PROBLEMS:
In previous problems we discussed Strum-Liouville Problems of following types defined on interval $0 \le x \le c$

$$X''(x) + \lambda X(x) = 0, \qquad X'(0) = 0, \qquad X'(c) = 0, \tag{1}$$

$$X''(x) + \lambda X(x) = 0, \qquad X(0) = 0, \qquad X(c) = 0, \tag{2}$$

Now consider linear homogeneous Strum-Liouville Problem

$$[r(x)X'(x)]' + [q(x) + \lambda p(x)]X(x) = 0, \qquad (a < x < b) \tag{3}$$

with a pair of homogeneous boundary conditions

$$a_1X(a) + a_2X'(a) = 0, \qquad b_1X(b) + b_2X'(b) = 0 \tag{4}$$

where, a_1 and a_2 are not both zero; and also b_1 and b_2 are not both zero. Here the real valued functions p, q and r in equation (3) are independent of λ.
Strum-Liouville Problem is said to be **Regular** whenever real valued functions p, q, r and r' are continuous on interval $a \le x \le b$ and $p(x) > 0$ and also $r(x) > 0$ when $a \le x \le b$.

EXAMPLES:

1. Strum-Liouville Problems in equation (1) and (2) are regular Strum-Liouville Problems.

2. $X''(x) + \lambda X(x) = 0, \qquad (0 < x < c)$
   $X'(0) = 0, \qquad hX(c) + X'(c) = 0$
   is regular Strum-Liouville Problem.

3. $[x^2X'(x)]' + \lambda X(x) = 0 \qquad (1 < x < b)$
   $X(1) = 0, \qquad X(b) = 0$
   is regular Strum-Liouville Problem.

4. $[xX'(x)]' + (\lambda/x)X(x) = 0 \qquad (1 < x < b)$
   $X(1) = 0, \qquad X(b) = 0$
   is regular Strum-Liouville Problem.

5. $[|x|X'(x)]' + (\lambda/x)X(x) = 0 \qquad (1 < x < b)$
   $X(1) = 0, \qquad X(b) = 0$
   is not regular Strum-Liouville Problem.

A value of λ for which problem $(3) - (4)$ has a non-trivial solution is called an **eigenvalue**; and the non-trivial solution is called an **eigenfunction**.
Spectrum: The set of eigenvalues of problem $(3) - (4)$ is called the spectrum of the problem.

SINGULAR STRUM-LIOUVILLE PROBLEMS:
Strum-Liouville Problem in $(3) - (4)$ is called singular Strum-Liouville Problem

1. if at least one of the regularity condition fails. or
2. if function q has infinite discontinuities at an end point of an interval $a \leq x \leq b$. or
3. if $p(x)$ or $r(x)$ vanishes at an end points.

EXAMPLES:

1. $$[xX'(x)]' + (-\frac{n^2}{x} + \lambda x)X(x) = 0 \qquad (0 < x < c)$$
   where $n = 0, 1, \cdots$ with $X(c) = 0$ is a singular Strum-Liouville Problem because the functions $p(x) = x$ and $r(x) = x$ vanishes at $x = 0$.
2. $$[(1 - x^2)X'(x)]' + \lambda X(x) = 0 \qquad (-1 < x < 1)$$
   is a singular Strum-Liouville Problem because the function $r(x) = 1 - x^2$ vanishes at both ends of the interval (-1,1.)

ORTHOGONALITY OF EIGENFUNCTIONS:

A set $\{\psi_n(x)\}$ is orthogonal on an interval $a < x < b$ with respect to a weight function $p(x)$ which is peicewise continuous and positive on that interval if

$$\int_a^b p(x)\psi_m(x)\psi_n(x)dx = 0, \text{ when } m \neq n$$

Here above integral represents the inner product $< \psi_m(x), \psi_n(x) >$ with respect to weight function $P(x)$.

