A STUDY OF THE LOGISTIC DISTRIBUTION

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CHAPTER 1

INTRODUCTION

The purpose of this dissertation is to study the logistic family of distributions. The logistic function was first used as a model for human population growth by Verhulst (1945). The use of the logistic function as a growth curve is due to its impressive differential equation.

$$\frac{dF}{dy} = a\,[F(y)-b]\;[C-F(y)] \text{ where a,b,c are constants with } a > 0,\ c > b .$$

Recently Leach (1981) has re-examined the logistic curve as a model for growth of human population. The logistic curve was used as an alternative to the integrated normal curve in the treatment of binomial response data in bio-assay; The problem of estimation of parameters of logistic function was discussed by Berkson (1955). The logistic regression analysis is widely applicable to epidemiologic studies like follow up study and case-control study. Kleinbaum, Kupper and Chambless (1982) have illustrated it's use in quantifying an association between an exposure variable and a disease status. The logistic distribution arises as an asymptotic distribution of standardized mid-range of a random sample taken from a continuous symmetric distribution with mean zero. Logistic distribution closely resembles the normal distribution in shape and hence can be used as an alternative to normal distribution. However unlike the bivariate normal distribution there is no unique way of defining bivariate logistic distribution.

Here we have divided this dissertation into the following parts:

(i) Properties of logistic distribution and related distributions.

(ii)Estimation of parameters of logistic distribution.

(iii) Estimation of logistic function in the treatment of binomial response data.

(iv)Bivariate logistic distributions.

A systematic study of the logistic distribution has been done in the present dissertation in the light of the points mentioned above.

Chapter 2 is regarding the properties and related distributions. Moments of the logistic distribution are obtained using cumulant generating function. The expressions for mean, mode and variance of k-th order statistic from a random sample drawn from logistic distribution with mean μ and variance σ^2 are derived. It reveals how logistic distribution belongs to Perk's family of distributions. In this chapter logistic function and its applications are covered.

In Chapter 3, maximum likelihood estimators, estimators based on sample quantiles and some other estimators of the parameters of the logistic distribution, when complete sample is available, are given. Different methods of estimation of the parameters from symmetrically censored, one sided censored and censored samples are described. It also includes tolerance limits and confidence limits for the parameters of logistic distribution.

Chapter 4 deals with estimation of logistic function used as a model for probability of response in binomial response data. In this Chapter, the maximum likelihood estimators, minimum logit χ^2 estimators and minimum χ^2 estimators are discussed with illustrative examples.

Properties of Gumbel's bivariate logistic distribution, it's generalization and another bivariate logistic distribution are studied in Chapter 5.

Each Chapter of this dissertation consists of serially numbered sections. The section a.b represents Section b of Chapter a while the equation (a.b) represents the equation b of Chapter a.

CHAPTER 2

PROPERTIES AND RELATED DISTRIBUTIONS

2.0 Introduction :

In this Chapter, we discuss the genesis and properties of the logistic distribution. The logistic distribution with mean zero and variance one, called standard logistic distribution, is compared with other continuous distributions.

Section 2.1 gives the genesis of the logistic distribution from the asymptotic distribution of standardized mid-range of a random sample taken from a symmetric continuous distribution with mean zero.

In Section 2.2, the properties of the distribution are studied while in Section 2.3 moment generating function of k-th order statistic and distribution of the range in the random sample taken from logistic distribution are obtained.

Section 2.4 reveals the relation between Champernown distribution and logistic distribution. The comparative study of the nature of the curves of standard logistic, standard normal and Laplace distribution with mean zero and variance unity is done in Section 2.4.

In Section 2.5, we discuss the form of logistic function and its use as a model in different situations.

We now define logistic distribution through its probability density function (p.d.f.).

Definition :

A continuous random variable Y is said to follow logistic distribution (or sech-square distribution) with location parameter μ and scale parameter σ if its p.d.f. is

$$f(y;\mu,\sigma) = \frac{\pi}{\sigma\sqrt{3}} \frac{\exp\{-\frac{\pi}{\sigma\sqrt{3}}(y-\mu)\}}{[1+\exp\{\frac{-\pi}{\sigma\sqrt{3}}(y-\mu)\}]^2} \tag{2.1}$$

$$-\infty < y < \infty$$

$$-\infty < \mu < \infty$$

$$\sigma > 0 .$$

We shall denote this logistic distribution by $L(\mu,\sigma)$. In the above form μ is the mean and σ^2 is the variance of the distribution.

The distribution function of $L(\mu,\sigma)$ is given by

$$F(y;\mu,\sigma) = \frac{1}{1+\exp\{-\frac{\pi}{\sigma\sqrt{3}}(y-\mu)\}} \; ; \quad -\infty < y < \infty, \quad \sigma > 0, \quad -\infty < \mu < \infty . \tag{2.2}$$

It should be noted that logistic distribution (2.1) is symmetric about μ.

The logistic distribution with $\mu = 0$ and $\sigma = 1$ is called standard logistic distribution and written as $L(0,1)$.

The probability density function of L(0,1) is

$$f(y) = \frac{\pi}{\sqrt{3}} \frac{\exp\{\frac{-\pi y}{\sqrt{3}}\}}{[1 + \exp\{\frac{-\pi y}{\sqrt{3}}\}]^2} ; -\infty < y < \infty \qquad (2.3)$$

and the distribution function is

$$F(y) = \frac{1}{[1+\exp\{\frac{-\pi}{\sqrt{3}} y\}]} ; -\infty < y < \infty . \qquad (2.4)$$

Sometimes the probability density function of logistic distribution is expressed in the form

$$f(y;\alpha,\beta) = \frac{1}{\beta} \frac{\exp\{-(\frac{y-\alpha}{\beta})\}}{[1+\exp\{-(\frac{y-\alpha}{\beta})\}]^2} \qquad (2.5)$$

$$\beta \in (0,\infty)$$

$$-\infty < y < \infty$$

$$-\infty < \alpha < \infty .$$

$$\text{OR} \quad g(y;\alpha,\beta) = \frac{1}{4\beta} \operatorname{sech}^2 \{\tfrac{1}{2} (\tfrac{y-\alpha}{\beta})\} . \qquad (2.6)$$

Here α is the location parameter and β is the scale parameter of the distribution.

The distribution functions corresponding to (2.5) and (2.6) are

$$F(y;\alpha,\beta) = \frac{1}{[1+\exp\{-(\frac{y-\alpha}{\beta})\}]} ; \quad -\infty < y < \infty$$
$$\beta \varepsilon (0,\infty)$$
$$-\infty < \alpha < \infty . \quad (2.7)$$

and

$$G(y;\alpha,\beta) = \frac{\{1+\tanh[\frac{1}{2}(\frac{y-\alpha}{\beta})]\}}{2} ; \quad -\infty < y < \infty$$
$$\beta \varepsilon (0,\infty)$$
$$-\infty < \alpha < \infty . \quad (2.8)$$

(2.1) Genesis of the distribution :

The logistic distribution arises as an asymptotic distribution of standardized mid-range of random sample taken from a continuous symmetric distribution with mean zero.

Let $X_{1:n}, X_{2:n} \ldots X_{n:n}$ be order statistics corresponding to a random sample of size n drawn from a continuous symmetric distribution having probability density function $\phi(x)$ with mean zero.

The m^{th} mid range is defined as :

$$m^{th} \text{ mid range} = X_{m:n} + X_{n-m+1:n}$$
$$m = 1,2,\ldots,n .$$

If m=1, the mid range is called standardized mid range .

$\therefore$ Standardized mid range $= X_{1:n} + X_{n:n}$.

We shall denote m^{th} and $(n-m+1)^{th}$ order statistic by ${}_mX$ and X_m respectively. Hence m^{th} mid range $= {}_mX + X_m = V_m$ (say).
Define $\alpha_m = \frac{n}{m}\phi(u_m)$ where u_m is the mode of the distribution of X_m. The probability density function $f_m(x_m)$ of x_m for large n and small m is

$$f_m(x_m) = \alpha_m \frac{m^m}{(m-1)!} \quad e^{-my_m - m\, e^{-y_m}} \tag{2.9}$$

where $y_m = \alpha_m (x_m - u_m)$.

Define

$${}_mf({}_mx) = f_m(-x_m) \text{ and}$$

$${}_mf(v_m - x_m) = \frac{\alpha_m\, m^m}{(m-1)!} \quad e^{m\, \alpha_m(v_m - x_m + u_m)}$$

$$e^{-m\, e^{\alpha_m(v_m - x_m + u_m)}} .$$

Gumbel (1944) has obtained the distribution of m^{th} mid range v_m as

$$g(v_m) = \int_{-\infty}^{\infty} f_m(x_m)\ {}_mf\ (v_m - x_m)\ dx_m \tag{2.10}$$

$$\therefore\ g(v_m) = \frac{\alpha_m^2\, m^{2m}}{[(m-1)!]^2} \int_{-\infty}^{\infty} e^{-my_m + m\,\alpha_m(u_m - x_m + u_m) - m\, e^{-y_m}}$$

$$e^{-m\, e^{+\alpha_m(u_m - x_m + u_m)}} \qquad dx_m$$

putting $e^{-y_m} = Z$, the integral reduces to

$$e^{m\alpha_m v_m} \int_0^\infty e^{-mz(1+e^{\alpha_m v_m})}\, Z^{2m-1}\, dZ \, .$$

Thus the distribution of m^{th} mid-range becomes

$$g(v_m) = \frac{\alpha_m (2m-1)!}{[(m-1)!]^2} \cdot \frac{e^{m\alpha_m v_m}}{[1+e^{\alpha_m v_m}]^{2m}} \, . \qquad (2.11)$$

If $m = 1$, we get

$$g(v) = \frac{\alpha\, e^{\alpha v}}{[1+e^{\alpha v}]^2} \qquad (2.12)$$

which can be looked upon as the probability density function of logistic distribution.

