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Department of Mathematics

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Subject: REAL ANALYSIS

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Chapter 1. MEASURE THEORY

Preliminaries

A point $x \in \mathbb{R}^d$ consists of a d-tuple of real numbers

$$x = (x_1, x_2, \cdots, x_d)\text{, where each } x_i \in \mathbb{R}\text{, for } i = 1, 2, \cdots, d.$$

NOTE:

1. For $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}$
   Let, $x = (x_1, x_2, \cdots, x_d)$ and $y = (y_1, y_2, \cdots, y_d)$, addition, subtraction and scalar multiplication are defined as

   $$\begin{aligned} x + y &= (x_1 + y_1, x_2 + y_2, \cdots, x_d + y_d), \\ x - y &= (x_1 - y_1, x_2 - y_2, \cdots, x_d - y_d), \\ \alpha x &= (\alpha x_1, \alpha x_2, \cdots, \alpha x_d). \end{aligned}$$

   So, addition, subtraction and scalar multiplication are defined componentwise.

2. The norm of x is denoted by $|x|$ and is defined by

   $$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}.$$

3. The distance between two points x and y in $\mathbb{R}^d$ is denoted by $|x - y|$ and defined as

   $$|x - y| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_d - y_d)^2}.$$

DEFINITIONS
Let E is subset of $\mathbb{R}^d$

1. **Compliment of a set:**
   The complement of set E in $\mathbb{R}^d$ is denoted by E^c and defined by

   $$E^c = \left\{x \in \mathbb{R}^d | x \notin E\right\}.$$

   If E and F are two subsets of $\mathbb{R}^d$, we denote the compliment of F in E by $E - F$ and defined by

   $$E - F = \left\{x \in \mathbb{R}^d | x \in E \text{ and } x \notin F\right\}.$$

   Distance between two set E and F defined by

   $$d(E, F) = \inf |x - y|,$$

   where, infimum is taken over all $x \in E, y \in F$.

2. **Open ball in $\mathbb{R}^d$:**
   Open ball in $\mathbb{R}^d$ centered at x and of radius r is defined by

$$B(x,r) = \left\{y \in \mathbb{R}^d | \ |y - x| < r\right\}.$$

3. **Open set in $\mathbb{R}^d$:**
   A subset $E \subset \mathbb{R}^d$ is open set if for every $x \in E$ there exist $r > 0$ such that $B(x,r) \subset E$.
   E is a closed subset of $\mathbb{R}^d$ if E^c is open subset of $\mathbb{R}^d$.

4. **Bounded set in $\mathbb{R}^d$:**
   A set E is bounded if it is contained in some ball of finite radius.

5. **Compact set in $\mathbb{R}^d$:**
   A set E is compact if and only if E is closed and bounded in $\mathbb{R}^d$.

6. **Limit point in $\mathbb{R}^d$:**
   A point $x \in \mathbb{R}^d$ is a limit point of the set E if for every $r > 0$, the ball $B(x,r)$ contains points of E other than x.

7. **Isolated point in $\mathbb{R}^d$:**
   A point $x \in E$ is an isolated point of set E if there exists an $r > 0$, where $B(x,r) \cap E = \{x\}$.

8. **Interior point in $\mathbb{R}^d$:**
   A point $x \in E$ is an interior point of set E if there exist $r > 0$ such that $B(x,r) \subset E$.
   Set of all points in E is denoted by E^o.
   A point x is exterior point of a set E if there exist $r > 0$, where $B(x,r) \cap E = \phi$.

9. **Closure of set E:**
   It is denoted by $\overline{E}$ and defined as, union of set E and its limit points.

10. **Boundary of set E:**
    It is denoted by δE and defined as set of points which are in closure of E but not in interior of E.

$$\delta E = \overline{E} - E^o$$

11. **Perfect Set:**
    A closed set E is perfect if E does not have any isolated points.
    NOTE:
    (a) Closure of set is closed set.
    (b) Set E is closed if and only if it contains all its limit points.

EXAMPLES:

1. For set $E = \mathbb{Q} \cap \mathbb{R}$, Here set E is set of rationals.
   Find limit points, interior points, closure of E, boundary of E, is E closed?
   Answer : Set of limit points of E is $\mathbb{R}$, it has no interior points, closure of E is $\mathbb{R}$, boundary of E is $\mathbb{R}$, E is not closed because it does not contains all its limit points.

2. Let $E = \mathbb{Z}$.
   Find limit points, interior points, closure of E, boundary of E, Is E closed?

3. Let $S = \{(x, y)\,|y > x^2\}$
   Is S bounded? Find boundary points of S and closure of S.
   Answer: Set S is not bounded,
   Boundary of $S = \delta S = \{(x, y)\,|y = x^2\}$
   Closure of $S = \overline{S} = \{(x, y)\,|y \geq x^2\}$

4. Let $S = \{(x, y)\,|y > 1/x\}$
   Is S bounded?, Find boundary points of S and closure of S.

5. Let $S = \{(x, \sin x)\,|x \in [0, \pi]\}$
   Is S open? Is S bounded? Is S compact?

6. Show that set of integers is closed.
   Answer: To show set $\mathbb{Z}$ is closed it is enough to show its complement $\mathbb{Z}^c$ is open.
   $\mathbb{Z}^c = \cdots \cup (-2, -1) \cup (-1, 0) \cup (0, 1) \cup \cdots$
   As countable union of open sets is open.
   Hence $\mathbb{Z}^c$ is open. Implies $\mathbb{Z}$ is closed set.

THE CANTOR SET:
Cantor set is a set of real numbers in $[0, 1]$ whose ternary expansion contains either 0 or 2.
CONSTRUCTION OF THE CANTOR SET :
STEP 0: Consider closed unit interval $C_0 = [0, 1]$.
STEP 1: Let C_1 denote the set obtained from deleting the middle one third open interval from $[0, 1]$.
Hence, $C_1 = [0, 1/3] \cup [2/3, 1]$.
STEP 2: Repeat this proces s for each subinterval in C_1.
We get $C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$.
STEP 3: Repeat this process for each subinterval in C_2 and so on. So This

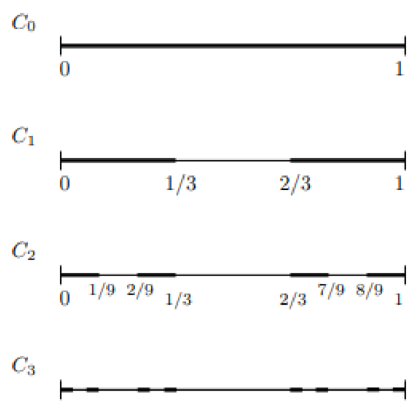

Figure 1: Construction of Cantor set

process gives us a sequence $C_k, k = 0, 1, 2, \cdots$ of compact sets with

$$C_0 \supset C_1 \supset C_2 \supset \cdots C_k \supset C_{k+1} \supset \cdots$$