Theorem 1: If λ_m and λ_n are distinct eigenvalues of the Sturm Liouville problem then corresponding eigenfunctions $X_m(x)$ and $X_n(x)$ are orthogonal with respect to weight function $p(x)$ on the interval $a < x < b$. the orthogonality also holds in each of the following cases:

1. when $r(a) = 0$ and the first of boundary condition (4) is dropped from the problem;
2. when $r(b) = 0$ and the second of boundary condition (4) is dropped from the problem;
3. $r(a) = r(b)$ and conditions (4) are replaced by the conditions

$$X(a) = X(b), \qquad X'(a) = X'(b)$$

Example 1: Verify the theorem 1 above for the following regular Strum-Liouville Problem

$$X''(x) + \lambda X(x) = 0, \qquad X(0) = 0, \qquad X(c) = 0.$$

By solving $X''(x) + \lambda X(x) = 0$, we get $X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$. For $\lambda > 0$ we get non-trivial eigenvectors $X_n(x) = \sin(n\pi x/c)(n = 1, 2, \cdots)$, and they correspond to the distinct eigenvalues $\lambda_n(x) = (n\pi/c)^2$.
The functions $X_n(x) = \sin(n\pi x/c)$ are orthogonal on interval $0 < x < c$ with weight function $p(x) = 1$ because;

$$\int_0^c \sin\left(\frac{m\pi x}{c}\right)\sin\left(\frac{n\pi x}{c}\right)dx = 0$$

Corollary 1: If λ is an eigenvalue of Strum-Liouville Problem (3)-(4), then it must be real number; and the same is true in cases 1. 2. 3. treated in Theorem 1.
Uniquness of Eigenfunction:
Corollary 2: If λ is an eigenvalue of a Strum-Liouville Problem (3)-(4), and if conditions $q(x) \leq 0 (a \leq x \leq b)$ and $a_1 a_2 \leq 0, b_1 b_2 \geq 0$ are satisfied, then $\lambda \geq 0$.

METHOD OF SOLUTION:
Example 1: Let us solve the regular Strum-Liouville Problem

$$X''(x) + \lambda X(x) = 0, \qquad (0 < x < c) \tag{5}$$

$$X'(0) = 0, \qquad hX(c) + X'(c) = 0, \tag{6}$$

where h is positive constant.

From Corollary 2. there are no negative eigenvalues. If the general solution of equation (5) is $X(x) = Ax + B$, where A and B are constants; and it follows from boundary conditions (6) that $A = 0$ and $B = 0$. This gives problem has only the possibity that $\lambda > 0$.
If $\lambda > 0$, we consider $\lambda = \alpha^2, (\alpha > 0)$; hence general solution of equation (5) is $X(x) = c_1 \cos\alpha x + c_2 \sin\alpha x$. From first boundary condition in (6), we get

$$C_1(h\cos\alpha c - \alpha\sin\alpha c) = 0$$

Now, from second boundary condition in (6), we get

$$\tan(\alpha c) = \frac{h}{\alpha} \tag{7}$$

Hence by solving (7) we get $\lambda_n = \alpha_n^2$, where $\tan(\alpha_n c) = \frac{h}{\alpha_c}$. $\qquad (\alpha_n > 0)$.
so corresponding eigenfunctions are

$$X_n(x) = \cos\alpha_n x.(n = 1, 2, \cdots)$$

So, according to Theorem 1. $X_n(x)$ are orthogonal on the interval $0 < x < c$ with weight function $p(x) = 1$.
Hence,

$$||X_n||^2 = \frac{1}{2}\left(c + \frac{\sin 2\alpha_n c}{2\alpha_n}\right).$$

Then by using $\sin(2\alpha_n c) = 2\sin(\alpha_n c)\cos(\alpha_n c)$ and $\alpha_n = \frac{h}{\tan(\alpha_n c)}$ we get;

$$||X_n||^2 = \frac{hc + \sin^2\alpha_n c}{2h}.$$

Hence normalized eigenfunctions are,

$$\phi_n(x) = \sqrt{\frac{2h}{hc + \sin^2\alpha_n c}}\cos(\alpha_n x) \qquad (n = 1, 2, \cdots).$$

Example 2: Solve the regular Strum-Liouville Problem

$$X''(x) + \lambda X(x) = 0, \qquad X(0) = 0, \qquad X'(1) = 0.$$

Example 3: Let us solve the regular Strum-Liouville Problem

$$[xX'(x)]' + \frac{\lambda}{x} X(x) = 0, \qquad (1 < x < b) \tag{8}$$

$$X'(1) = 0, \qquad hX(b) + X'(b) = 0, \tag{9}$$

where h is positive constant.