2.2. Cumulant generating function and moments :

The probability density function of logistic variable Y with mean μ and variance σ^2 is

$$f(y;\mu,\sigma) = \frac{h}{\sigma} \cdot \frac{\exp\{-\frac{h}{\sigma}(y-\mu)\}}{[1+\exp\{-\frac{h}{\sigma}(y-\mu)\}]^2} \qquad (2.13)$$

where $\qquad h = \frac{\pi}{\sqrt{3}}$.

The moment generating function of Y with density (2.13) is

$$M_Y(t) = E[e^{tY}]$$

$$= \int_{-\infty}^{\infty} e^{ty} \frac{n}{\sigma} \frac{\exp\{-\frac{n}{\sigma}(y-\mu)\}}{[1+\exp\{-\frac{n}{\sigma}(y-\mu)\}]^2} dy$$

put $Z = \dfrac{1}{1+\exp\{\frac{n}{\sigma}(y-\mu)\}}$ then

$$y = \mu + \frac{\sigma}{n} \ln \left(\frac{1-Z}{Z}\right)$$

$$\therefore M_Y(t) = \int_0^1 e^{\mu t} Z^{-\frac{\sigma}{n}t} (1-Z)^{\frac{\sigma}{n}t} dZ$$

$$= e^{\mu t} B(1-\frac{\sigma}{n}t, 1+\frac{\sigma}{n}t) \; ; \; t < \frac{n}{\sigma}$$

$$= e^{\mu t} \Gamma(1-\frac{\sigma}{n}t) \Gamma(1+\frac{\sigma}{n}t) . \qquad (2.14)$$

Taking logrithms of both sides of (2.14) the cumulant generating function is given by,

$$K_Y(t) = \mu t + \ln [\Gamma(1+\frac{\sigma}{n}t)] + \ln [\Gamma(1-\frac{\sigma}{n}t)]; \; t < \frac{n}{\sigma} . \qquad (2.15)$$

Differentiating (2.15) with respect to t, we get,

$$\frac{d}{dt} K_Y(t) = \mu + \frac{\sigma}{n}[\frac{d}{dt} \ln \Gamma(1+\frac{\sigma}{n}t) - \frac{d}{dt} \ln \Gamma(1-\frac{\sigma}{n}t)]$$

$$= \mu + \frac{\sigma}{n}[\Psi(1+\frac{\sigma}{n}t) - \Psi(1-\frac{\sigma}{n}t)] \qquad (2.16)$$

where $\Psi(.)$ is digamma function. Putting $t = 0$ in (2.16)

$$\frac{d}{dt} K_Y(t)\Big|_{t=0} = E(Y) = \mu . \tag{2.17}$$

Differentiating (1.16) with respect to t and putting $t = 0$, we have the second cumulant as

$$K_2 = Var(Y) = \frac{d^2 K_Y(t)}{dt^2}\Big|_{t=0}$$

$$= \frac{\sigma^2}{h^2} [\Psi'(1+\frac{\sigma}{h} t) + \Psi'(1-\frac{\sigma}{h} t)]\Big|_{t=0}$$

with $\Psi'(.)$ as trigamma function

$$= \frac{\sigma^2}{h^2} [2\Psi'(1)]$$

$$= \sigma^2 . \tag{2.18}$$

Proceeding in this manner, the r^{th} cumulant $K_r : (r \neq 1)$ is obtained as

$$K_r = \frac{\sigma^r}{h^r} [\Psi^{r-1}(1) + (-1)^r \Psi^{r-1}(1)]. \tag{2.19}$$

If r is an odd number then r^{th} cumulant is zero.

The coefficient of Skewness β_1 is zero and the coefficient of kurtosis $\beta_2 = \frac{\mu_4}{\mu_2^2} \doteq 4.2$. Hence the logistic distribution is leptokurtic.

2.3. Order Statistics :

Let $X_{1:n} < X_{2:n} < \ldots < X_{k:n} < \ldots < X_{n:n}$ be order statistics corresponding to a random sample from standard logistic distribution with density

$$f(x) = h \frac{e^{-hx}}{[1+e^{-hx}]^2} \; ; \; -\infty < x < \infty \quad \text{with } h = \frac{\pi}{\sqrt{3}} \; .$$

The probability density function of K^{th} order statistic $X_{K:n}$ is

$$P_{X_{K:n}}(x) = \frac{n!\,h}{(k-1)!\,(n-k)!} \left[\frac{1}{1+e^{-hx}}\right]^{k-1} \left[\frac{e^{-hx}}{1+e^{-hx}}\right]^{n-k} \frac{e^{-hx}}{[1+e^{-hx}]^2}$$

$$= \frac{n!\,h}{(k-1)!\,(n-k)!} \; e^{-(n-k+1)hx} \, (1+e^{-hx})^{-(n+1)}$$

$$k = 1,2,\ldots, n \; . \qquad (2.20)$$

The moment generating function of $X_{K:n}$ is

$$E[e^{tX_{k:n}}] = \frac{\Gamma(n+1)\,h}{\Gamma(k)\,\Gamma(n-k+1)} \int_{-\infty}^{\infty} \frac{e^{-(n-k-\frac{t}{h}+1)hx}}{(1+e^{-hx})^{n+1}} \, dx$$

$$= \frac{\Gamma(n+1)}{\Gamma(k)\,\Gamma(n-k+1)} \, B\left(n-k-\frac{t}{h}+1, \; k+\frac{t}{h}\right)$$

$$= \frac{\Gamma(k+\frac{t}{n})\ \Gamma(n-k+1-\frac{t}{n})}{\Gamma(k)\ \Gamma(n-k+1)} \tag{2.21}$$

where $-nk < t < n\ (n-k+1)$.

Gupta and Shah (1965) have shown that the moment generating function of k^{th} order statistic from standard logistic distribution is given by

$$M(t) = \frac{(-1)^{n-k}}{(k-1)!\ (n-k)!}$$

$$\cdot \Big[\sum_{i=1}^{n} \sum_{j=0}^{i} S(i,n)\ \binom{i}{j}\ \left(\frac{t}{n}\right)^{j-1} (k-1)^{i-j}$$

$$+ 2 \sum_{p=1}^{\infty} \sum_{i=1}^{n} \sum_{j=0}^{i} \frac{(-1)^{p-1}(2^{2p-1}-1)}{(2p)!}\ \pi^{2p} \binom{i}{j}\ (k-1)^{i-j}$$

$$B_{2p}\ S(i,n)\ \left(\frac{t}{n}\right)^{j+2p-1} \Big] \tag{2.22}$$

where $B_n = B_n(0)$ with $B_n(x)$ as coefficient of

$$\frac{t^n}{n!} \text{ in } \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t\, e^{xt}}{e^t - 1}$$

and S(i,n) stirling number of the first kind.

$$E[\,e^{tx}\,] = \frac{\Gamma(k+\frac{t}{n})\ \Gamma(n-k+1-\frac{t}{n})}{\Gamma(k)\ \Gamma(n-k+1)}$$

$$= \left[(k-1+\tfrac{t}{h})\ (k-2+\tfrac{t}{h}) \ldots (1+\tfrac{t}{h})\ \tfrac{t}{h} \right]$$

$$(-1)^{n-k} \left[(k-n+\tfrac{t}{h})\ (k-n+1+\tfrac{t}{h}) \ldots (-1+\tfrac{t}{h}) \right]$$

$$\frac{\Gamma(\frac{t}{h})\ \Gamma(1-\frac{t}{h})}{(k-1)!\ (n-k)!}$$

$$= \frac{(-1)^{n-k}\ (k-1+\frac{t}{h})^{(n)}\ \pi \operatorname{cosec}(\frac{\pi t}{h})}{(k-1)!\ (n-k)!}$$

where $x^{(n)} = x(x-1) \ldots (x-n+1)$

$$= \frac{(-1)^{n-k}\ \pi}{(k-1)!\ (n-k)!} \left[2 \sum_{p=1}^{\infty} \frac{(-1)^{p-1}(2^{2p-1}-1)\pi^{2p-1}}{(2p)!} B_{2p} (\tfrac{t}{h})^{2p-1} + \frac{h}{\pi t} \right]$$

$$\left[\sum_{i=1}^{n} \sum_{j=0}^{i} \binom{i}{j} (k-1)^{i-j} (\tfrac{t}{h})^{j}\ S(i,n) \right].$$

$$= \frac{(-1)^{n-k}}{(k-1)!\ (n-k)!} \left[\sum_{i=1}^{n} \sum_{j=0}^{i} S(i,n) \binom{i}{j} (\tfrac{t}{h})^{j-1} (k-1)^{i-j} \right.$$

$$+ 2 \sum_{p=1}^{\infty} \sum_{i=1}^{n} \sum_{j=0}^{i} \frac{(-1)^{p-1}(2^{2p-1}-1)}{(2p)!} \pi^{2p} \binom{i}{j} (k-1)^{i-j}$$

$$\left. B_{2p}\ S(i,n)\ (\tfrac{t}{h})^{j+2p-1} \right]. \qquad (2.23)$$

Collecting the coefficients of t^{2r-1} and t^{2r} from (2.23) we have

$$\mu'_{2r-1} = \frac{(-1)^{n-k}\ (2r-1)!}{h^{2r-1}\ (k-1)!\ (n-k)!}$$

$$\left[\sum_{i=2r}^{n} b_{i,2r} S(i,n) + 2 \sum_{j=1}^{r} \sum_{i=2(r-j)}^{n} \ell_j b_{i,2(r-j)} S(i,n) \right] \quad (2.24)$$

and

$$\mu'_{2r} = \frac{(-1)^{n-k} (2r)!}{n^{2r}(k-1)!\,(n-k)!}$$

$$\left\{ \sum_{i=2r+1}^{n} b_{i,2r+1} S(i,n) + 2 \sum_{j=1}^{r} \sum_{i=2(r-j)+1}^{n} \ell_j b_{i,2r-2j+1} S(i,n) \right\} \quad (2.25)$$

where $\ell_p = \dfrac{(-1)^{p-1}(2^{2p-1}-1)\, B_{2p}\, \pi^{2p}}{(2p)!}$

and $b_{ij} = \binom{i}{j} (k-1)^{i-j}$; $0 \leq j \leq i$.