The cantor set C is defined as intersection of all $C_k's$:

$$C = \bigcap_{k=0}^{\infty} C_k$$

The Cantor set C is non empty, because all end points of intervals in C_k belongs to C.
As C is closed and bounded , Hence compact.
C_k is disjoint union of 2^k intervals of length 3^{-k}, hence total length of cantor set is $(2/3)^k$ and $(2/3)^k \longrightarrow 0$ as $k \longrightarrow 0$.
Roughly, Cantor set has length 0.
Rectangles: A closed rectangle R in $\mathbb{R}^d$ is given by the product of d one dimensional closed and bounded intervals

$$R = [a_1, b_1] \times [a_2, b_2] \times ... \times [a_d, b_d],$$

where $a_j \leq b_j$ are real numbers, $j = 1, 2, \cdots, d$
Cube: It is rectangle in which all sides are same.
i.e. $b_1 - a_1 = b_2 - a_2 = b_3 - a_3 = ... = b_d - a_d = l$
Volume of Rectangle R: It is denoted by $|R|$, and defined to be

$$|R| = (b_1 - a_1)(b_2 - a_2) \cdots (b_d - a_d).$$

1. Volume of rectangle in $\mathbb{R}$ is nothing but length of interval. If $R = [a, b]$ then $|R| = b - a$
2. Volume of rectangle in $\mathbb{R}^2$ is equal to area of that rectangle. If $R = [a, b] \times [x, y]$ then $|R| = (b - a)(y - x)$.
3. Volume of rectangle in $\mathbb{R}^3$ is equal to volume of that parallelogram in $\mathbb{R}^3$.
4. If $Q \subset \mathbb{R}^d$ is cube of common side length l then $|Q| = l^d$.

Almost Disjoint : A union of rectangles is said to be almost disjoint if interior of rectangles are disjoint.
Interior of rectangle $R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$ is $(b_1 - a_1) \times (b_2 - a_2) \times \cdots \times (b_d - a_d)$
Lemma 1.1: If a rectangle is almost disjoint union of finitely many other rectangles, say $R = \bigcup_{k=1}^{N} R_k$ then

$$|R| = \sum_{k=1}^{N} |R_k|.$$

Proof: Consider the grid formed by extending infinitely the sides of all rectangles $R_1, R_2, ..., R_N$.
So that we get finitely many rectangles $\tilde{R}_1, \tilde{R}_2, \cdots, \tilde{R}_M$ and partition $J_1, J_2, \cdots, J_N$ of integers between 1 and M such that below unions are almost disjoint.

$$R = \bigcup_{j=1}^{M} \tilde{R}_j \quad \text{and} \quad R_k = \bigcup_{j \in J_k} \tilde{R}_j, \quad \text{for } k = 1, 2, 3, \cdots, N.$$

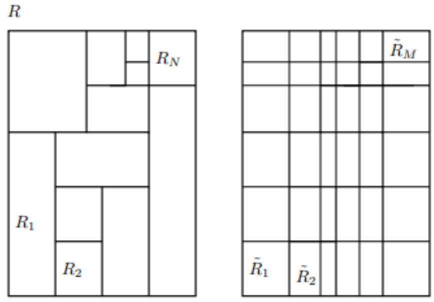

Figure 2: The grid formed by rectangles R_k

Hence, $|R| = \sum_{j=1}^{M} |\tilde{R}_j| = \sum_{j=1}^{N} \sum_{j \in J_k} |\tilde{R}_j| = \sum_{k=1}^{N} |R_k|.$

Lemma 1.2: If $R_1, R_2, \cdots, R_N$ are rectangles, and $R \subset \bigcup_{k=1}^{N} R_k$ then

$$|R| \leq \sum_{k=1}^{N} |R_k|.$$

Proof: We know that union of rectangles need not be a rectangle (As below

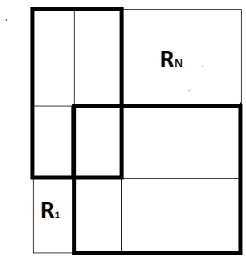

Figure 3:

diagram). Now, Extend the sides of all rectangles $R_1, R_2, ..., R_N$ so that we get grid and finitely many rectangles $\tilde{R}_1, \tilde{R}_2, \cdots, \tilde{R}_M$ and partition $J_1, J_2, \cdots, J_M$ of integers 1 and M such that

$$R_i = \bigcup_{j \in J_k} \tilde{R}_j \text{ , } J_k \subset \{1, 2, ..., M\} \text{ and } i \in \{1, 2, ..., N\}$$

Let $R \subset \bigcup_{k=1}^{N} R_k$ then $\widetilde{\widetilde{\widetilde{R_j}}} = R \cap R_j$ where $\widetilde{\widetilde{\widetilde{R_j}}}$ is some rectangle among $\tilde{R}_1, \tilde{R}_2, ..., \tilde{R}_M$.

Hence $R = \bigcup_{j=1}^{N} \widetilde{\widetilde{\widetilde{R_j}}}$

This implies R is almost disjoint union of finitely many other rectangles.

Hence , By Lemma 1.1, $|R| = \sum_{j=1}^{N} |\widetilde{\widetilde{R_j}}| \leq \sum_{j=1}^{N} |R_j|$.

Hence $|R| \leq \sum_{j=1}^{N} |R_j|$.

THEOREM 1.3: Every open subset $\mathcal{O}$ of $\mathbb{R}$ can be written as countable union of disjoint open intervals.

Proof: Let $\mathcal{O}$ is open subset of $\mathbb{R}$, for each $x \in \mathcal{O}$. Since, $\mathcal{O}$ is open, x is contained in some small interval.

Suppose that, I_x denote the largest open interval containing x and contained in $\mathcal{O}$. If

$$a_x = \inf \{a < x | (a, x) \subset \mathcal{O}\} \quad \text{and} \quad b_x = \sup \{x < b | (x, b) \subset \mathcal{O}\}$$

Therefore, $a_x < x < b_x$, $I_x = (a_x, b_x)$ this is our required largest interval containing x and contained in $\mathcal{O}$

Hence

$$\mathcal{O} = \bigcup_{x_j \in \mathcal{O}} I_{x_j}$$

claim 1: $I_x = I_y$ or $I_x \cap I_y = \phi$, for $x, y \in \mathcal{O}$.

