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Chapter 2. Integration Theory

Canonical form of simple function:
Let ϕ be a simple function given by

$$\phi(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x),$$

then ϕ is said to be canonical form if all family of $c_i, i = 1, 2...n$ are distinct and $\{E_i\}_{i=1}^{n}$ are distinct measurable sets.
Note: Any simple function can be expressed in its canonical form by changing the sets $\{E_i\}_{i=1}^{n}$ and constants c_i.
Question: Convert the following simple functions in its canonical form:

1. $\phi(x) = \chi_{[0,1]} + 3\chi_{(1,2]} + \chi_{(2,3]} - \chi_{(3,4]}$
2. $\phi(x) = 2\chi_{[0,3]} + 3\chi_{(1,4]}$

Answer: In problem 1. intervals are disjoint but coefficient are not disjoint so we take union of those intervals whose coefficients are same and check that it will give us canonical form of 1. as,

$$\phi(x) = \chi_{[0,1]\cup(2,3]} + 3\chi_{(1,2]} - \chi_{(3,4]}$$

In problem 2. coefficients are disjoint but intervals are not, so find coefficient of common interval by taking sum of coefficients of intersecting intervals, Here common interval is $(1, 3]$ and its coefficient will become $2 + 3 = 5$, we get connonical form of problem 2. as,

$$\phi(x) = 2\chi_{[0,1]} + 5\chi_{(1,3]} + 3\chi_{(3,4]}$$

STAGE I : SIMPLE FUNCTIONS

Lebesgue Integral of Simple function:
Let $\phi(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x)$ be a simple function in canonical form then legesgue integral of ϕ is given by

$$\int \phi = \sum_{i=1}^{n} c_i m(E_i)$$

Note: If $E \subset \mathbb{R}^d$ be measurable subset with finite measure then $\phi(x)\chi_E(x)$ is also simple function then

$$\int \phi(x) = \int \phi(x)\chi_E(x)dx.$$

Proposition 2.1: The integral of simple function satisfies following properties:

(I) **Independence of representation:** If $\phi = \sum_{k=1}^{N} a_k \chi_{E_k}$ is any representation of ϕ then

$$\int \phi = \sum_{k=1}^{N} a_k m(E_k).$$

(II) **Linearity:** If ϕ and χ are simple and $a, b \in \mathbb{R}$ then

$$\int (a\phi + b\psi) = a \int \phi + b \int \psi.$$

(III) **Additivity:** If E and F are disjoint subsets of $\mathbb{R}^d$ with finite measure then

$$\int_{E \cup F} f = \int_E f + \int_F f.$$

(IV) **Monotonicity:** If $\phi \leq \psi$ are simple functions then $\int \phi \leq \int \psi$.

(V) **Triangle Inequality:** If ϕ is a simple function then $|\phi|$ is also simple function and

$$|\int \phi| \leq \int |\phi|.$$

Proof:

(I) Let $\phi = \sum_{k=1}^{N} a_k \chi_{E_k}$

Case (i) E_k are disjoint and a_k are not disjoint and non-zero. For each distinct non-zero value a among $\{a_k\}$ we define,

$$E'_a = \cup E_k$$

Where union is taken over indices k such that $a_k = a$ now, E'_a are disjoint so

$$m(E'_a) = \sum_k m(E_k)$$

Then we get,

$$\int \phi = \sum a m(E'_a) = \sum_{k=1}^{N} a_k m(E_k)$$

Case (ii) E_k are not disjoint and a_k are disjoint and non-zero. Then we can refine the decomposition $\cup E_k$ by finding disjoint sets $E_1^*, E_2^*, \cdots, E_n^*$ with proprerty

$$\bigcup_{k=1}^{N} E_k = \bigcup_{j=1}^{n} E_j^*$$

For each j, $a_j^* = \sum a_k$ where sum is over all k such that E_k contains E_j^* so $\phi = \sum_{j=1}^{n} a_j^* \chi_{E_j^*}$ Hence,

$$\int \phi = \sum a_j^* m E_j^* = \sum_{k=1}^{N} a_k m(E_k)$$

Hence, $\int \phi = \sum_{k=1}^{N} a_k m(E_k)$.