Equation (8) can be written as

$$x^2 \frac{d^2X}{dx^2} + x\frac{dX}{dx} + \lambda X(x) = 0 \tag{10}$$

By using Chain rule of differentiation and considering substitution $x = e^s$; we get

$$\frac{dX}{dx} = e^{-s}\frac{dX}{ds} \quad \text{and} \quad \frac{d^2X}{ds^2} = e^{2s}\frac{d^2X}{dx^2} + e^s\frac{dX}{dx}$$

Substitute these value in equation (10),

$$\frac{d^2X}{ds^2} + \lambda X = 0, \qquad (0 < s < \log(b)) \tag{11}$$

By using first boundary condition at $s = 0$ we obtain

$$\frac{dX}{ds} = 0 \tag{12(a)}$$

also at $s = \log b$ we get

$$hX + \frac{1}{b}\frac{dX}{ds} = 0 \tag{12(b)}$$

Hence equation (11) with [12(a)] and [12(b)] is reduced Strum-Liouville Problem. Hence by From Corollary 2. there are no negative eigenvalues. If the general solution of equation (11) is

$$X(s) = As + B,$$

where A and B are constants; and it follows from boundary conditions $[12(a)]$ that $A = 0$ and $B = 0$. This gives problem has only the possibity that $\lambda > 0$. If $\lambda > 0$, we consider $\lambda = \alpha^2, (\alpha > 0)$; hence general solution of equation (11) is

$$X(s) = c_1 \cos \alpha s + c_2 \sin \alpha s.$$

From boundary condition in [12(a)], we get $c_2 = 0$ and then solving for [12(b)] we get

$$hb = \alpha \tan(\alpha s).$$

Hence, $\lambda_n = \alpha_n^2$, $hb = \alpha_n \tan(\alpha_n \log b)$ and corresponding eigenfunctions are $X_n(s) = \cos(\alpha_n \log x)$. So,

$$||X_n|| = \sqrt{\frac{hb \log b + \sin^2(\alpha_n \log b)}{2hb}}$$

Hence normalized eigenfunctions of problem (8) and (9) is

$$\phi_n(x) = \sqrt{\frac{2hb}{hb \log b + \sin^2(\alpha_n \log b)}} \cos(\alpha_n \log x). \quad (n = 1, 2, \cdots)$$

SURFACE HEAT TRANSFER:
Let $u(x,t)$denotes temperature in a slab $0 \leq x \leq 1$, initially slab has temperature $f(x)$ at face $x = 0$.
The boundary value problem is

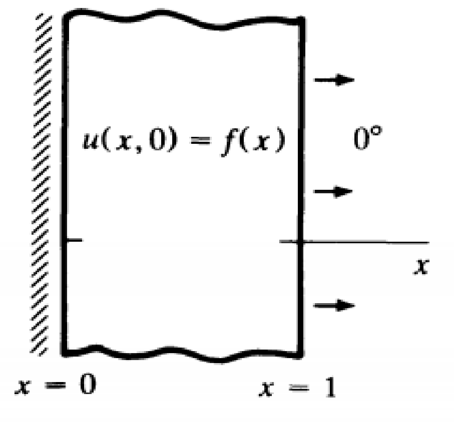

Figure 1:

$$u_t(x,t) = ku_{xx}(x,t) \qquad (0 < x < 1, t > 0) \qquad (13)$$

$$u_x(0,t) = 0, \qquad u_x(1,t) = -hu(1,t), \qquad u(x,0) = f(x). \qquad (14)$$

where h is positive constant.
Consider $u(x,t) = X(x)T(t)$ is any non-trivial soluion of given problem, take its derivatives with respect to t,x and substitute in equation (13), we get system of two boundary value problems;

$$X''(x) + \lambda X(x) = 0, \qquad X'(0) = 0, \qquad hX(1) + X'(1) = 0. \qquad (15)$$

$$T'(t) + \lambda kT(t) = 0 \qquad (16)$$

For boundary value problem in equation (15) has non-trivial solution $X_n(x) = \cos \alpha_n x$ for negative λ_n only where $\tan \alpha_n = h/\alpha_n, (\alpha_n > 0)$.
Norm of given eigenfunction is

$$||X_n||^2 = \frac{h + \sin^2 \alpha_n}{2h}.$$

Hence the normalized eigenfunctions are

$$\phi_n(x) = \sqrt{\frac{2h}{h+\sin^2\alpha_n}}\cos\alpha_n x \quad (n = 1, 2, 3, \cdots)$$