Suppose that $I_x \cap I_y \neq \phi$ and $x \in I_x \cap I_y$ then $x \in I_x \cup I_y$ and $I_x \cup I_y \subset \mathcal{O}$, since I_x is maximal, we must have $I_x = (I_x \cup I_y)$. Similarly $I_y = (I_x \cup I_y)$, Hence $I_y = I_x$ or $I_x \cap I_y = \phi$

claim 2: $\{I_x\}_{x \in \mathcal{O}}$ is countable.

We know that $\mathcal{O} = \bigcup_{x \in \mathcal{O}} I_x = \bigcup_{r_x \in I_x \in \mathcal{O}} I_x$, where r_x are rational numbers in I_x.

But rational numbers are countable, hence union contains countable intervals.

THEOREM 1.4 : Every open subset $\mathcal{O}$ of $\mathbb{R}^d, d \geq 1$ can be written as countable union of almost disjoint closed cubes .

Proof: Let $\mathcal{O}$ is open subset of $\mathbb{R}^d, d \geq 1$

claim: There is countable collection $\mathcal{A}$ of closed cubes whose interiors are disjoint and $\mathcal{O} = \bigcup_{Q \in \mathcal{A}} Q$.

Procedure for collecting closed cubes of to form $\mathcal{A}$ is as follows:

1. Consider grid in $\mathbb{R}^d$ formed by taking all closed cubes of side length 1 whose vertices have integer co-ordinates.
2. Accept cubes as part of $\mathcal{A}$ if cube Q is entirely contained in $\mathcal{O}$.
3. Tentatively accept cube Q if it intersect both $\mathcal{O}$ and $\mathcal{O}^c$.
4. Reject cube Q if it is entirely contained in $\mathcal{O}^c$.
5. Bisect tentatively accepted cubes into 2^d cubes of side length $1/2$.
6. Then accept those smaller cubes or reject them or tentatively accept them as earlier.
7. Repeat this procedure infinitely many times, we get collection $\mathcal{A}$ of accepted cubes Q.

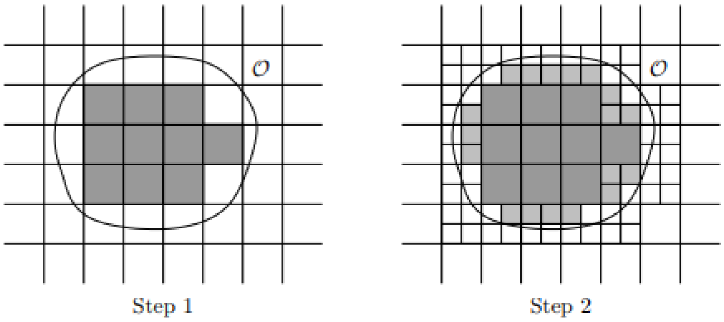

Figure 4: Decomposition of $\mathcal{O}$ into almost disjoint cubes

As every cube contains a point with rational co-ordinates implies that collection $\mathcal{A}$ is countable and consists of almost disjoint cubes.
Claim: $\mathcal{O} = \bigcup_{Q\in\mathcal{A}} Q$

We are considering cube Q contained in $\mathcal{O}$ then $\mathcal{O} \supset \bigcup_{Q\in\mathcal{A}} Q$ [1]

Now, Let $x \in \mathcal{O}$, there exists a cube of side length 2^{-N} containing x and that is entirely contained in $\mathcal{O}$. hence $\mathcal{O} \subset \bigcup_{Q\in\mathcal{A}} Q$ [2]

Hence from [1] and [2], $\mathcal{O} = \bigcup_{Q\in\mathcal{A}} Q$

Exterior Measure:
Definition: Let $E \subset \mathbb{R}^d$, the exterior measure of E is denoted by m_*E and defined as $m_*E = \inf \sum_{j=1}^{\infty} |Q_j|$, where infimum is taken over all countable covering $E \subset \cup_{j=1}^{\infty} Q_j$ by closed cubes in $\mathbb{R}^d$.
Note: $0 \leq m_*E \leq \infty$
EXAMPLE 1: The exterior measure of a point in $\mathbb{R}$ is zero.
Let, $E = \{x\}, x \in \mathbb{R}$ and $r > 0$, $\{x\} \subset [x-r, x+r] = Q$ then, $m_*E = \inf\{2r | E \subseteq Q\}$ where, infimum taken over non-negative $2r$, Hence $m_*E = 0$ as r tends to 0.
EXAMPLE 2: The exterior measure of empty set is zero.
EXAMPLE 3: The exterior measure of a point in $\mathbb{R}^d$ is zero.
EXAMPLE 4: The exterior measure of closed cube is equal to its volume in $\mathbb{R}^d$.
Let $Q = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$ be a closed cube in $\mathbb{R}^d$, then $(b_1 - a_1) = (b_2 - a_2) = \cdots = (b_d - a_d) = k$, for some constant k. Hence volume of cube $= |Q| = k^d$. Q covers itself, we must have $m_*Q \leq |Q|$. Therefore, it suffices to prove the reverse inequality.

Consider an arbitrary covering $Q \subset \bigcup_{j=1}^{\infty} Q_j$ by cubes, and note that it suffices to prove that

$$|Q| \leq \sum_{j=1}^{\infty} |Q_j|$$

For a fixed $\epsilon > 0$ we choose for each j an open cube S_j which contains Q_j, and such that

$$|S_j| \leq (1+\epsilon)\,|Q_j|.$$

From the open covering $\bigcup_{j=1}^{\infty} S_j$ of the compact set Q, we may select a finite sub-covering like $Q \subset \bigcup_{j=1}^{N} S_j$. Taking the closure of the cubes S_j, we may apply Lemma 1.2 to conclude $|Q| \leq \sum_{j=1}^{\infty} |S_j|$

Hence,

$$|Q| \leq (1+\epsilon)\sum_{j=1}^{N}|Q_j| \leq (1+\epsilon)\sum_{j=1}^{\infty}|Q_j|$$

.