It is easy to obtain moments of k^{th} order statistic using cumulant generating function.

The cumulant generating function of k^{th} order statistic can be obtained taking logritum of (2.21) as

$$\ln\left[\Gamma\left(k+\frac{t}{n}\right)\right] + \ln\left[\Gamma\left(n-k+1-\frac{t}{n}\right)\right]$$

$$- \ln\left[\Gamma(k)\right] - \ln\left[\Gamma(n-k+1)\right] . \quad (2.26)$$

Differentiating (2.26) with respect to t and putting $t=0$, we get the first cumulant

$$k_{1}; X_{k:n} = \frac{1}{h} [\Psi(k) - \Psi(n-k+1)] \qquad (2.27)$$

where $\psi(m) = \frac{d}{dm} [\ln \Gamma(m)]$.

Differentiating (2.26) with respect to t r times and putting $t = 0$, the r^{th} cumulant is given as

$$k_{r;X_{k:n}} = \frac{1}{h^r} [\Psi^{r-1}(k) + (-1)^r \Psi^{r-1} (n-k+1)]$$

where $\psi^{r-1}(m) = \frac{d^{r-1}}{dm^{r-1}} [\Psi(m)]$

$$= \frac{(r-1)! \ (-1)^r}{n^r}$$

$$\left[\sum_{j=1}^{\infty} \frac{1}{(j+k-1)^r} + (-1)^r \sum_{j=1}^{\infty} \frac{1}{(j+n-k)^r} \right] r \geq 2. \qquad (2.28)$$

Gupta and Shah (1965) have tabulated the values of first four raw moments of k^{th} order statistic in a random sample of size n from a standard logistic distribution for n = 1 to 10.

If the distribution function of the k^{th} order statistic from standard logistic distribution is $H_{k;n}(x)$ then

$$H_{k;n}(x) + H_{n-k+1;n}(-x) = 1.$$

In table No.1., the numerical values of means and standard deviations of order statistics from standard logistic distribution computed by

Birnbaum and Dudman (1963) are given for n= 1 to 10, 15, 20, 50, 100 .

The expected value and variance of k^{th} order statistic in a random sample from logistic distribution with mean μ and variance σ^2 can be obtained as follows :

$$E [X_{k:n} / \mu , \sigma^2] = \mu + \sigma E [X_{k:n} / \mu = 0, \sigma^2 = 1]$$

and

$$Var [X_{k:n} / \mu , \sigma^2] = \sigma^2 \ Var [X_{k:n} / \mu = 0, \sigma^2 = 1] .$$

Table No.1.

Mean and standard deviations of order statistics from standard logistic distribution

n	k	$E[X_{k:n}]$	S.D. $[X_{k:n}]$
1	1	0	1.0000
2	1	0.5513	0.8343
3	1	0.8270	0.7374
3	2	0.0000	0.6262
4	1	1.0108	0.7657
4	2	0.2757	0.5622
5	1	1.1486	0.7532
5	2	0.4594	0.5313
5	3	0.0000	0.4900
6	1	1.2589	0.7451
6	2	0.5973	0.5131
6	3	0.1838	0.4542
7	1	1.3508	0.7394
7	2	0.7075	0.5012
7	3	0.3216	0.4328
7	4	0.0000	0.4154
8	1	1.4295	0.7352
8	2	0.7994	0.4927
8	3	0.4314	0.4182
8	4	0.1378	0.3918

n	k	$E[X_{k:n}]$	$S.D.[X_{k:n}]$
9	1	1.4934	0.7319
9	2	0.8782	0.4863
9	3	0.5238	0.4083
9	4	0.2481	0.3760
9	5	0.0000	0.3668
10	1	1.5597	0.7294
10	2	0.9471	0.4814
10	3	0.6025	0.4006
10	4	0.3340	0.3646
10	5	0.1103	0.3498

Source : Birnbaum and Dudman (1963) .

Mode of k^{th} order Statistic

The density function of k^{th} order statistic in a random sample of size n from standard logistic distribution is given as

$$h_{k,n}(x) = \frac{\pi}{\sqrt{3}\, B(k,n-k+1)} \; \frac{e^{-\frac{\pi}{\sqrt{3}}(n-k+1)x}}{[1+\exp\{-\frac{\pi}{\sqrt{3}}x\}]^{n+1}} \; ; -\infty < x < \infty$$

Gupta and Shah (1965) have given the expression of mode of k^{th} order statistic. The mode of k^{th} order statistic can be obtained by differentiating $h_{k,n}(x)$ with respect to x and equating the expression to zero.

$$\frac{\partial h_{k,n}(x)}{\partial x} = \frac{\pi}{\sqrt{3}B(k,n-k+1)} \left[\frac{\pi}{\sqrt{3}} e^{-\frac{\pi}{\sqrt{3}}(n-k+1)x} (1+e^{-\frac{\pi}{\sqrt{3}}x})^{n} \right]$$

$$\left[\frac{(n+1)\, e^{-\frac{\pi}{\sqrt{3}}x} - (n-k+1)\,(1+e^{-\frac{\pi}{\sqrt{3}}x})}{(1 + e^{-\frac{\pi}{\sqrt{3}}x})^{2(n+1)}} \right]$$

$$\frac{\partial h_{k,n}(x)}{\partial x} = 0$$

$$\Rightarrow (n-k+1)\,(e^{-\frac{\pi}{\sqrt{3}}x} + 1) = (n+1)\, e^{-\frac{\pi}{\sqrt{3}}x}$$

$$\Rightarrow (n-k+1) = k\, e^{-\frac{\pi}{\sqrt{3}}x}$$

$$\Rightarrow x = \frac{\sqrt{3}}{\pi} \, \ln \left(\frac{k}{n-k+1}\right) . \qquad (2.29)$$

From (2.29) , we see that the mode of sample median is zero while modes of the minimum and maximum observations in the samples are $-\frac{\sqrt{3}}{\pi} \ln n$ and $\frac{\sqrt{3}}{\pi} \ln n$ respectively.

2.4. Relation with other distributions :

The general form of probability density function of Perks' family of distributions with parameters α_t, β_t and λ is

$$f_X(x) = [\sum_{t=0}^{m} \alpha_t e^{-t\lambda x}] [\sum_{t=0}^{n} \beta_t e^{-t\lambda x}]^{-1}. \qquad (2.30)$$

In (2.30) taking $\lambda = 1$

$\alpha_t = 0$ for $m = 0,2,3\ldots,$

$\alpha_1 = \beta_0$

$$\beta_t = \begin{cases} 2\beta_0 & \text{if } n = 1 \\ \beta_0 & \text{if } n = 2 \\ 0 & \text{if } n = 3,4,\ldots, \end{cases}$$

Then,

$$f_X(x) = \frac{\beta_0 e^{-x}}{\beta_0 + 2\beta_0 e^{-x} + \beta_0 e^{-2x}}$$

$$= \frac{e^{-x}}{(1+e^{-x})^2},$$

which represents probability density function of standard logistic distribution. We now define champernowne distribution. The distribution function of Champernowne variable T is given by

$$F_T(t) = 1 - \frac{1}{\theta} \tan^{-1} \left\{ \frac{\sin\theta}{\cos\theta + (\frac{t}{t_o})^\alpha} \right\}, \tag{2.31}$$

$t > 0$, $0 < \theta < \pi$ with t_o and α as median and shape parameter respectively. Taking limit of (2.31) as $\theta \to 0$

$$F_T(t) = 1 - \frac{t_o^\alpha}{t_o^\alpha + t^\alpha} \; ; \; t > 0 \; . \tag{2.32}$$

Differentiating (2.32) with respect to t, the density function is given as

$$f_T(t) = \frac{\alpha(\frac{t}{t_o})^{\alpha-1}}{t_o[1 + (\frac{t}{t_o})^\alpha]^2} \; ;$$

After making the transformation

$$(\frac{T}{t_o})^\alpha = e^Y \; , \quad \text{we get}$$

$$P_Y(y) = \frac{e^y}{[1 + e^y]^2}$$

$$= \frac{e^{-y}}{(1+e^{-y})^2} \; ; \; -\infty < y < \infty . \tag{2.33}$$

(2.33) is the density of the standard logistic distribution. Suppose that continuous r.v. X has extreme value distribution with distribution function

$$P[X \le x] = \exp\{-e^{-(\frac{x-\alpha}{\beta})}\} \ ; \ \beta > 0 \ .$$

The form of generalized extreme value distribution after introducing an extra parameter τ defined by Dubey (1969) is given by

$$P[X \le x/\tau] = \exp\{-\tau\beta \exp[-(\frac{x-\alpha}{\beta})]\}$$
$$= \exp\{-e^{-(\frac{x-\xi}{\beta})}\} \ ,$$

where $\xi = \alpha + \beta \ \ln(\tau\beta)$.