By solving equation (16) we get,

$$T_n(t) = e^{-k\alpha_n^2 t}$$

Hence formal solution of given temperature problem is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha_n^2 kt}\phi_n(x).$$

So, $u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$, and c_n can be evaluated by Fourier cosine coefficient formula for $f(x)$,

$$c_n = \int_0^1 f(x)\phi_n(x)dx = \sqrt{\frac{2h}{h+\sin^2\alpha_n}}\int_0^1 f(x)\cos\alpha_n x dx.$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty}\frac{2h}{h+\sin^2\alpha_n}(e^{-\alpha_n^2 kt})\cos\alpha_n x\int_0^1 f(x)\cos(\alpha_n x)dx.$$

BOUNDARY VALUE PROBLEMS:

Example 1: let u(x,y) denote the bounded steady-state temperature in a semi-infinite slab bounded by the planes $x = 0$ and $x = \phi$, and $y = 0$ with the face $x = 0$ is insulated and face $x = \phi$ kept at zero temperature, and the flux is inwards toward the face in the plane $y = 0$ is prescribed function $f(x)$.

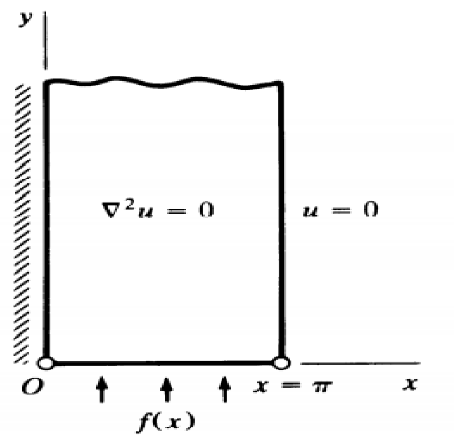

Figure 2:

Hence boundary value problem is

$$u_{xx}(x,y) + u_{yy}(x,y) = 0 \qquad (0 < x < \pi, y > 0), \qquad (17)$$
$$u_x(0,y) = 0, \quad u(\pi,y) = 0 \qquad (y > 0), \qquad (18)$$
$$-Ku_y(x,0) = f(x) \qquad (0 < x < \pi), \qquad (19)$$

where K is a positive constant.
Let, $u(x,y) = X(x)Y(y)$ be non-trivial solution of given booundary value problem take its derivatives with respect to y,x and substitute in equation (17), we get system of two boundary value problems;

$$X''(x) + \lambda X(x) = 0, \quad X'(0) = 0, \quad X(\pi) = 0. \qquad (20)$$
$$Y''(y) - \lambda Y(y) = 0. \qquad (21)$$

By solving boundary value problem in equation (20), we get its nontrivial solution for $\lambda < 0$ as
$\lambda_n = \alpha_n^2, \qquad X_n(x) = \cos(\alpha_n x), \qquad (n = 1, 2, \cdots)$
where

$$\alpha_n = \frac{2n-1}{2} \quad \text{and} \quad ||X_n|| = \sqrt{\frac{\pi}{2}}$$

Hence normalized eigenfunctions of given problem is

$$\phi_n(x) = \sqrt{\frac{2}{\pi}} \cos(\alpha_n x)$$

The corresponding boounded solution of equation (21) are constant multiple of the functions

$$Y_n(x) = \exp(-\alpha_n y)$$

Consequently,

$$u(x,y) = \sum_{n=1}^{\infty} c_n \exp(-\alpha_n y)\phi_n(x).$$

Now applying non-homogeneous condition (19) to this expression,

$$f(x) = \sum_{n=1}^{\infty} (Kc_n\alpha_n)\phi_n(x).$$

Where, $(Kc_n\alpha_n) =< f, \phi_n >$.
Example 2: Solve the boundary value problem
$$u_{xx}(x,y) + u_{yy}(x,y) = 0 \qquad (0 < x < a, 0 < y < b),$$
$$u_x(0,y) = 0, \quad u_x(a,y) = -hu(a,y) \qquad (0 < y < b),$$
$$u(x,0) = 0, \quad u(x,b) = f(x) \qquad (0 < x < a),$$
where h is a positive constant.