Since ϵ is arbitrary ,we get $m_*Q \geq |Q|$.
EXAMPLE 5: The exterior measure of $\mathbb{R}$ is infinite.
EXAMPLE 6: If Q is an open cube then $m_*(Q) = |Q|$.
As Q is covered by its closure $\bar{Q}$ and $|\bar{Q}| = |Q|$, implies $m_*(Q) \leq |Q|$. To prove reverse inequality, If $Q_0 \subseteq Q$ and Q_0 is closed cube then $m_*(Q_0) \leq m_*(Q)$. Since any covering of Q by a countable number of closed cubes is also a covering of Q_0, hence $|Q_0| \leq m_*(Q)$.
EXAMPLE 7: The exterior measure of rectangle R is equal to its volume.
EXAMPLE 8: The exterior measure of $\mathbb{R}$ is infinite.
EXAMPLE 9: Cantor set has exterior measure zero.
Remark: For every $\epsilon > 0$, there exists a covering $E \subset \bigcup_{j=1}^{\infty} Q_j$ with

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon.$$

Properties of the exterior measure:
Property 1: (Monotonicity) If $E_1 \subseteq E_2$ then $m_*(E_1) \leq m_*(E_2)$.
Let $\{Q_j\}$ be a covering of E_2 by closed cubes,

$$E_1 \subseteq E_2 \subseteq \bigcup_{j=1}^{\infty} Q_j.$$

Hence $\{Q_j\}$ is also covering of E_1, implies $m_*(E_1) \leq \sum_{j=1}^{\infty} |Q_j|$. Now taking infimum on RHS of above inequality over all such $\{Q_j\}$ covering E_2 we get,

$$m_*(E_1) \leq \inf\left\{ \sum_{j=1}^{\infty} |Q_j| | E_2 \subseteq \bigcup_{j=1}^{\infty} Q_j \right\} \leq m_*(E_2)$$

Hence $m_*(E_1) \leq m_*(E_2)$

Property 2: (Countable sub-additivity) If $E = \bigcup_{j=1}^{\infty} E_j$, then $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$.

First, we may assume that each $m_*(E) < \infty$, for otherwise the inequality clearly holds. For any $\epsilon > 0$, the definition of the exterior measure yields for each j a covering $E_j \subset \bigcup_{j=1}^{\infty} Q_{k,j}$ by closed cubes with

$$\sum_{j=1}^{\infty} |Q_j| \leq m_*(E_j) + \frac{\epsilon}{2^j}.$$

. Then, $E \subset \bigcup_{j,k=1}^{\infty} Q_{k,j}$ is a covering of E by closed cubes, and therefore

$$m_*(E) \leq \sum_{j,k} |Q_{k,j}| = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |Q_{k,j}|$$

Since this holds true for every $\epsilon > 0$, the second observation is proved.

Property 3: If $E \subset R^d$, then $m_*(E) = \inf m_*(\mathcal{O})$, where the infimum is taken over all open sets $\mathcal{O}$ containing E.

By monotonicity on $E \subset \mathcal{O}$, it is clear that the inequality $m_*(E) \leq \inf m_*(\mathcal{O})$ holds. For the reverse inequality, let $\epsilon > 0$ and choose cubes Q_j such that $E \subset \bigcup_{j=1}^{\infty} Q_j$, with

$$\sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \frac{\epsilon}{2}.$$

Let Q_j^0 denote an open cube containing Q_j, and such that $|Q_j^0| \leq |Q_j| + \epsilon/2^{j+1}$. Then $\mathcal{O} = \bigcup_{j=1}^{\infty} Q_j^0$ is open, and by property 2

$$\begin{aligned} m_*(\mathcal{O}) &\leq \sum_{j=1}^{\infty} m_*(Q_j^0) \\ &= \sum_{j=1}^{\infty} |Q_j^0| \\ &\leq \sum_{j=1}^{\infty} (|Q_j| + \epsilon/2^{j+1}) \\ &\leq \sum_{j=1}^{\infty} |Q_j| + \epsilon/2 \\ &\leq m_*(E) + \epsilon \end{aligned}$$

. Hence $\inf m_*(\mathcal{O}) \leq m_*(E)$.

Property 4: If $E = E_1 \cup E_2$, and $d(E_1, E_2) > 0$, then $m_*(E) = m_*(E_1) + m_*(E_2)$.

By property 2, we know that, $m_*(E) \leq m_*(E_1) + m_*(E_2)$, so it suffices to prove the reverse inequality. We first select δ such that $d(E_1, E_2) > \delta > 0$ and choose a covering $E \subset \bigcup_{j=1}^{\infty} Q_j$ by closed cubes, with

$$\sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \epsilon.$$

Now subdividing the cubes Q_j, assume that each Q_j has a diameter less than δ. In this case, each Q_j can intersect at most one of the two sets E_1 or E_2. If we denote by J_1 and J_2 the sets of those indices j for which Q_j intersects E_1 and E_2 respectively, then $J_1 \cap J_2$ is empty, and we have $E_1 \subset \bigcup_{j \in J_1} Q_j$ as well as $E_2 \subset \bigcup_{j \in J_2} Q_j$. Therefore,

$$\begin{aligned} m_*(E_1) + m_*(E_2) &\leq \sum_{j \in J_1} |Q_j| + \sum_{j \in J_2} |Q_j| \\ &\leq \sum_{j=1}^{\infty} |Q_j| \\ &\leq m_*(E) + \epsilon. \end{aligned}$$

Since ϵ is arbitrary, we get $m_*(E_1) + m_*(E_2) \leq m_*(E)$
. **Property 5:** If a set E is the countable union of almost disjoint cubes $E = \bigcup_{j=1}^{\infty} Q_j$, then

$$m_*(E) = \sum_{j=1}^{\infty} |Q_j|.$$

. By property 2, $m_*(E) \leq \sum_{j=1}^{\infty} m_*(Q_j)$, here Q_j are closed cubes, so $m_*(Q_j) = |Q_j|$ hence,

$$m_*(E) \leq \sum_{j=1}^{\infty} |Q_j|$$

For reverse inequality consider $\epsilon > 0$ and $\widetilde{Q_j}$ be a cube strictly contained in Q_j such that

$$|Q_j| \leq |\widetilde{Q_j}| + \frac{\epsilon}{2^j}$$

where ϵ is arbitrary but fixed. Then, for every N, the cubes $\widetilde{Q_1}, \widetilde{Q_2}, \cdots, \widetilde{Q_N}$ are disjoint, hence at a finite distance from one another, and repeated applications of Property 4 imply

$$\begin{aligned} m_*(\bigcup_{j=1}^{N} \widetilde{Q_j}) &= \sum_{j=1}^{N} m_*(\widetilde{Q_j}) \\ &= \sum_{j=1}^{N} |\widetilde{Q_j}| \\ &\geq \sum_{j=1}^{N} \left\{ |Q_j| - \frac{\epsilon}{2^j} \right\}. \end{aligned}$$

As limit N tends to infinity we get, $m_*(\bigcup_{j=1}^{\infty} \widetilde{Q_j}) \geq \sum_{j=1}^{\infty} |Q_j| - \epsilon$ and monotonicity on $\bigcup_{j=1}^{\infty} \widetilde{Q_j} \subset E$ gives, $m_*(E) + \epsilon \geq \sum_{j=1}^{\infty} |Q_j|$
So as $\epsilon \to 0$ we get,

$$m_*(E) \geq \sum_{j=1}^{\infty} |Q_j|.$$

Measurable Sets and the Lebesgue Measure:

Definition: A subset E of $\mathbb{R}^d$ is **Lebesgue measurable** or **measurable**, if for any $\epsilon > 0$ there exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m_*(\mathcal{O} - E) \leq \epsilon$.
If E is measurable, then $m(E) = m_*(E)$.