(II) Let $\phi = \sum_{k=1}^{N} a_k \chi_{E_k}$ and $\psi = \sum_{k=1}^{M} b_k \chi_{F_k}$
consider $P = \max\{M, N\}$ then

$$\begin{aligned}\int (a\phi + b\psi) &= \sum_{k=1}^{P} [a.a_k m(E_k) + b.b_k m(F_k)] \\ &= a \sum_{k=1}^{N} a_k m(E_k) + b \sum_{k=1}^{M} b_k m(F_k) \\ &= a \int \phi + b \int \psi\end{aligned}$$

(III) Let E and F are disjoint subsets of $\mathbb{R}^d$ with finite measure and observe that

$$\chi_{E \cup F} = \chi_E + \chi_F$$

Let f be a simple function then,

$$\begin{aligned}\int_{E \cup F} f = \int_{\mathbb{R}} f.\chi_{E \cup F} &= \int_{\mathbb{R}} f(\chi_E + \chi_F) \\ &= \int_{\mathbb{R}} f.\chi_E + \int_{\mathbb{R}} f.\chi_F\end{aligned}$$

Hence, $\int_{E \cup F} f = \int_E f + \int_F f$.

(IV) Firstly we will prove that if $\phi \geq 0$ be a simple function then $\int \phi \geq 0$.

Let $\phi \geq 0$ and $\phi(x) = \sum_{i=1}^{N} c_i \chi_{E_i}(x)$ then $c_i \geq 0$ and

$$\int \phi = \sum_{i=1}^{N} c_i m(E_i) \geq 0$$

$\implies \int \phi \geq 0$
Now, $\phi \leq \psi$ are a simple functions then $\psi - \phi \geq 0$ is also simple function, hence $\int \psi - \phi = \int \psi - \int \phi \geq 0$ so we get that $\int \psi \geq \int \phi$. then $\int \phi \leq \int \psi$.

(V) Let ϕ is simple function and $\phi(x) = \sum_{i=1}^{N} c_i\chi_{E_i}(x)$ then $|\phi(x)| = \sum_{i=1}^{N} |c_i|\chi_{E_i}(x)$ and $|\phi|$ is simple function with

$$\begin{aligned} |\int \phi| &= |\sum_{i=1}^{N} c_i m(E_i)| \\ &\leq \sum_{i=1}^{N} |c_i| m(E_i) \\ &= \int |\phi| \end{aligned}$$

Hence $|\int \phi| \leq \int |\phi|$.

STAGE II: Bounded Functions Supported on a Set of Finite Measure:

Support of a function: Let f be a measurable function on $\mathbb{R}^n$ then set of all points where f does not vanish is called support of function

$$\text{supp}(f) = \{x | f(x) \neq 0\}$$

f is supported on a set E, if $f(x) = 0$ whenever $x \notin E$.
Question: Find support of following functions.
$\sin(x)$ and $f(x, y) = x^2 + y^2 - 9$

Lemma 2.2: Let f be a bounded function supported on a set E of finite measure. If $\{\phi_n\}_{n=1}^{\infty}$ is any sequence of simple functions bounded by M supported on E,and with $\phi_n(x) \to f(x)$ for almost every x, then

1. The limit $\lim\limits_{n\to\infty} \int \phi_n$ exists.
2. If $f = 0$ almost everywhere then the limit $\lim\limits_{n\to\infty} \int \phi_n = 0$

For part 1. Let $I_n = \int_E \phi_n$, it is enough to prove that $\{I_n\}_{n=1}^{\infty}$ is cauchy sequence in $\mathbb{R}$.
Let $\epsilon > 0$ and consider

$$|I_n - I_m| = |\int_E \phi_n - \int_E \phi_m| = |\int_E (\phi_n - \phi_m)| \leq \int_E |\phi_n - \phi_m|$$