Let τ have an exponential distribution with density function

$$p(t) = \beta \ e^{-\beta t} \quad (t > 0) \ .$$

The resulting compound distribution has distribution function is given by

$$P[X \le x] = \beta \int_0^\infty \exp\{-\beta t[1 + \exp(-(\frac{x-\alpha}{\beta}))]\} \ dt$$

$$= \frac{\Gamma(1)}{[1 + \exp\{-(\frac{x-\alpha}{\beta})\}]}$$

$$= \frac{1}{[1 + \exp\{-(\frac{x-\alpha}{\beta})\}]} \ ,$$

which is distribution function of logistic distribution with mean α and variance $\frac{\pi^2\beta^2}{3}$. Comparision between the standard logistic distribution and standard normal distribution is done with the help of distribution function and ordinates at different values. They indicate that the tails of standard logistic distribution are longer than those of standard normal distribution.

On the other hand in a comparative study of the nature of standard logistic distribution and Laplace distribution with mean zero and variance one, it is observed that the tails of this Laplace distribution are longer than those of standard logistic distribution.

The table no.2 gives the figures corresponding to the distribution function and ordinates at different values of the variable for standard normal distribution, standard logistic distribution and Laplace distribution with mean zero and variance one.

TABLE No.2

Z : Standard Normal Variate, L : Standard logistic variable,

T : Laplace Variable with mean Zero and variance one.

Value of x	Ordinate for Z	Ordinate for L	Ordinate for T
0.0	0.3989423	0.4534498	0.3535533
0.1	0.3969525	0.4497812	0.3294167
0.2	0.3910427	0.4388408	0.3069278
0.3	0.3813878	0.4214739	0.2859743
0.4	0.3682701	0.3986478	0.2664513
0.5	0.3520653	0.3716492	0.248261
0.6	0.3332246	0.341842	0.2313126
0.7	0.3122539	0.3105517	0.2155212
0.8	0.2896916	0.2789654	0.2008079
0.9	0.2660852	0.2430632	0.187099
1.0	0.2419707	0.2186158	0.174326
1.2	0.1941861	0.1847812	0.1513364
1.4	0.1497275	0.1229697	0.1313788
1.6	0.1109208	0.0894956	0.114053
1.8	0.0789502	0.0642868	0.0990121
2.0	0.0539910	0.0457469	0.0859547
2.3	0.0283270	0.0271349	0.0695251
2.6	0.0135830	0.0159502	0.0562359
2.9	0.0059525	0.00932589	0.0454869
3.0	0.0044318	0.00779267	0.0423815
3.4	0.0012322	0.00378871	0.0319403

Z : Standard Normal Variate, L : Standard Logistic Variable,

T : Laplace Variable with mean zero and variance one.

Values of x	P[Z ≤ x]	P[L ≤ x]	P[T ≤ x]
0.0	0.5	0.5	0.5
0.1	0.53983	0.5449731	0.5341343
0.2	0.57926	0.5897083	0.5659383
0.3	0.61791	0.6327749	0.5955712
0.4	0.65542	0.6738213	0.6231808
0.5	0.6914625	0.7123654	0.6489057
0.6	0.72575	0.7480576	0.6728744
0.7	0.75804	0.7806844	0.6952068
0.8	0.78814	0.810583	0.7160145
0.9	0.81595	0.8365008	0.7354019
1.0	0.8413447	0.8598204	0.7534656
1.2	0.88493	0.8981246	0.7859777
1.4	0.919243	0.9268526	0.8142022
1.6	0.945291	0.9479491	0.8387046
1.8	0.96407	0.9632026	0.8599757
2.0	0.9772499	0.9741082	0.8784416
2.3	0.989276	0.9848089	0.9016765
2.6	0.9953388	0.9911274	0.9204703
2.9	0.9981342	0.9948316	0.9356718
3.0	0.9986501	0.9956852	0.9400634
3.4	0.9996631	0.9979067	0.9548294

2.5. Logistic function.

The logistic function is given by

$$P(y; \alpha, \beta) = \frac{1}{1 + \exp\{-(\alpha + \beta y)\}} \tag{2.34}$$

where $-\infty < y < \infty$, $\alpha = \frac{-\pi\mu}{\sigma\sqrt{3}}$ and $\beta = \frac{\pi}{\sigma\sqrt{3}}$.

Thus $P(y; \alpha, \beta)$ is also a function of μ and σ. It can be viewed as distribution function of logistic random variable.

The straight line transform of (2.34) with α as the intercept and β as slope can be written as

$$\text{logit } P = \ln\left(\frac{P}{Q}\right) = \alpha + \beta y \tag{2.35}$$

with $Q = 1 - P$, $P \equiv P(y; \alpha, \beta)$.

The differential equation of logistic function (2.34) is

$$\frac{dP}{dy} = -\beta P(1-P) .$$

In bioassay the function (2.34) is taken as a model for true probabilities of response at different doses. It is discussed in detail in chapter 4.

Logistic function is also applicable as a model in epidemiologic studies like case-control study or follow-up study. In follow-up study suppose that study period of specific disease development in an individual, disease-free at time t_0, is from time t_0 to time t_1. Y is a variable taking value one if the disease develops in the individual during study period and value zero otherwise. Then the probability of disease development is a function of independent variables $X_1, X_2, \ldots, X_k$ measured at t_0 which are related to disease development process. This probability of

$$\{ Y = 1/x_1, x_2, \ldots, x_k \} \equiv P(\underline{X}) \text{ (say)}$$

is modelled using logistic function as

$$P(\underline{X}) = \frac{1}{1 + \exp\{-(\alpha+\beta'\underline{X})\}} \tag{2.36}$$

where $\beta' = (\beta_1, \beta_2, \ldots, \beta_k)$

$\underline{X}' = (X_1, X_2, \ldots, X_k)$.

Kleinbaum, Kupper, Chambless (1982) have used logistic regression analysis to epidemiologic studies concerned with quantifying an association between a study factor (i.e. an exposure variable) and a health outcome (i.e. disease status). If there is a positive association between the factor and the disease, those exposed will tend to develop the disease while those not exposed will tend not to develop it. Techniques for fitting logistic models to data and for making statistical inferences concerning ratio measures of association (i.e. risk ratio and odds ratio) were also discussed. Particular attention was given to conditional and unconditional maximum likelihood (ML) procedures and specific numerical applications were presented. The use of logistic function as a model for human population grouth is re-examined by Leach (1981). He has used the model

$$P_t = \frac{P_s}{1 + (\frac{P_s}{P_o} - 1)\exp(-rt)} \qquad (2.37)$$

where P_t = the population after time t

P_s = the asymptotic maximum or saturation level of the population

P_o = Population at t = 0

r = growth rate constant .

It is observed that the generalization of the logistic function fits substentially better than the model (2.37) for the population of Great Britain from 1801 to 1971.

Generalization of logistic function as a model for human population growth is

$$P_t = d + \frac{\lambda_s}{1 + (\frac{\lambda_s}{\lambda_o} - 1) \exp\{-rt\}}$$

where population at $t = 0$ is $d + \lambda_o$ and saturation level is $d + \lambda_s$.

CHAPTER 3

ESTIMATION OF PARAMETERS OF THE LOGISTIC DISTRIBUTION

3.0 Introduction :

In this chapter different methods of estimation of parameters of logistic distribution from complete sample and censored sample are discussed.

The method of finding maximum likelihood estimator when complete sample is available is described in Section 3.1. Minimal sufficient statistic for the family of logistic distribution is obtained in the same section.

In Section 3.2, the method of estimation of the parameters based on sample quantiles is considered. It also gives review of L, R_n and W_n estimators for the location parameter. Tolerance limits and confidence intervals for the parameters are obtained in Section 3.3 and 3.4 respectively.

Method of estimation of the parameters from symmetrically censored and one sided censored samples using simple estimators is described in Section 3.5. The estimators proposed by Plackett (1958) and Gupta, Qureishi, Shah (1967) for estimating the parameters from censored sample are discussed in Section 3.6.