Property 1: Every open set in $\mathbb{R}^d$ is measurable.
Hint: Consider $\mathcal{O} = E$.

Property 2: If $m_*(E) = 0$ then E is measurable. In particular, if F is a subset of a set of exterior measure 0, then F is measurable.
We know for any $\epsilon > 0$ there exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m_*(\mathcal{O} - E) \leq \epsilon$.

Property 3: A countable union of measurable sets is measurable.
Let $E_1, E_2, \cdots$ are measurable sets and $E = \bigcup_{j=1}^{\infty} E_j$. Given $\epsilon > 0$ there exist open set $\mathcal{O}_j$ with $E_j \subset \mathcal{O}_j$ and $m_*(\mathcal{O}_j - E_j) \leq \epsilon/2^j$. Then the union $\mathcal{O} = \bigcup_{j=1}^{\infty} \mathcal{O}_j$ is open, as $\bigcup_{j=1}^{\infty} E_j \subset \bigcup_{j=1}^{\infty} \mathcal{O}_j$ hence $E \subset \mathcal{O}$ and $(\mathcal{O} - E) \subset \bigcup_{j=1}^{\infty} (\mathcal{O}_j - E_j)$,
so monotonicity and sub-additivity of the exterior measure imply,

$$m_*(\mathcal{O} - E) \subset \sum_{j=1}^{\infty} m_*(\mathcal{O}_j - E_j) \leq \epsilon.$$

Hence, E is measurable set.

Property 4: Closed sets are measurable.
Every closed set F can be written as the union of compact sets, say
$F = \bigcup_{j=1}^{\infty}(F \cap B_k)$, where B_k denotes the closed ball of radius k centered at the origin; so, it is enough to prove every compact set is measurable.
So, suppose F is compact set, let $\epsilon > 0$ then there exist open set $\mathcal{O}$ with $F \subset \mathcal{O}$ and $m_*(\mathcal{O}) \leq m_*(F) + \epsilon$. Since F is closed, $\mathcal{O} - F$ is open, and by Theorem 1.4 we may write this difference as a countable union of almost disjoint closed cubes. Hence, $\mathcal{O} - F = \bigcup_{j=1}^{\infty} Q_j$. Now for a fixed $N \in \mathbb{N}$, the finite union $K = \bigcup_{j=1}^{N} Q_j$ is compact; therefore $d(K, F) > 0$. Since $(K \cup F) \subset \mathcal{O}$ Observations $1, 4$, and 5 of the exterior measure imply

$$\begin{aligned} m_*(\mathcal{O}) &\geq m_*(F) + m_*(K) \\ &= m_*(F) + \textstyle\sum_{j=1}^{N} m_*(Q_j). \end{aligned}$$

Hence $\sum_{j=1}^{N} m_*(Q_j) \leq m_*(\mathcal{O}) - m_*(F) \leq \epsilon$ implies F is measurable set.
Lemma 1.5: If F is closed, K is compact, and these sets are disjoint, then $d(F, K) > 0$.
Proof: As F is closed, for each point $x \in K$, there exists $\delta_x > 0$ so that $d(x, F) > 3\delta_x$. Since $\bigcup_{x \in K} B_{2\delta_x}(x)$ covers K, and K is compact, so we may find a finite subcover of K,

$$K \subseteq \bigcup_{j=1}^{N} B_{2\delta_j}(x_j)$$

Let $\delta = min(\delta_1, \delta_2, \cdots, \delta_N)$, then our claim is $d(K, F) \geq \delta > 0$. If $x \in K$ then $x \in B_{2\delta_i}(x_i)$, for some i and $y \in F$, then for all i we have $|x_i - x| \leq 2\delta_j$, and by construction $|y - x_i| \geq 3\delta_i$. Therefore

$$|y - x| \geq |y - x_i| - |x_j - x| \geq 3\delta_i - 2\delta_j$$

$$|y - x| \geq \delta$$

Hence $d(F, K) > 0$.

EXAMPLE 1: Give an example of two subsets E and F of $\mathbb{R}$ such that $E \cap F = \phi$ and $d(E, F) = 0$.
EXAMPLE 2: Give an example of two sets E and F such that $E \cap F = \phi$ and both are bounded but still $d(E, F) = 0$.
Hint: $E = \{0\}$ and $F = \{1/n | n \in \mathbb{N}\}$
EXAMPLE 3: Give an example of two sets E and F such that $E \cap F = \phi$ and both are closed but still $d(E, F) = 0$.

Property 5: The complement of a measurable set is measurable.
If E is measurable, then for every positive integer n we may choose an open set $\mathcal{O}_n$ with $E \subseteq \mathcal{O}_n$ and

$$m_*(\mathcal{O}_n - E) \leq 1/n.$$

The complement $\mathcal{O}_n^c$ is closed set, hence by property (4), $\mathcal{O}_n^c$ measurable, which implies that the union $S = \bigcup_{j=1}^{\infty} \mathcal{O}_n^c$ is also measurable by Property (3). Now $E \subseteq \mathcal{O}_n$ implies $S \subseteq E^c$ and S is measurable.

$$(E^c - S) \subset (\mathcal{O}_n - E),$$

such that

$$m_*(E^c - S) \leq \frac{1}{n}, \qquad \text{for all } n.$$

Therefore, $m_*(E^c - S) = 0$, Hence by Property (2) $E^c - S$ is measurable. We know union of two measurable sets is measurable,

$$(E^c - S) \cup S = E^c.$$

Hence, E^c is measurable.

Property 6: A countable intersection of measurable sets is measurable.

Let for each j, E_j be measurable set then by property (5), E_j^c is also measurable set. Hence by property (3) there union $\bigcup E_j^c$ is also measurable. Again by property (5),

$$(\bigcup E_j^c)^c = \cap E_j$$

is measurable.

Theorem 1.6: If $E_1, E_2, \cdots$ are disjoint measurable sets and $E = \bigcup_{j=1}^{\infty} E_j$ then

$$m(E) = \sum_{j=1}^{\infty} m(E_j)$$

Notations:

1. $E_k \nearrow E$: $E_1, E_2, \cdots$ is countable collection of subsets of $\mathbb{R}^d$ that increases to E, and $E_k \subseteq E_{k+1}, \forall k$ and $E = \bigcup_{k=1}^{\infty} E_k$.
2. $E_k \searrow E$: $E_1, E_2, \cdots$ is countable collection of subsets of $\mathbb{R}^d$ that decreses to E, and $E_k \supseteq E_{k+1}, \forall k$ and $E = \bigcap_{k=1}^{\infty} E_k$.