By Egorov's theorem there exist closed set $A_\epsilon \subset E$ such that $m(E - A_\epsilon) \leq \epsilon$ and $\phi_n \to f$ uniformly converges on A_ϵ, means

$$|\phi_n - \phi_m| < \epsilon, \forall m, n \geq M$$

where M is large natural number. Now splite E into two sets A_ϵ and $E - A_\epsilon$,

$$\begin{aligned} |I_n - I_m| &\leq \int_{A_\epsilon} |\phi_n - \phi_m| + \int_{E-A_\epsilon} |\phi_n - \phi_m| \\ &\leq \int_{A_\epsilon} \epsilon + 2\int_{E-A_\epsilon} M \\ &\leq \epsilon m(A_\epsilon) + 2M\epsilon \end{aligned}$$

RHS tends to zero as $\epsilon \to 0$, hence $|I_n - I_m| \to 0$, as $m, n \to \infty$ and $\epsilon \to 0$. Hence $\{I_n\}_{n=1}^{\infty}$ is cauchy sequence in $\mathbb{R}$ and every cauchy sequence in $\mathbb{R}$ is convergent thats implies $\{I_n\}_{n=1}^{\infty}$ is convergent sequence in $\mathbb{R}$.
Hence, the limit $\lim\limits_{n\to\infty} I_n = \lim\limits_{n\to\infty} \int \phi_n$ exists.
For part 2. Let $f = 0$ almost everywhere with $I_n = \int\limits_E \phi_n$ and $\epsilon > 0$ then

$$|I_n| = |\int\limits_E \phi_n| \leq \int\limits_E |\phi_n|$$

By Egorov's theorem there exist closed set $A_\epsilon \subseteq E$ such that $m(E - A_\epsilon) \leq \epsilon$ and $\phi_n \to 0$ uniformly converges on A_ϵ, means

$$|\phi_n - 0| < \epsilon, \forall n \geq M$$

where M is large natural number. Now splite E into two sets A_ϵ and $E - A_\epsilon$ then,

$$\begin{aligned} |I_n| &= \int\limits_{A_\epsilon} |\phi_n| + \int\limits_{E-A_\epsilon} |\phi_n| \\ &\leq \epsilon m(A_\epsilon) + \int\limits_{E-A_\epsilon} M \\ &= \epsilon m(A_\epsilon) + Mm(E - A_\epsilon) \\ &\leq \epsilon m(A_\epsilon) + M\epsilon \end{aligned}$$

note RHS tends to zero as ϵ tends to zero, hence $I_n \to 0$.
Integration of bounded function supported on set of finite measure: It is defined as

$$\int f(x)dx = \lim_{n\to\infty} \int \phi_n(x)dx.$$

where $\{\phi_n\}$ is sequence of simple functions satisfying

(a) $|\phi_n| \leq M$,

(b) Each function ϕ_n is supported on support of f

(c) $\phi_n \to f$ for almost every x.

Proposition 2.3: Let f and g be bounded function supported on set of finite measure then they satisfies following properties:

(I) **Linearity:** If $a, b \in \mathbb{R}$ then

$$\int(af + bg) = a\int f + b\int g.$$

(II) **Additivity:** If E and F are disjoint subsets of $\mathbb{R}^d$ with finite measure then

$$\int\limits_{E\cup F} f = \int\limits_E f + \int\limits_F f.$$

(III) **Monotonicity:** If $f \leq g$ then $\int f \leq \int g$.

(IV) **Triangle Inequality:** $|f|$ be bounded function supported on set of finite measure and

$$|\int f| \leq \int |f|.$$

All these properties follows by approximation by simple functions and the properties of integral of simple functions given in Proposition 2.1.