3.1. Maximum Likelihood Estimators:

Let $X_1, X_2, \ldots, X_n$ be independent random variables each having $L(\mu, \sigma)$ distribution with p.d.f.

$$f(x; \mu, \sigma) = \frac{\exp\{-\frac{\pi}{\sigma\sqrt{3}}(x-\mu)\}}{[1+\exp\{-\frac{\pi}{\sigma\sqrt{3}}(x-\mu)\}]^2} \tag{3.1}$$

$$-\infty < x < \infty, \; -\infty < \mu < \infty, \sigma > 0.$$

After observing $\underset{\sim}{X} = \underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$, the likelihood function of (μ, σ) is given by

$$L_{\underset{\sim}{x}}(\mu, \sigma) = \left(\frac{\pi}{\sigma\sqrt{3}}\right)^n e^{-\frac{\pi}{\sigma\sqrt{3}} \sum_{i=1}^{n}(x_i-\mu)}$$

$$\prod_{i=1}^{n}[1+\exp\{-\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\}]^{-2}. \tag{3.2}$$

Taking logarithms of both sides of (3.2)

$$\text{Ln } L_{\underset{\sim}{x}}(\mu, \sigma) = n \,\ell n\left(\frac{\pi}{\sigma\sqrt{3}}\right) - \frac{\pi}{\sigma\sqrt{3}} \sum_{i=1}^{n}(x_i-\mu) - 2\sum_{i=1}^{n} \ell n[1+\exp\{-\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\}]. \tag{3.3}$$

Differentiating $\ell n \, L_{\underset{\sim}{x}}(\mu, \sigma)$ with respect to μ and σ and equating to zero, we have

$$\frac{\partial}{\partial \mu} \ln L_{\underline{X}}(\mu,\sigma) = \frac{n\pi}{\sigma\sqrt{3}} - \frac{2\pi}{\sigma\sqrt{3}} \sum_{i=1}^{n} \left[1-\exp\left\{-\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\right\}\right]^{-1} \exp\left\{-\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\right\}$$

$$= \frac{\pi}{\sigma\sqrt{3}}\left(n-2\sum_{i=1}^{n}\left[1+\exp\left\{\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\right\}\right]^{-1}\right) \qquad (3.4)$$

$$\frac{\partial \ln L_{\underline{X}}(\mu,\sigma)}{\partial \mu} = 0$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} \frac{1}{1+\exp\left\{\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\right\}} = \frac{1}{2} \quad . \qquad (3.5)$$

Now

$$\frac{\partial \ln L_{\underline{X}}(\mu,\sigma)}{\partial \sigma} = -\frac{n}{\sigma} + \frac{\pi}{\sigma^2\sqrt{3}} \sum_{i=1}^{n}(x_i-\mu)$$

$$-2\sum_{i=1}^{n}\left[1+\exp\left\{-\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\right\}\right]^{-1}\exp\left\{-\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\right\}$$

$$\frac{\pi}{\sigma^2\sqrt{3}}(x_i-\mu)$$

$$= \frac{\pi}{\sigma^2\sqrt{3}}\left[\sum_{i=1}^{n}(x_i-\mu)\left(\frac{\exp\left\{\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\right\}-1}{1+\exp\left\{\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\right\}}\right)\right] - \frac{n}{\sigma} \qquad (3.6)$$

$$\frac{\partial \ln L_{\underline{X}}(\mu,\sigma)}{\partial \sigma} = 0$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n} \frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\left\{\frac{\exp\left\{\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\right\}-1}{1+\exp\left\{\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\right\}}\right\} = 1. \qquad (3.7)$$

Thus in order to get the maximum likelihood estimates, we have to solve equations (3.5) and (3.7). Antle, Klimko and Harkness (1970) have shown that the likelihood equations have a unique set of solutions. Hence the Newton-Raphson method can be applied to solve these equations.

Let $\theta = (\mu, \sigma)$ and $\hat{\theta}(0) = (\hat{\mu}(0), \hat{\sigma}(0))$ be the first approximation to maximum likelihood estimates. The second approximation to θ is given by

$$\hat{\theta}(1) = \hat{\theta}(0) - D(\hat{\theta}(0))\, B^{-1}(\hat{\theta}(0))$$

where

$$D(\hat{\theta}(0)) = \left[\frac{\partial \ln L_{\underline{X}}(\mu,\sigma)}{\partial \mu}, \frac{\partial \ln L_{\underline{X}}(\mu,\sigma)}{\partial \sigma} \right]_{[\hat{\theta}(0)=(\hat{\mu}(0),\hat{\sigma}(0))]}$$

$$B(\hat{\theta}(0)) = \begin{bmatrix} \dfrac{\partial^2 L_{\underline{X}}(\mu,\sigma)}{\partial \mu^2} & \dfrac{\partial^2 \ln L_{\underline{X}}(\mu,\sigma)}{\partial \mu \partial \sigma} \\ \dfrac{\partial^2 \ln L_{\underline{X}}(\mu,\sigma)}{\partial \mu \partial \sigma} & \dfrac{\partial^2 \ln L_{\underline{X}}(\mu,\sigma)}{\partial \sigma^2} \end{bmatrix}_{\hat{\theta}(0)} .$$

This iterative procedure is continued taking k^{th} step as

$$\hat{\theta}(k) = \hat{\theta}(k-1) - D(\hat{\theta}(k-1))\, B^{-1}(\hat{\theta}(k-1)). \tag{3.8}$$

We stop at r^{th} iteration if

$| \hat{\theta}(r,j) - \hat{\theta}(r-1,j) | < \varepsilon_j$ for $j = 1,2$ where $\underset{\sim}{\varepsilon}$ is the desired level of accuracy.

We can start with initial solution $\hat{\theta}(0)$ as $(\bar{x}, s)$ where

$$s = +\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2} .$$

Then $\quad D(\hat{\theta}(k-1)) = [\ d_1,\ d_2\ ]$

where

$$d_1 = \frac{\pi}{\hat{\sigma}(k-1)\sqrt{3}}\ [n-2\sum_{i=1}^{n}(1+\exp\{-\frac{\pi\ (x_i-\hat{\mu}(k-1))}{\hat{\sigma}(k-1)\sqrt{3}}\})^{-1}] ,$$

$$d_2 = \frac{\pi}{\hat{\sigma}^2(k-1)\sqrt{3}}[\ \sum_{i=1}^{n}(x_i-\hat{\mu}(k-1))\{\frac{\exp\{-\frac{\pi}{\hat{\sigma}(k-1)\sqrt{3}}(x_i-\hat{\mu}(k-1))\}-1}{1+\exp\{-\frac{\pi}{\hat{\sigma}(k-1)\sqrt{3}}(x_i-\hat{\mu}(k-1))\}}\}]$$
$$-\frac{n}{\hat{\sigma}(k-1)} .$$

In order to obtain $B(\hat{\theta}(k-1))$, we must find second order derivatives of $\ell n\ L_{\underset{\sim}{x}}(\mu,\sigma)$ with respect to μ and σ .

Differentiating both sides of (3.4) with respect to μ, we have

$$\frac{\partial^2 \ell n\ L_{\underset{\sim}{x}}(\mu,\sigma)}{\partial\mu^2} = -\frac{2\pi^2}{\sigma^2}\sum_{i=1}^{n}\frac{\exp\{-\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\}}{[1+\exp\{-\frac{\pi}{\sigma\sqrt{3}}\ (x_i-\mu)\}]^2} \qquad (3.9)$$

Differentiating both sides of (3.6) with respect to σ, we get

$$\frac{\partial^2 \ln L_{\underline{x}}(\mu,\sigma)}{\partial\sigma^2} = -\frac{2\pi}{\sigma^2\sqrt{3}}\left[\sum_{i=1}^{n}(x_i-\mu)\left\{\frac{\exp\{\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\}-1}{1+\exp\{\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\}}\right\}\right]$$

$$-\frac{2\pi^2}{3\sigma^4}\left[\sum_{i=1}^{n}\frac{(x_i-\mu)^2\exp\{\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\}}{[1+\exp\{\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\}]^2}\right] + \frac{n}{\sigma^2}\,. \qquad (3.10)$$

Differentiating (3.4) with respect to σ we have

$$\frac{\partial^2 \ln L_{\underline{x}}(\mu,\sigma)}{\partial\mu\partial\sigma} = -\frac{n\pi}{\sigma^2\sqrt{3}} + \frac{2\pi}{\sigma^2\sqrt{3}}\sum_{i=1}^{n}\left[1+\exp\{\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\}\right]^{-1}$$

$$+\frac{\pi^2}{3\sigma^3}\sum_{i=1}^{n}\frac{\exp\{\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\}}{[1+\exp\{\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\}]^2}(x_i-\mu). \qquad (3.11)$$

Then,

$$B(\hat{\theta}(k-1)) = \begin{bmatrix} b_{11} & b_{12} \\ & \\ b_{21} & b_{22} \end{bmatrix} \qquad \text{where}$$

$$b_{11} = \left.\frac{\partial^2 \ln L_{\underline{x}}(\mu,\sigma)}{\partial\mu^2}\right]_{\hat{\theta}(k-1)}$$

$$b_{12} = b_{21} = \left.\frac{\partial^2 \ln L_{\underline{x}}(\mu,\sigma)}{\partial\mu\partial\sigma}\right]_{\hat{\theta}(k-1)}$$

$$b_{22} = \left. \frac{\partial^2 \ell n\, L_{\underset{\sim}{x}}(\mu,\sigma)}{\partial\sigma^2} \right]_{\hat{\theta}(k-1)} .$$

Thus maximum likelihood estimators of μ and σ can be found by iterative procedure of desired level of accuracy with k^{th} iteration as

$$[\hat{\theta}(k),\hat{\sigma}(k)] = [\hat{\mu}(k-1),\hat{\sigma}(k-1)] - [d_1,d_2] \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}^{-1} . \qquad (3.12)$$

But this method suffers from spider webb effect (see: Discrete Statistical Models Social Science Applications page No.49-50). Hence some times this method does not converge to the maximum likelihood estimators.