Theorem 1.7: Suppose $E_1, E_2, \cdots$ are measurable subsets of $\mathbb{R}^d$

1. If $E_k \nearrow E$, then $m(E) = \lim_{N\to\infty} m(E_N)$.
2. If $E_k \searrow E$ and $m(E_k) < \infty$ for some K, then $m(E) = \lim_{N\to\infty} m(E_N)$.

Proof 1: Let $G_1 = E_1, G_2 = E_2 - E_1 = E_2 \cap E_2^c$, and in general

$$G_k = E_k - E_{k-1} = E_k \cap E_{k-1}^c, \quad \text{for } k \geq 2.$$

By their construction, the sets G_k are measurable, disjoint, and

$$E = \bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} G_k.$$

Hence by theorem (1.6),

$$m(E) = \sum_{j=1}^{\infty} m(G_j) = \lim_{N\to\infty} \sum_{j=1}^{N} m(G_j) = \lim_{N\to\infty} m(\bigcup_{j=1}^{N} G_k)$$

$$\therefore m(E) = \lim_{N\to\infty} m(E_N).$$

2. Let $E_k \searrow E$ and $m(E_1) < \infty$ and

$$G_k = E_k - E_{k+1}, \qquad \forall k$$

G_k and E are disjoint measurable sets. and

$$E_1 = E \cup \bigcup_{k=1}^{\infty} G_k$$

By theorem (1.6)

$$\begin{aligned} m(E_1) &= m(E) + \sum_{k=1}^{\infty} m(G_k) = m(E) + \sum_{k=1}^{\infty} m(E_k - E_{k+1}) \\ &= m(E) + \lim_{N\to\infty} \sum_{k=1}^{N} (mE_k - mE_{k+1}) \\ &= mE + mE_1 - \lim_{N\to\infty} mE_{N+1} \end{aligned}$$

Hence,

$$m(E) = \lim_{N\to\infty} m(E_N).$$

Give counterexample for theorem 1.7.2
Symmetric Difference: Let E and F be two sets, then symmetric difference between E and F is denoted by $E \triangle F$ and defined as

$$E \triangle F = (E - F) \cup (F - E)$$

Theorem 1.8: Suppose E is a measurable subset of $\mathbb{R}^d$. Then, for every $\epsilon > 0$:

1. There exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m(\mathcal{O} - E) \leq \epsilon$.
2. There exists a closed set F with $F \subset E$ and $m(E - F) \leq \epsilon$.
3. If $m(E)$ is finite, there exists a compact set K with

$$K \subset E \text{ and } m(E - K) \leq \epsilon.$$

4. If $m(E)$ is finite, there exists a finite union $F = \bigcup_{j=1}^{N} Q_j$ of closed cubes such that $m(E \triangle F) \leq \epsilon$.

Proof 1.: E is measurable subset of $\mathbb{R}^d$, hence by defination of measurable set there exist open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m(\mathcal{O} - E) \leq \epsilon$.
2. E is measurable subset of $\mathbb{R}^d$, then E^c is also measurable. Apply part 1. on E^c, there exist open set $\mathcal{O}$ with $E^c \subset \mathcal{O}$ and $m(\mathcal{O} - E^c) \leq \epsilon$. If we let $F = \mathcal{O}^c$ then F is a closed set such that $F \subset E$ and $E - F = \mathcal{O} - E^c$, Hence $m(E - F) \leq \epsilon$.
3. Let E is measurable subset of $\mathbb{R}^d$ and $m(E) < \infty$, according to part 2. we get closed set F containing E such that $m(E - F) \leq \epsilon$.
For each n, we let $B_n(0)$ denote the ball centered at the origin of radius n, we approximate F by compact sets define as $K_n = F \cap B_n$. Then $E - K_n$ is a sequence of measurable sets that decreases to $E - F$,

$$\lim_{n\to\infty} m(E - K_n) = m(E - F) \leq \epsilon.$$

and since $m(E) < \infty$. So $K = K_n$ is our required compact set.
4. Let E is measurable subset of $\mathbb{R}^d$, choose a family of closed cubes $\{Q_j\}_{j=1}^{\infty}$ such that $E \subseteq \bigcup_{j=1}^{\infty} Q_j$ and For $\epsilon > 0$,

$$\sum_{j=1}^{\infty} |Q_j| \leq m(E) + \epsilon/2.$$

Since$m(E) < \infty$, the series converges and there exists $N > 0$ such that

$$\sum_{j=1}^{\infty} |Q_j| < \epsilon/2.$$

If $F = \bigcup_{j=1}^{N} Q_j$, then

$$m(E \triangle F) = m(E - F) + m(F - E) = m(\bigcup_{j=N+1}^{\infty} Q_j) + m(\bigcup_{j=1}^{N} Q_j - E)$$

Hence,

$$m(E \triangle F) = \textstyle\sum_{j=N+1}^{\infty} |Qj| + \sum_{j=1}^{\infty} |Qj| - m(E) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$\therefore m(E \triangle F) \leq \epsilon.$$

Invarience Properties of Lebesgue Measure:

1. **Translationn Invarience:** If E is measurable set and $h \in \mathbb{R}^d$, then the set $E_h = E + h = x + h | x \in E$ is also measurable, and $m(E) = m(E_h)$.
   Let E be a measurable set, then for $\epsilon > 0$ by Theorem 1.8.1 , there exist open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m(\mathcal{O} - E) \leq \epsilon$, then for set $E + h$ consider open set $\mathcal{O} + h$ such that

$$m(\mathcal{O} + h - (E + h)) = m(\mathcal{O} - E) < \epsilon.$$

which gives $E+h$ set is measurable. Now we will prove that $m(E+h) = m(E)$,as $E+h$ set is measurable $m(E+h) = m_*(E)$

$$m(E+h) = \inf\{\sum_{j=1}^{\infty} |Q_j + h||E + h \subseteq \bigcup_{j=1}^{\infty} Q_j + h\}$$
$$= \inf\{\sum_{j=1}^{\infty} |Q_j||E \subseteq \bigcup_{j=1}^{\infty} Q_j\}$$

Hence, $m(E_h) = m_*(E) = m(E)$.