Theorem 2.4: Bounded Convergence Theorem: Suppose that $\{f_n\}$ is a sequence of measurable functions that are all bounded by M are supported on set E of finite measure, and $f_n(x) \to f(x)$ almost every x as $n \to \infty$. Then f is measurable, bounded, supported on E for almost every x, and

$$\int |f_n - f| \to 0 \text{ as } n \to \infty$$

Consequently,

$$\int f_n \to f \text{ as } n \to \infty.$$

Proof: $\{f_n\}$ is a sequence of measurable functions and $f_n \to f$ hence f is measurable function then,
By Egorov's theorem there exist closed set $A_\epsilon \subseteq E$ such that $m(E - A_\epsilon) \leq \epsilon$ and $f_n \to f$ uniformly on A_ϵ, means

$$|f_n - f| < \epsilon, \forall n \geq M,$$

where M is large natural number. Now splite E into two sets A_ϵ and $E - A_\epsilon$ then,

$$\begin{aligned} |\int_E (f_n - f)| &\leq \int_E |f_n - f| \\ &= \int_{A_\epsilon} |f_n - f| + \int_{E-A_\epsilon} |f_n - f| \\ &\leq \epsilon m(A_\epsilon) + 2M\epsilon \end{aligned}$$

note RHS tends to zero as ϵ tends to zero, hence $\int_E |f_n - f| \to 0$ as ϵ tends to zero.
Consequently, $\int f_n \to f$ as $n \to \infty$.

Lemma 2.5: Let $f \geq 0$ be bounded function supported on set of finite measure E and $\int f = 0$ then $f = 0$ almost everywhere.

Connection in Riemann Integration and Lebesgue Integration:
Theorem 2.6: Suppose f is Riemann integrable on closed interval $[a, b]$ then f is measurable and

$$\overset{\mathcal{R}}{\int_{[a,b]}} f(x)dx = \overset{\mathcal{L}}{\int_{[a,b]}} f(x)dx$$

[This theorem says that, Every Riemann integrable function is Lebesgue integrable.]

STAGE III: NON-NEGATIVE FUNCTIONS

Let,f be a measurable, non-negative function but not necessarily bounded then Lebesgue integral of f is defined by

$$\int f(x)dx = \sup_g \int g(x)dx,$$

where supremum is taken over all measurable function g such that $0 \leq g \leq f$, and g is bounded and supported on a set of finite measure.

Proposition 2.7: The integral of non-negative measurable functions satisfy following properties:

(I) **Linearity:** If $f, g \geq 0$ and a, b are positive real numbers then

$$\int (af + bg) = a \int f + b \int g.$$

(II) **Additivity:** If E and F are disjoint subsets of $\mathbb{R}^d$ and $f \geq 0$ then

$$\int_{E \cup F} f = \int_E f + \int_F f.$$

(III) **Monotonicity:** If $0 \leq f \leq g$, then $\int f \leq \int g$.

(IV) If g is integrable and $0 \leq f \leq g$ then f is integrable.

(V) If f is integrable ,than $f(x) < \infty$,for almost every x.

(VI) If $\int f = 0$, then $f(x) = 0$ for almost every x.

• Is convergence Theorem holds for non-negative functions?
Here we have to check if $f_n \geq 0$ and $f_n(x) \to f(x)$ for almost every x then

$$\int f_n(x)dx \to \int f(x)dx.$$

Consider function $f(x) = \begin{cases} n & 0 < x < 1/n, \\ 0 & otherwise \end{cases}$ then $f_n(x) \to 0$ for all x, yet $\int f_n(x)dx = 1$ for all n. In this example the limit of the integrals is greater than the integral of the limit function which is $\int f = 0$.