Illustrative Example 3.1. :

We are interested in obtaining the maximum likelihood estimators of the following random sample generated from standard logistic distribution.

1.4550166, -0.3115278, 1.0301834, -0.301224,

0.1063156, 0.6935024, 0.4276079, -1.0350541,

-2.0440766, -0.7087509 .

Suppose that desired level of accuracy

$$\underline{\varepsilon} = (0.001,\ 0.001).$$

Here,

$$\hat{\theta}(0) = (\bar{x}, s) = (-0.0687992,\ 1.0406104)$$

$$D(\hat{\theta}(0)) = [\ 0.3707531,\ 0.0218486\ ]$$

$$B(\hat{\theta}(0)) = \begin{bmatrix} -9.8350563\ , & -0.1462987 \\ -0.1462987\ , & -14.314756 \end{bmatrix}.$$

Hence,

$$\hat{\theta}(1) = [\ -0.0687992,\ 1.0406104\ ] - [\ 0.3707531,\ 0.0218486\ ] \begin{bmatrix} -0.1016925\ , & 0.0010393 \\ 0.0010393\ , & -0.0698686 \end{bmatrix}$$

$$= [\ -0.031124,\ 1.0417516\ ].$$

$$\therefore\ |\hat{\theta}(1) - \hat{\theta}(0)| = [\ 0.0376752\ ,\ 0.0011412\ ].$$

Since $|\hat{\theta}(1) - \hat{\theta}(0)| > \underline{\varepsilon}$, we go to the next iteration.

$$\hat{\theta}(2) = \hat{\theta}(1) - D(\hat{\theta}(1))\ B^{-1}(\hat{\theta}(1)).$$

$$D(\hat{\theta}(1)) = [\ -0.0055005,\ -0.0097009\ ].$$

$$B(\hat{\theta}(1)) = \begin{bmatrix} -7.2510843 & 0.1261233 \\ 0.1261233 & -14.028511 \end{bmatrix}$$

$$\therefore \quad \hat{\theta}(2) = [-0.031124,\ 1.0417516] -$$

$$[-0.0055005,\ -0.0097009] \begin{bmatrix} -0.1379319 & -0.00124 \\ -0.00124 & -0.0712945 \end{bmatrix}$$

$$= [-0.0318956,\ 1.0410532].$$

$$\therefore |\hat{\theta}(2) - \hat{\theta}(1)| = [0.0007706\ ,\ 0.0006984].$$

As $|\hat{\theta}(2) - \hat{\theta}(1)| < \varepsilon$, we stop taking iterations. Thus the required maximum likelihood estimates of μ and σ are

-0.0318956 and 1.0410532 respectively.

Minimal Sufficient Statistic for the family of logistic distribution :

Consider the joint density of sample drawn from $L(\mu,1)$ which is given by

$$\mathbb{P} \equiv \mathbb{P}_{(\mu,1)}(x) = \left(\frac{\pi}{\sqrt{3}}\right)^n \exp\left\{-\frac{\pi}{\sqrt{3}} \sum_{i=1}^{n}(x_i - \mu)\right\}$$

$$\prod_{i=1}^{n} \left[1+\exp\left\{-\frac{\pi}{\sqrt{3}}(x_i - \mu)\right\}\right]^{-2}.$$

Let $\mathbb{P}_0$ be subfamily of $\mathbb{P}$ with $\mu_0 = 0, \mu_1, \ldots, \mu_k$. As per theorem 5.3. in Lehmann (1983), we have that $T(\underset{\sim}{X}) = [T_1(X), T_2(X) \ldots T_k(X)]$ where

$$T_r(x) = e^{n\mu_r} \prod_{i=1}^{n} \left[\frac{1 + e^{-\frac{\pi}{\sqrt{3}} x_i}}{1 + \exp\{-\frac{\pi}{\sqrt{3}}(x_i - \mu_r)\}} \right]^2 ; \; r = 1, 2, \ldots, k,$$

is minimal sufficient for $\mathbb{P}_0$. It will be minimal sufficient for $\mathbb{P}$ provided $T(\underset{\sim}{X})$ is equivalent to order statistics.

That is if $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)$, $\underset{\sim}{y} = (y_1, y_2, \ldots, y_n)$ and $k = n+1$, then $T_r(\underset{\sim}{x}) = T_r(\underset{\sim}{y})$ for $r = 1, 2, \ldots, n+1$,

or

$$\prod_{i=1}^{n} \left[\frac{1 + \exp\{-\frac{\pi}{\sqrt{3}} x_i\}}{1 + \exp\{-\frac{\pi}{\sqrt{3}}(x_i - \mu_r)\}} \right]^2 = \prod_{i=1}^{n} \left[\frac{1 + \exp\{-\frac{\pi}{\sqrt{3}} y_i\}}{1 + \exp\{-\frac{\pi}{\sqrt{3}}(y_i - \mu_r)\}} \right]^2, r = 1, 2, \ldots n+1.$$

$\therefore$ It is equivalent to show that

$$\prod_{i=1}^{n} \frac{1 + \phi u_i}{1 + u_i} = \prod_{i=1}^{n} \frac{1 + \phi_r v_i}{1 + v_i} \text{ for } \phi_r = \phi_1 \ldots \phi_{n+1} ;$$

where $\phi_r = e^{-\frac{\pi}{\sqrt{3}} \mu_r}$, $u_i = e^{-\frac{\pi}{\sqrt{3}} x_i}$,

$v_i = e^{-\frac{\pi}{\sqrt{3}} y_i}$.

Both sides of above equation are polynomials of degree n provided coefficients of ϕ_r^p on both sides are equal.

If p = 0, then $\prod_{i=1}^{n}(1+u_i) = \prod_{i=1}^{n}(1+v_i)$.

Hence,

$$\prod_{i=1}^{n} 1+\phi u_i = \prod_{i=1}^{n} 1+\phi v_i \text{ for } \phi = \phi_1 \ldots \phi_{n+1}$$

suppose $\xi = -\frac{1}{\phi}$, then

$$\prod_{i=1}^{n}(\xi+u_i) = \prod_{i=1}^{n}(\xi+v_i) \text{ for } \xi = \xi_1 \ldots \xi_{n+1} .$$

So that two polynomials in ξ have same roots. This means that x_i's and y_i's have the same order statistics.

Hence $T(\underline{X})$ is minimal sufficient for $\mathbb{P}$.

3.2. Method of estimation based on sample quantiles :

Let $x_1, x_2, \ldots, x_n$ be a random sample from a logistic distribution with distribution function

$$F(x;\mu,\sigma) = \frac{1}{1+\exp\{-\frac{\pi}{\sigma\sqrt{3}}(x-\mu)\}} ; \quad -\infty < x < \infty$$

$$-\infty < \mu < \infty$$

$$\sigma > 0 .$$

suppose that $x_{(n_1)} < x_{(n_2)} < \ldots < x_{(n_k)}$ are K sample quantiles in a sample of size n from logistic distribution.

Let us define

$$\lambda_i = \frac{n_i}{n} \text{ for } i = 1,2,\ldots, k. \quad \lambda_o \equiv 0 \quad \lambda_{k+1} = 1$$

$$u_i = \ln \left\{\frac{\lambda_i}{1-\lambda_i}\right\} \; ; \; i=1,2,\ldots,k$$

$$f_i = \lambda_i(1-\lambda_i) \; ; \; f_o \equiv f_{k+1} \equiv 0$$

$$X = \sum_{i=1}^{k+1} (1-\lambda_i-\lambda_{i-1}) \left\{\lambda_i(1-\lambda_i)\, x_{(n_i)} - \lambda_i(1-\lambda_i) x_{(n_{i-1})}\right\}$$

$$Y = \sum_{i=1}^{k+1} \left\{ \frac{\left[\lambda_i(1-\lambda_i) \ln \left\{\frac{\lambda_i}{1-\lambda_i}\right\} - \lambda_{i-1}(1-\lambda_{i-1}) \ln \left\{\frac{\lambda_{i-1}}{1-\lambda_{i-1}}\right\}\right] \left[\lambda_i(1-\lambda_i) x_{(n_i)} - \lambda_{i-1}(1-\lambda_{i-1})\, x_{(n_{i-1})}\right]}{\lambda_i - \lambda_{i-1}} \right\}$$

$$C_1 = \sum_{i=1}^{k+1} (\lambda_i-\lambda_{i-1})\,(1-\lambda_i-\lambda_{i-1})^2$$

$$C_2 = \sum_{i=1}^{k+1} \frac{\left[\lambda_i(1-\lambda_i) \ln \left\{\frac{\lambda_i}{1-\lambda_i}\right\} - \lambda_{i-1}(1-\lambda_{i-1}) \ln \left\{\frac{\lambda_{i-1}}{1-\lambda_{i-1}}\right\}\right]^2}{\lambda_i - \lambda_{i-1}}$$

$$C_3 = \sum_{i=1}^{k+1} (1-\lambda_i-\lambda_{i-1}) \left[\lambda_i(1-\lambda_i) \ln \left\{\frac{\lambda_i}{1-\lambda_i}\right\} - \lambda_{i-1}(1-\lambda_{i-1}) \ln\left\{\frac{\lambda_{i-1}}{1-\lambda_{i-1}}\right\}\right].$$

Estimation of parameters in three different cases viz., estimation of μ when σ is known, estimation of σ when μ is known and estimation of both μ and σ is considered :

(i) Estimation of μ with σ known.