Example 3: Consider Dirichlet problem for function $u(\rho, \phi)$, in polar coordinates which satisfies Laplace's equation
$$\rho^2 u_{\rho\rho}(\rho,\phi) + \rho u_\rho(\rho,\phi) + u_{\phi\phi}(\rho,\phi) = 0, \qquad (1 < \rho < b, 0 < \phi < \pi) \qquad (22)$$
and boundary conditions are
$$u(\rho,0) = 0, \quad u(\rho,\pi) = u_0 \qquad (1 < \rho < b) \quad (23),$$
$$u(1,\phi) = 0, \quad u(b,\phi) = 0 \qquad (0 < \phi < \pi) \quad (24),$$
where u_0 is a constant.

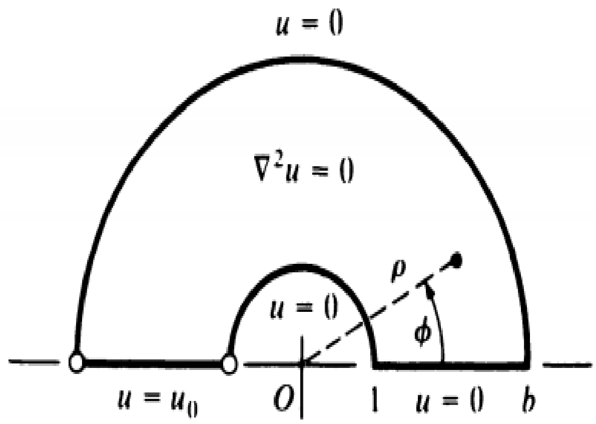

Figure 3:

Let $u = R(\rho)\Phi(\phi)$ be non-trivial solution of given problem, take its suitable derivatives and substitute in equations $(22) - (24)$ so that we get reduced boundary value problems

$$[\rho R'(\rho)]' + \frac{\lambda}{\rho}R(\rho) = 0, \qquad R(1) = 0, \qquad R(b) = 0 \tag{25}$$

$$\Phi''(\phi) - \lambda\Phi(\phi) = 0, \qquad \Phi(0) = 0. \tag{26}$$

To solve Strum-Liouville problem in equation (25) substitute $\rho = e^s$, then by applying chain rule we get reduced boundary value problem for problem in equation (25) as,

$$\frac{d^2R}{ds^2} + \lambda R = 0, \qquad 0 < s < \log b \tag{27}$$

$$R(a) = 0 \quad \text{and} \quad R(\log b) = 0 \tag{28}$$

this system in (27)-(28) will give non-trivial solution for $\lambda > 0$ which is

$$\lambda_n = \alpha_n^2 \text{ where } \alpha_n = \frac{n\pi}{\log b}$$

with corresponding eigen-functions

$$R_n(\rho) = \sin(\alpha_n \log \rho)$$

So normalized eigenfunctions are

$$\phi_n(\rho) = \sqrt{\frac{2}{\log b}}\sin(\alpha_n \log \rho)$$

Now solving (26) we get $\phi_n(\phi) = \sinh(\alpha_n\phi)$
Hence,

$$u(\rho, \phi) = \sum_{n=1}^{\infty} c_n \sinh(\alpha_n\phi)\phi_n(\rho). \tag{29}$$

For non-homogeneous condition put $\phi = \pi$ in equation (29) so that we obtain

$$u(\rho, \pi) = u_0 = \sum_{n=1}^{\infty} [c_n \sinh(\alpha_n \pi)] \sqrt{\frac{2}{\log b}} \sin(\alpha_n \log \rho) \qquad (1 < \rho < b)$$

then $c_n \sinh(\alpha_n \pi)$ is fourier sine coefficient of $\sinh(\alpha_n \pi)$ Hence,

$$c_n \sinh(\alpha_n \pi) = u_0 \sqrt{\frac{2}{\log b}} \int_1^b \frac{1}{\rho} \sin(\alpha_n \log \rho) d\rho.$$

after thesesubstitution in (29) we get

$$u(\rho, \phi) = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \frac{\sinh(\alpha_n \phi)}{\sinh(\alpha_n \pi)} \sinh(\alpha_n \log \rho).$$

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