Lemma 2.8 [FATOU'S LEMMA]: Suppose $\{f_n\}$ is a sequence of measurable functions with $f_n \geq 0$. If $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every x then

$$\int f =\leq \liminf_{n\to\infty} \int f_n.$$

proof: Let $0 \leq g \leq f$, where g function which in bounded on supported on set E of finite measure, If we set $g_n(x) = \min(g(x), f_n(x))$, then $g_n(x) \leq f_n$ and $g_n(x) \leq g$.
If $g_n(x) \leq g$ and g is bounded and finitely supported then g_n is bounded and finitely supported and $f_n \to f$ hence $g_n \to g$. Hence by bounded convergence theorem,

$$\lim_{n\to\infty} \int g_n = \int \lim_{n\to\infty} g_n = \int g.$$

Since $g_n \leq f_n$ then $\int g_n \leq \int f_n$ and hence

$$\liminf_{n\to\infty} \int g_n = \lim_{n\to\infty} \int g_n \le \liminf_{n\to\infty} \int f_n$$

then we get, $\int g \le \liminf_{n\to\infty} \int f_n$, now take supremum then we obtain,

$$\int f \le \liminf_{n\to\infty} \int f_n.$$

Corollary 2.9: Suppose f is a non-negative measurable function and $\{f_n\}$ is sequence of non-negative measurable functions with $f_n(x) \le f(x)$ and $f_n(x) \to f(x)$ for almost every x, then

$$\lim_{n\to\infty} \int f_n = \int f.$$

Here $f_n(x) \le f(x)$, almost everywhere then for all n $\int f_n \le \int f$ hence

$$\limsup_{n\to\infty} \int f_n \le \int f$$

by Fatou's lemma,

$$\int f =\le \liminf_{n\to\infty} \int f_n.$$

Hence, $\int f = \lim_{n\to\infty} \int f_n$.
Question: Give an example to show that strict inequality may occure in Fatou's lemma.
Hint: Sequence of function $f_n(x) = \begin{cases} n & 0 < x < 1/n, \\ 0 & otherwise \end{cases}$.

Notation:

(I) $f_n \nearrow f$ means $\{f_n\}_{n=1}^{\infty}$ is sequence of measurable functions that satisfies $f_n(x) \le f_{n+1}(x)$ for almost every x, and for all $n \in \mathbb{R}$ with $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every x.

(II) $f_n \searrow f$ means $\{f_n\}_{n=1}^{\infty}$ is sequence of measurable functions that satisfies $f_n(x) \ge f_{n+1}(x)$ for almost every x and for all $n \in \mathbb{R}$ with $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every x.

Theorem 2.10 [Monotone Convergence Theorem]: Suppose $\{f_n\}$ is a sequence of non-negative measurable functions with $f_n \nearrow f$, then

$$\lim_{n\to\infty} \int f_n = \int f = \int \lim_{n\to\infty} f_n.$$

Given $f_n \ge 0$, is sequence of non-negative measurable function and $\{f_n\}$ is monotonically increasing sequence converging to f with

$$0 \le f_1 \le f_2 \le \cdots \le f$$

hence $f_n \le f$ then by considering limsup of their integrals on both sides we get,

$$\limsup_{n\to\infty} \int f_n \le \limsup_{n\to\infty} \int f \le \int f.$$

Now by Fatou's lemma we know that, $\int f \le \liminf_{n\to\infty} \int f_n$. From these two inequalities we can observe that;

$$\int f = \liminf_{n\to\infty} \int f_n = \limsup_{n\to\infty} \int f_n = \lim_{n\to\infty} \int f_n.$$

Hence, $\lim_{n\to\infty} \int f_n = \int f = \int \lim_{n\to\infty} f_n$.

Corollary 2.11: Consider a series $\sum_{k=1}^{\infty} a_k(x)$, where $a_k(x) \geq 0$ is measurable for every $k \geq 1$, then

$$\int \sum_{k=1}^{\infty} a_k(x)dx = \sum_{k=1}^{\infty} \int a_k(x)dx.$$

If $\sum_{k=1}^{\infty} \int a_k(x)dx$ is finite, then series $\sum_{k=1}^{\infty} a_k(x)dx$ converges for almost every x.