When the scale parameter σ is known, the optimum spacing of the sample quantiles for estimating the location parameter μ is obtained for a given number of quantiles. The general expressions for the best linear unbiased estimator were derived by Ogawa (1951). These expressions are

$$\mu^* = \frac{X-\sigma C_3}{C_1} \tag{3.13}$$

for a fixed number of quantiles k and for fixed values of λ_i's and

$$\operatorname{var}(\mu^*) \doteq \frac{\sigma^2}{n} \frac{1}{C_1} \quad . \tag{3.14}$$

Gupta and Gnanadesikan (1966) have shown that $\operatorname{var}(\mu^*)$ is minimized at $\lambda_i = \frac{i}{k+1}$.

For $\lambda_i = \frac{i}{k+1}$,

$$\left.\begin{array}{l} \lambda_i + \lambda_{k-i+1} = 1, \; u_i + u_{k-i+1} = 0 \\ f_i = f_{k-i+1} \text{ for } i = 1,2,\ldots, k \end{array}\right\} \quad . \tag{3.15}$$

From (3.15) it can be seen that the spacing is symmetric and

$$C_3 = \sum_{i=1}^{k+1}(1-\lambda_i-\lambda_{i-1})\ \{f_i u_i - f_{i-1}u_{i-1}\}$$

$$= \sum_{i=1}^{k+1}\{f_i u_i - f_{i-1}\ u_{i-1}\} - \sum_{i=1}^{k+1}$$

$$\{\lambda_i f_i u_i + \lambda_{i-1}\ f_i u_i - \lambda_i\ f_{i-1}u_{i-1} - \lambda_{i-1}\ f_{i-1}\ u_{i-1}\}$$

$$= \sum_{i=1}^{k+1} \lambda_i\ f_{i-1}\ u_{i-1} - \sum_{i=1}^{k+1} \lambda_{i-1}\ f_i u_i$$

$$= \frac{2}{k+1} \sum_{i=1}^{k} f_i u_i .$$

If k is an even number,

$$f_1 u_1 = -f_k\ u_k$$

$$f_2 u_2 = -f_{k-1}\ u_{k-1}$$

$$\vdots$$

$$f_{\frac{k}{2}}\ u_{\frac{k}{2}} = -\ f_{\frac{k+2}{2}}\ \ u_{\frac{k+2}{2}}$$

$$\therefore \sum_{i=1}^{k} f_i u_i = 0 \quad . \tag{3.16}$$

If k is an odd number,

$$f_1 u_1 = -f_k u_k$$

$$\vdots$$

$$f_{\frac{k-1}{2}} \; u_{\frac{k-1}{2}} = - f_{\frac{k+3}{2}} \; u_{\frac{k+3}{2}}$$

$$f_{\frac{k+1}{2}} \; u_{\frac{k+1}{2}} = 0$$

$$\therefore \quad \sum_{i=1}^{k} f_i u_i = 0 \; . \tag{3.17}$$

From (3.16) and (3.17), we have

$$C_3 = 0 \quad \forall \, k \quad \text{if} \quad \lambda_i = \frac{i}{k+1} \; .$$

In this case, $\mu^* = \frac{X}{C_1}$, and X simplifies to

$$X = \sum_{i=1}^{k+1} (1-\lambda_i-\lambda_{i-1}) \; \{\lambda_i(1-\lambda_i)X_{(n_i)} - \lambda_{i-1}(1-\lambda_{i-1})X_{(n_{i-1})}\}$$

$$= \sum_{i=1}^{k+1} \frac{(k+2-2i)}{k+1} \{ \frac{i}{k+1} \left(\frac{k+1-i}{(k+1)}\right) X_{(n_i)} - \left(\frac{i-1}{k+1}\right)\left(\frac{k+2-i}{k+1}\right) X_{(n_{i-1})}\}$$

$$= \frac{1}{(k+1)^3} \sum_{i=1}^{k+1} (k+2-2i) \; \{i(k+1-i)X_{(n_i)} - (i-1)(k+2-i) \; X_{(n_{i-1})}\} .$$

Now, $C_1 = \sum_{i=1}^{k+1} (\lambda_i - \lambda_{i-1}) \; (1-\lambda_i - \lambda_{i-1})^2$

$$= \sum_{i=1}^{k+1} \frac{1}{k+1} \; \frac{(k+2-2i)^2}{(k+1)^2}$$

$$= \frac{1}{(k+1)^3} \sum_{i=1}^{k+1} (k+2-2i)^2 .$$

Hence the best linear unbiased estimator of the unknown parameter μ is given by

$$\mu^* = \frac{K[k\, X_{(n_1)} - 0] + (k-2)\, [2(k-1)X_{(n_2)} - k\, X_{(n_1)}] + (k-4)[3(k-2)X_{(n_3)} - 2(k-1)X_{(n_2)}] + \ldots + (-k)[0 - k\, X_{(n_k)}]}{\sum_{i=1}^{k+1} (k+2-2i)^2}$$

$$= \frac{12 \sum_{i=1}^{k} i\ (k+1-i)\ X_{(n_i)}}{(k+1)\ (k+2)\ [6k + 12 - 12k - 24 + 8k + 12]}$$

$$= \frac{6 \sum_{i=1}^{k} i\ (k+1-i)\ X_{(n_i)}}{K(k+1)\ (k+2)} . \qquad (3.18)$$

The expression for approximate var(μ^*) can be obtained from (3.14) as

$$\mathrm{var}(\mu^*) = \frac{\sigma^2}{n} \frac{(k+1)^3}{\sum_{i=1}^{k+1} (k+2-2i)^2}$$

$$= \frac{\sigma^2}{n} \left[\frac{(k+1)^3}{\{(k+1)\ (k+2)^2 - 2\ (k+1)\ (k+2)^2} \right.$$

$$\left. + \frac{4(k+1)\ (k+2)\ (2k+3)}{6} \} \right]$$

$$= \frac{\sigma^2}{n} \quad \frac{3(k+1)^3}{k(k+1)\ (k+2)}$$

$$= \frac{\sigma^2}{n} \quad \frac{(k+1)^2}{k(k+2)} \quad . \tag{3.19}$$

Gupta and Gnanadesikan (1966) have obtained the exact variance of μ^* as

$$\text{var}(\mu^*) = \frac{9}{\pi^2} \frac{\sigma^2}{n} \frac{(k+1)^2}{k(k+2)} \, . \tag{3.20}$$

The Cramer-Rao lower bound for variance of an unbiased estimator of μ with σ known is $\frac{9\sigma^2}{\pi^2 n}$.

Hence the relative efficiency of the unbiased estimator μ^* when σ is known based on optimum spacing of sample quantiles

$$= \frac{\text{Cramer-Rao lower bound of } \mu^*}{\text{var } (\mu^*)}$$

$$= \frac{k(k+2)}{(k+1)^2}$$

$$= [\, 1 - \frac{1}{k^2+2k+1} \,]. \tag{3.21}$$

From (3.21) we observe that with increase in number of sample quantiles k, relative efficiency of μ^* also increases. For k=4, the efficiency is 0.96 while for k=6, the efficiency is approximately 0.98.

Illustrative Example 3.2.

Suppose that following are the observations in a random sample from logistic distribution with standard deviation one we want to estimate mean of the distribution using equal and unequal spacing of sample quantiles.

0.0668 ,	1.4644,	0.8532,	-0.3125,	-1.4761,
-0.9175 ,	0.6479,	-0.2314,	-1.3012,	0.3232,
1.4576 ,	0.1257.			

The parameter μ can be estimated as follows :

Ordered Observations

-1.4761

-1.3012

-0.9175

-0.3125

-0.2314

0.0668

0.1257

0.3232

0.6479

0.8352

1.4576

1.4644.

(i) Estimation of μ using equal spacing of sample quantiles.

Suppose we take k=3 with equal spacings. Then $n_1=3$, $n_2=6$, $n_3=9$

$$X_{(n_1)} = -0.9175,\ X_{(n_2)} = 0.0668,\ X_{(n_3)} = 0.6479.$$

Thus,

$$\mu^* = \frac{6 \sum_{i=1}^{3} i\ (4-i)\ X_{(n_i)}}{60}$$

$$= -0.05416\ .$$

(ii) Estimation of μ using unequal spacing of sample quantiles.

Suppose we consider first three observations only to estimate the value of μ.

i.e. $n_1=1$, $n_2=2$, $n_3=3$, with $\lambda_1 = \frac{1}{12}$, $\lambda_2 = \frac{2}{12}$, $\lambda_3 = \frac{3}{12}$.

Here $C_1 = \sum_{i=1}^{4} (\lambda_i - \lambda_{i-1})\ (1-\lambda_i-\lambda_{i-1})^2$ where $\lambda_0 = 0$ and $\lambda_4 = 1$

$$= 0.1921296.$$

$$X = \sum_{i=1}^{4} (1-\lambda_i-\lambda_{i-1})\ \{\lambda_i(1-\lambda_i)X_{(n_i)} - \lambda_{i-1}(1-\lambda_{i-1})X_{(n_{i-1})}\}$$

$$= -0.2206602.$$

$$C_3 = \sum_{i=1}^{4} (1-\lambda_i-\lambda_{i-1})[\lambda_i(1-\lambda_i)\ \ln\{\frac{\lambda_i}{1-\lambda_i}\} - \lambda_i(1-\lambda_{i-1})\ \ln\{\frac{\lambda_{i-1}}{1-\lambda_{i-1}}\}]$$

$$= -0.2394423\ .$$

Hence $\mu^* = \frac{X-C_3}{C_1}$

$= 0.0977574$.