2. **Dilation Invarience:**If E is measurable set in $\mathbb{R}^d$ and $\delta > 0$ then

$$m(\delta E) = \delta^d m(E).$$

Let E is measurable set in $\mathbb{R}^d$ then for $\epsilon > 0$ by Theorem 1.8.1 , there exist open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m(\mathcal{O} - E) \leq \epsilon$,then for set δE consider open set $\delta\mathcal{O}$ such that

$$m(\delta\mathcal{O} - \delta E) < \epsilon. \text{ and } \delta E \subseteq \delta\mathcal{O}$$

which gives δE set is measurable.

$$\begin{aligned} m(\delta E) &= m_*(\delta E) \\ &= \inf\{\sum_{j=1}^{\infty} |\delta\mathcal{O}_j||\delta E \subseteq \bigcup_{j=1}^{\infty} \delta\mathcal{O}_j\} \\ &= \inf\{\sum_{j=1}^{\infty} \delta^d|\mathcal{O}_j||\delta E \subseteq \bigcup_{j=1}^{\infty} \delta\mathcal{O}_j\} \\ &= \delta^d \inf\{\sum_{j=1}^{\infty} |\mathcal{O}_j||E \subseteq \bigcup_{j=1}^{\infty} \mathcal{O}_j\} \\ &= \delta^d m_*(\delta E) \\ m(\delta E) &= \delta^d m(\delta E) \end{aligned}$$

3. **Reflection Invarience:** If E is measurable set in $\mathbb{R}^d$ then $-E$ is also measurable set and $m(-E) = m(E)$.

σ-Algebra and Borel Sets:
A σ algebra of a set is collection of subset of $\mathbb{R}^d$ that is closed under countable unions , countable intersection and complements.
Question 1.:Check collection of open sets in $\mathbb{R}$ is σ algebra?
Question 2.:Check collection of all measurable sets in $\mathbb{R}^d$ is σ algebra?
Borel σ-Algebra: Smallest σ-algebra on $\mathbb{R}^d$ which contains all open set in $\mathbb{R}^d$, or Intersection of all σ-algebra that contain the open sets.
Elements of this σ-algebra are called as Borel sets.
G_δ **Sets:** A set G in $\mathbb{R}^d$ is said to be G_δ set if G can be expressed as intersection of countable number of open sets.
F_σ **Sets:** A set $F \subseteq \mathbb{R}^d$ is said to be F_σ , if F_σ can be expressed as countable union of closed sets.
Every F_σ and G_δ sets are Borel sets.

MEASURABLE FUNCTIONS:

Characteristic function on set E : Let $E \subseteq X$ then characteristic function on E is defined as

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E. \end{cases}$$

Step Function: A step function $f(x)$ is finite linear characteristic function on rectangle in $\mathbb{R}^d$,

$$f(x) = \sum_{j=1}^{N} c_i \chi_{R_i}(x).$$

Simple Function: Let $E_1, E_2, \cdots, E_n$ be measurable set in $\mathbb{R}^d$ and $c_1, c_2, \cdots, c_n$ be real constants then simple function ϕ is defined as

$$\phi(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x).$$

Note: Each step function is simple function.
Measurable function: A function $f : E \to \mathbb{R}$ where $E \subseteq \mathbb{R}^d$ is said to be measurable function if

$$f_a = \{x \in E | f(x) < a\} = f^{-1}[-\infty, a)$$

is measurable set for every $a \in \mathbb{R}$.
Example 1. $f(x) = x^2, x \in \mathbb{R}$ is a measurable function.
Example 2. Characteristic function on interval $[0, 1]$ is a measurable function.
Example 3. Every continuous function from $\mathbb{R}$ to $\mathbb{R}$ is a measurable function.
Example 4. Constant functions are measurable functions.
Lemma 1.9: Following statements are equivalents:

(i) f is a measurable function.

(ii) $\forall a; \{x | f(x) \leq a\}$ is measurable set.

(iii) $\forall a; \{x | f(x) > a\}$ is measurable set.

(iv) $\forall a; \{x | f(x) \geq a\}$ is measurable set.

$(i) \implies (ii)$
Let $E \subseteq \mathbb{R}^d$ and $f : E \to \mathbb{R}$ be a measurable function. implies set $\forall a; \{x | f(x) < a\}$ is a measurable set. Hence for each $n \in \mathbb{N}$ set

$$E_n = \{x \in E | f(x) < a + \tfrac{1}{n}\}$$

is a measurable set, and so there intersection is also measurable set, and intersection is

$$\bigcap_{n=1}^{\infty} E_n = \{x \in E | f(x) \leq a\}, \text{ for all } a$$

$(ii) \implies (iii)$
$\{x \in E | f(x) \leq a\}$, for all a is measurable set, so there complement is also measurable,

$$\{x \in E | f(x) \leq a\}^c = \{x \in E | f(x) > a\}.$$

$(iii) \implies (iv)$
Suppose $\{x \in E | f(x) > a\}$ is a measurable set, so for each n the set

$$E_n = \{x \in E | f(x) > a - \tfrac{1}{n}\}$$

is measurable set, so their intersection is also measurable and their intersection is

$$\forall a; \quad \bigcap_{n=1}^{\infty} \{x \in E | f(x) > a - \frac{1}{n}\} = \{x | f(x) \geq a\}$$

$(iv) \implies (i)$
Let $\{x | f(x) \geq a\}$ is a measurable set, then its complement is also measurable set and its complement is

$$\{x | f(x) \geq a\}^c = \{x | f(x) < a\}$$

Hence, f is measurable function.
Remark: Let f be a measurable function then inverse image of interval (a, b) is also measurable.
Let f be a measurable function, then

$$\begin{aligned} f^{-1}(a, b) &= \{x | f(x) \in (a, b)\} \\ &= \{x | a < f(x)\} \cap \{x | f(x) < b\} \end{aligned}$$

Hence, $f^{-1}(a, b)$ is intersection of measurable sets, so $f^{-1}(a, b)$ is measurable.
property 1: The finite-valued function f is measurable if and only if $f^{-1}(\mathcal{O})$ is measurable for every open set $\mathcal{O}$, and if and only if $f^{-1}(F)$ is measurable for every closed set F.
property 2: Any continuous function is measurable,
Let $S \subseteq \mathbb{R}^d$ and $f : S \to \mathbb{R}$ be continuous function and $\mathcal{O}$ be open set in $\mathbb{R}^d$. If f is continuous function then inverse image of $\mathcal{O}$ is open in $\mathbb{R}^d$. but every open set is measurable in $\mathbb{R}^d$. Hence, $f^{-1}(\mathcal{O})$ is measurable set hence by property 1, f is measurable function. But, every measurable function need not be continuous. Counterexample is charachteristic function on interval $[0, 1]$.
property 3: If f is continuous on $\mathbb{R}^d$ then f is measurable. If f is measurable and finite-valued, and Φ is continuous, then $\Phi \circ f$ is measurable.
Let $f : S \to T$ and $\phi : T \to R$ then $\phi \circ f : S \to R$ Let $\mathcal{O}$ be open set in R then $\phi^{-1}(\mathcal{O})$ is open set in T and measurable set. Then by prooperty (1) we get

$$f^{-1}(\phi^{-1}(\mathcal{O})) = (\phi \circ f)^{-1}(\mathcal{O}), \quad \forall \mathcal{O} \subset R$$

is measurable set in S. Hence $\phi \circ f$ is a measurable function.
property 4: Suppose $\{f_n\}$ is a sequence of measurable functions then

(a) $\sup_{n \in \mathbb{N}} f_n(x)$ and $\inf_{n \in \mathbb{N}} f_n(x)$ are measurable functions.