STAGE IV: GENERAL CASE

Let f be real valued any measurable function on $\mathbb{R}^d$, f is lebesgue integrable if the non-negative measurable function $|f|$ is integrable.

Definition: $f^+(x) = \max(f(x), 0)$ and $f^-(x) = \max(-f(x), 0)$

Note that both functions f^+ and f^- are non-negative functions and observe that we can express functions f and $|f|$ as

$$f = f^+ - f^- \text{ and } |f| = f^+ + f^-.$$

• If f and g are measurable functions then $\int f + g = \int f + \int g$.

Proposition 2.12: The integral of Lebesgue integrable functions is linear, additive, monotonic, and satisfies the triangle inequality.

Proposition 2.13: Suppose f is integrable function on $\mathbb{R}^d$, then for every $\epsilon > 0$

(i) There exist a set of finite measured a ball B such that

$$\int_{B^c} |f| < \epsilon.$$

(ii) There is a $\delta > 0$ such that

$$\int_E |f| < \epsilon. \qquad \text{whenever } m(E) < \delta.$$

Proof: Without loss of generality, assume that $f \geq 0$.

For part(i) Let B_N is ball centered at origin of radius N, and note that if $f_N(x) = f(x)\chi_{B_N}(x)$, then $f_N \geq 0$ is measurable, $f_N(x) \leq f_{N+1}(x)$, and $\lim_{n\to\infty} f_N(x) = f(x)$.

By monotone convergence theorem, we must have

$$\lim_{N\to\infty} \int f_N = \int f.$$

In particular, for some large N,

$$0 \leq \int f - \int f\chi_{B_N}\epsilon,$$

and since $1 - \chi_{B_N} = \chi_{B_N^c}$, this implies $\int_{B_N^c} f < \epsilon$.

For part (ii), Without loss of generality, assume that $f \geq 0$ and $\epsilon > 0$.

define, $E_N = \{x|f(x) \leq N\}$ and $E_N \subseteq E_{N+1}$ and define

$$f_N(x) = f(x)\chi_{E_N}(x)$$

then $f_N(x) \leq f_{N+1}(x)$. For given $\epsilon > 0$, there is an integer N such that,

$$\int (f - f_N) < \frac{\epsilon}{2}.$$

We now choose $\delta > 0$ so that $N\delta < \epsilon/2$, then

$$\begin{aligned}\int_E f &= \int_E (f - f_N) + \int_E f_N \\ &\leq \int_E (f - f_N) + \int_E f_N \\ &\leq \int_E (f - f_N) + Nm(E) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

This proves the given proposition.

Proposition 2.14: [Dominated Convergence Theorem] Suppose $\{f_n\}$ is sequence of measurable functions such that $f_n(x) \to f(x)$ almost everywhere on x, as $n \to \infty$. If $|f_n(x)| \leq g(x)$, where g is integrable, then

$$\int |f_n - f| \to 0 \text{ as } n \to \infty$$

and consequently

$$\int f_n \to \int f \text{ as } n \to \infty.$$

Proof: For each $N > 0$ let $E_N = \{x : |x| \leq N, g(x) \leq N\}$. Given $\epsilon > 0$, by previous lemma, there exist N so that $\int_{B^c} g < \epsilon$, then the function $f_n\chi_{E_N}$ are bounded and supported on set of finite measure, so that by bounded convergence theorem ,

$$\int_{E_N} |f_n - f| < \epsilon \text{ for all large } n,$$

Hence, we get

$$\begin{aligned}\int_{E_N} |f_n - f| &= \int_{E_N} |f - f_N| + \int_{E_N^c} |f - f_N| \\ &\leq \int_{E_N} |f - f_N| + 2\int_{E_N^c} g \\ &\leq \epsilon + 2\epsilon \\ &\leq 3\epsilon \text{ for all large n.}\end{aligned}$$

Hence, $\int |f_n - f| \to 0$ as $n \to \infty$
Consequently, $\int f_n \to \int f$ as $n \to \infty$.

The space $\mathcal{L}^1$ of integrable functions: Definition: Let f be an $\mathcal{L}$-integrable function on $\mathbb{R}^d$ and norm is defined as

$$||f||_{\mathcal{L}^1} = \int_{\mathbb{R}^d} |f(x)|dx.$$

The space of all Lebesgue integrable functions with the above finite norm is called as $\mathcal{L}^1(\mathbb{R}^d)$.
• Equivaient: Two functions are equivalent if they agree almost everywhere.
• $\mathcal{L}^1(\mathbb{R}^d)$ is the space of equivalence classes of integrable functions. Because $f = g$ almost everywhere then $||f|| = ||g||$
• Integrable functions form a vector space.
Question: Find norm $||f||_{\mathcal{L}^1}$ of following function:
Let $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \dfrac{1}{1+x^2}$.
Proposition 2.15: Suppose f and g are two functions in $\mathcal{L}^1(\mathbb{R}^d)$ then they satisfies following properties:

(i) $||af||_{\mathcal{L}^1(\mathbb{R}^d)} = |a|.||f||_{\mathcal{L}^1(\mathbb{R}^d)}$

(ii) $||f+g||_{\mathcal{L}^1(\mathbb{R}^d)} \leq ||f||_{\mathcal{L}^1(\mathbb{R}^d)} + ||g||_{\mathcal{L}^1(\mathbb{R}^d)}$

(iii) $||f||_{\mathcal{L}^1(\mathbb{R}^d)} = 0$ if and only if $f = 0$ almost everywhere.

(iv) $d(f,g) = ||f-g||_{\mathcal{L}^1(\mathbb{R}^d)}$ defines a metric on $\mathcal{L}^1(\mathbb{R}^d)$

Complete Metric Space: A metric space (X, d) is said to be a complete if every cauchy sequence in x converges in X.

Theorem 2.16 (RIESZ FISCHER THEOREM) : The vector space $\mathcal{L}^1$ is complete in its metric.
Proof: Let $\{f_n\}$ be a cauchy sequence in $\mathcal{L}^1$. Hence by definition of cauchy sequence, for given $\epsilon > 0$ there exist N such that

$$||f_n - f_m|| < \epsilon \text{ for large } m, n$$

Consider subsequence $\{f_{n_k}\}$ of $\{f_n\}$ with

$$||f_{n_{k+1}} - f_{n_k}|| \leq 2^{-k}$$

Define

$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty}[f_{n_{k+1}(x)} - f_{n_k(x)}] \text{ and}$$
$$g(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty}|f_{n_{k+1}(x)} - f_{n_k(x)}|$$

then note that $|f(x)| \leq g(x)$. and $\int g < \infty$. hence by dominated convergence theorem, we get

$$\int |f| \leq \int g < \infty$$

Hence $\int f < \infty$ and f is integrable function. Now consider,

$$\begin{aligned} |f - f_{n_k}| &= |f_{n_1}(x) + (\sum_{k=1}^{\infty}[f_{n_{k+1}(x)} - f_{n_k(x)}]) - f_{n_k}| \\ &\leq |f_{n_1}(x)| + \sum_{k=1}^{\infty}|f_{n_{k+1}(x)} - f_{n_k(x)}| \\ \therefore |f - f_{n_k}| &\leq g \end{aligned}$$

then by deminated convergence theorem, as $k \to 0$, $|f - f_{n_k}| \to 0$, Therefore $||f_{n_k} - f|| \to 0$,

Hence $f_{n_k} \to f$ in $\mathcal{L}^1$. Simillarly we can show that $f_n \to f$ in $\mathcal{L}^1$. So here we proved every cauchy sequence is convergent in in $\mathcal{L}^1$, hence it is complete metric space.

• Families of simple functioons, step functions, continuous functions of compact support are dense in $\mathcal{L}^1$.

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