(b) $\limsup_{n \to \infty} f_n(x)$ and $\liminf_{n \to \infty} f_n(x)$ are measurable functions.

Let $h(x) = \sup_{n \in \mathbb{N}} f_n(x)$ so we have to prove in part (a) that h is measurable function, where f_n is sequence of measurable function, observe that

$$\{x \in \mathbb{R}^d | h(x) > a\} = \bigcup_{n=1}^{\infty} \{x \in \mathbb{R}^d | f_n(x) > a\}.$$

Let $E_n = \{x \in \mathbb{R}^d | f_n(x) > a\}$, then $\bigcup_{n=1}^{\infty} E_n = \{x \in \mathbb{R}^d | h(x) > a\}$
Each set E_n is measurable set, then their union is also measurable set, Hence

$$\{x \in \mathbb{R}^d | h(x) > a\}$$

is measurable set. Implies that h is a measurable function.
Let $g(x) = \inf_{n \in \mathbb{N}} f_n(x)$, and set below is measurable

$$E_n = \{x \in \mathbb{R}^d | f_n(x) < a\}$$

Then their intersection is also measurable,

$$\bigcap_{n=1}^{\infty} E_n = \{x \in \mathbb{R}^d | g(x) < a\}$$

Hence, function g is measurable. $\limsup_{n\to\infty} f_n(x) = \lim_{m \geq 1} \sup_{n \geq m} f_n(x)$ Let f_n be a sequence of measurable functions, then

$$t_i = \sup\{f_i(x), f_{i+1}(x), \cdots\}$$

are measurable sets, so $\inf(t_i)$ is also measurable function. Hence $\limsup_{n\to\infty} f_n(x)$ is a measurable function.
Simillarly, $\liminf_{n\to\infty} f_n(x)$ is a measurable function.
Property 5: Limit of measurable functions is measurable function.
Since

$$f(x) = \limsup_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x),$$

then by property 3. we get f is a measurable function.
Question: Show that $f_n(x) = \arctan(nx), x \in \mathbb{R}$ converges to measurable function.

Property 6: If f and g are measurable,then

(i) The integer powers $f^k, k \geq 1$ are measurable.

(ii) $f + g$ and fg are measurable if both f and g are finite-valued.

For part (i), Let f be a measurable function, If k is odd then

$$\{x | f^k(x) > a\} = \{x | f(x) > a^{1/k}\}$$

As f is measurable function, set $\{x | f(x) > a^{1/k}\}$ is measurable. Implies set $\{x | f^k(x) > a\}, \forall a$ is also measurable, hence $f^k, k \geq 1$ is a measurable function.
For part (ii), Let f and g are finite-valued measurable functions. to prove $f + g$ is a measurable function consider,

$$\begin{aligned}\{x | (f+g)(x) > a\} &= \{x | f(x) + g(x) > a\} \\ &= \{x | f(x) > a - g(x)\} \\ &= \bigcup_{r \in \mathbb{Q}} [\{x | f(x) > a - x\} \cap \{x | g(x) > r\}]\end{aligned}$$

Hence $\{x|(f+g)(x) > a\}$ is a countable union of measurable sets so measurable, from this we get $f+g$ is a measurable function.
Now to prove fg is a measurable function, we know that

$$fg = \frac{(f+g)^2 - (f-g)^2}{4}$$

If f and g are measurable functions then $-g, f+g$ and $f-g$ are measurable function. Use part (i) for $k=2$ we get $(f+g)^2$ and $(f-g)^2$ are measurable, hence fg is a measurable function.
Question: If f and g are measurable functions then show that

(a) $|f|$ is a measurable function.

(b) $\max\{f, g\}$ and $\min\{f, g\}$ are measurable functions.

Note: $f = g$ almost everywhere if and only if the set $\{x|f(x) \neq g(x)\}$ is set of measure zero.
property 7: Suppose f is measurable, and $f(x) = g(x)$ for almost every x then g is measurable.
Approximation by simple functions or step functions:
Theorem 1.10: Suppose f is a non-negative measurable function on $\mathbb{R}^d$ then there exist an increasing sequence of non-negative simple functions $\{\phi_k\}_{k=1}^{\infty}$ that converges pointwise to f, such that

$$\phi_k(x) \leq \phi_{k+1}(x) \text{ and } \lim_{k\to\infty} \phi_k(x) = f(x,) \text{ for all } x.$$

Theorem 1.11: Suppose f is a measurable function on $\mathbb{R}^d$ then there exist a sequence of simple functions $\{\phi_k\}_{k=1}^{\infty}$ that satisfies

$$|\phi_k(x)| \leq |\phi_{k+1}(x)| \text{ and } \lim_{k\to\infty} \phi_k(x) = f(x,) \text{ for all } x.$$

Theorem 1.12: Suppose f is measurable function on $\mathbb{R}^d$ then there exist a sequence of step functions $\{\chi_k\}_{k=1}^{\infty}$ that converges pointwise to $f(x)$, for almost every x.
Littlewood's three principles: For measurable sets and measurable functions,

(i) Every set in nearly a finite union of intervals.

(ii) Every function is nearly contiunous.

(iii) Every convergent sequence is nearly uniformly convergent.

Theorem 1.12 (Egorov's Theorem) Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set E with $m(E) < \infty$, and assume that $f_k \to f$ almost everywhere on E. Given $\epsilon > 0$, we can find a closed set $A_\epsilon \subset E$ such that $m(E - A_\epsilon) \leq \epsilon$ and $f_k \to f$ uniformly on A_ϵ.
Theorem 1.12 (Lusin's Theorem) Suppose f is measurable and finite valued on E with E of finite measure, then for every $\epsilon > 0$ there exist a closed set F_ϵ with

$$F_\epsilon \subset E \text{ and } m(E - F_\epsilon) \leq \epsilon$$

and such that $f|_{F_\epsilon}$ is continuous.
Theorem 5.1: Suppose A and B are measurable sets in $\mathbb{R}^d$ and their sum $A+B$ is also measurable.

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