Estimation of σ when μ is known :

Ogawa (1951) has shown that the best linear unbiased estimator of σ when μ is known on the basis of k sample quantiles is

$$\sigma^* = \frac{Y - \mu C_3}{C_2} \quad \text{and} \quad \text{var}(\sigma^*) = \frac{\sigma^2}{nC_2} \ .$$

In this case, the optimum spacing of sample quantiles for σ^* can be obtained by maximizing C_2. But it is not possible to solve the set of equations

$$\frac{\partial C_2}{\partial \lambda_i} = 0 \ ; \quad i = 1,2,\ldots, k$$

explicitly for λ_i .

Illustrative Example 3.3:

A random sample of 12 observations was generated from standard logistic distribution. The observations were found to be

-0.9994325, 0.2006645, 0.4411288, -0.452068,
-0.0884015, -0.493651 , 0.9563355, -0.2934264,
0.2466161, -0.52 0708, 2.1456726, 0.3656932 .

We want to estimate the value of the parameter σ on the basis of this sample.

The ordered observations are :

-0.9994325, -0.5207088, -0.493651, -0.4520288,
-0.2934264, -0.0884015, 0.2006645, 0.2463161,
0.3656932, 0.4411288, 0.9563355, 2.1456726 .

Let

$$n_1 = 1 \ , \ n_2 = 7, n_3 = 11$$

$$\lambda_1 = 1/12, \ \lambda_2 = \frac{7}{12} \ , \quad \lambda_3 = \frac{11}{12}$$

$$Y = \sum_{i=1}^{4} \frac{\left[\lambda_i(1-\lambda_i)\ln\{\frac{\lambda_i}{1-\lambda_i}\} - \lambda_{i-1}(1-\lambda_{i-1})\ln\{\frac{\lambda_{i-1}}{1-\lambda_{i-1}}\}\right][\lambda_i(1-\lambda_i)X_{(n_i)} - \lambda_{i-1}(1-\lambda_{i-1})x_{(n_{i-1})}]}{\lambda_i - \lambda_{i-1}}$$

$$= 0.6917395 \ .$$

$$C_2 = \sum_{i=1}^{4} \frac{[\lambda_i(1-\lambda_i)\ln(\frac{\lambda_i}{1-\lambda_i}) - \lambda_{i-1}(1-\lambda_{i-1})\ln(\frac{\lambda_{i-1}}{1-\lambda_{i-1}})]^2}{\lambda_i - \lambda_{i-1}}$$

$$= 0.9765202.$$

$$\therefore \ \sigma^* = \frac{Y}{C_2}$$

$$= 0.7083719.$$

Estimation of μ and σ :

In case both the parameters μ and σ are unknown then the [redacted] estimators given by Gupta and Gnanadesikan (1966) are

$$\mu^* = \frac{C_2 X - C_3 Y}{C_1 C_2 - C_3^2}$$

and

$$\sigma^* = \frac{C_1 Y - C_3 X}{C_1 C_2 - C_3^2} .$$

Also $\operatorname{var}(\mu^*) = \frac{\sigma^2}{n} \frac{C_2}{C_1 C_2 - C_3^2}$

and $\operatorname{var}(\sigma^*) = \frac{\sigma^2}{n} \frac{C_1}{C_1 C_2 - C_3^2}$

$$\operatorname{Cov}(\mu^*, \sigma^*) = - \frac{\sigma^2}{n} \frac{C_3}{C_1 C_2 - C_3^2} .$$

It is very complicated to obtain the optimum spacing of sample quantiles as the method of minimizing the generalized variance involves the simultaneous equations which cannot be solved explicitly.

Illustrative Example 3.4:

The observations given below are in a random sample of size 12 drawn from standard logistic distribution.

0.0725934,	0.7747002,	-0.9388869,	0.0952855,
-1.5669434,	0.2363472,	-0.8796116,	-0.388009
1.2588196,	0.8885079,	-1.9787879,	1.1961259

Ordered Observations :

-1.9787879,	-1.5669434,	-0.9388869,	-0.8796116,
-0.388009,	0.0725934,	0.0952855,	0.2363472,
0.7747002,	0.8885079,	1.1961259,	1.2588196.

We want to estimate the values of μ and σ.

Let $n_1 = 2, \quad n_2 = 7, \quad n_3 = 10$

$$\lambda_1 = \frac{1}{6}, \quad \lambda_2 = \frac{7}{12}, \quad \lambda_3 = \frac{5}{6}$$

$$X_{(n_1)} = -1.5669434, \quad X_{(n_2)} = 0.0952855,$$

$$X_{(n_3)} = -1.1961259.$$

$$X = -0.0600931$$

$$Y = 0.6906916$$

$$C_1 = 0.1898145$$

$$C_2 = 0.9036968$$

$$C_3 = 0.0172654$$

$$\mu^* = \frac{C_2 X - C_3 Y}{C_1 C_2 - C_3^2}$$

$$= -0.3854386.$$

$$\sigma^* = \frac{-C_3 X + C_1 Y}{C_1 C_2 - C_3^2}$$

$$= 0.769008 .$$

L estimator for location parameter μ:

Lehmann (1983) discusses L, R_n and W_n estimators for the location parameter μ. Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ be order statistics in a random sample of size n from logistic distribution with distribution function $F(x)$ and density $f(x)$. An estimator of the form $\sum_{i=1}^{n} W_{in} X_{i:n}$ with $\sum_{i=1}^{n} W_{in} = 1$ is called L estimator of the location parameter μ , we shall denote it by $\hat{\mu}_L$.

In order to find asymptotic distribution of $\hat{\mu}_L$, suppose that the weights are specified by means of a probability distribution Λ defined on (0,1) with density $\lambda(t)$ and $W_{in} \propto \lambda(\frac{i}{n+1})$.

Define $\gamma(x) = -\frac{f'(x)}{f(x)}$

and A(t) is a function with

$$A'(t) = \frac{\lambda(t)}{f[F^{-1}(t)]} .$$

It can be shown that [See : Lehmann (1983)] if F is symmetric about μ and $\lambda(t)$ is symmetric about $\frac{1}{2}$ then

$$\sqrt{n}\left(\sum_{i=1}^{n} W_{in} X_{i:n} - \int_0^1 \lambda(u) F^{-1}(u)du\right) \xrightarrow{d} N[0,\sigma^2(F,\lambda)]$$

where $\sigma^2(F,\lambda) = \int_0^1 A^2(t)\,dt - \left(\int_0^1 A(t)\,dt\right)^2$.

Further, the variance $\sigma^2(F,\lambda)$ is minimum when

$$\lambda_0(t) = \frac{\gamma'[F^{-1}(t)]}{\int_{-\infty}^{\infty} \gamma^2(x)f(x)dx} .$$

The L estimator corresponding to $\lambda_0(t)$ will be denoted by $\hat{\mu}_L(0)$.
For logistic distribution,

$$\gamma(x) = 2F(x) - 1 \quad \text{and} \quad \gamma'(x) = 2f(x)$$

$$\gamma'[F^{-1}(t)] = 2t(1-t) .$$

$\therefore$ $\lambda_0(t)$ is proportional to $t(1-t)$.

Hence $W_{in} \propto \frac{i}{n+1}(1 - \frac{i}{n+1})$ minimize the asymptotic variance.

If $\lambda(t)$ is taken to be the density function of uniform distribution $(k, 1-k)$ i.e. $\lambda(t) = \frac{1}{1-2k}$; $k < t < 1-k$ then $\hat{\mu}_L$ becomes equivalent to the trimmed mean with k observations censored from each end (Details are in Section 3.5.)

R_n and W_n estimators for the parameter μ.

Let

$$\bar{X}_{jk} = \frac{X_{j:n} + X_{k:n}}{2} \;; \quad j \leq k$$

$$j = 1,2,\ldots, n$$

$$k = 1,2,\ldots, n.$$

There are $\frac{n(n+1)}{2}$ values of $\bar{X}_{jk}$. Suppose that $d_{1:n}, d_{2:n}, \ldots, d_{n:n}$ are non-negative constants. If we assign probability

$$W_{jk} = \frac{d_{n-(k-j):n}}{\sum_{i=1}^{n} i\, d_{i:n}} \text{ to } \bar{X}_{jk}, \text{ then}$$

median of this probability distribution is called R_n estimator. We shall denote it by $\hat{\mu}_{R_n}$. In order to obtain an asymptotic distribution of $\hat{\mu}_{R_n}$ define a non-decreasing function R over (0,1) as

$$R(1-t) = R(t)$$

and

$$d_{i:n} = R\left(\frac{i+1}{2n+1}\right) - R\left(\frac{i}{2n+1}\right).$$

It can be shown that under regularity condition on R and F,

$$\sqrt{n}\,(R_n - \mu) \xrightarrow{\mathcal{L}} N\left[\,0, \sigma^2(F,R)\right]$$

where

$$\sigma^2\,(F,R) = \frac{\int_0^1 R^2(t)\,dt}{\left[\int_{-\infty}^{\infty} R'[F(x)]f^2(x)dx\right]^2}$$

$\hat{\mu}_{R_n}$ corresponding to $R(t) = \gamma[F^{-1}(t)]$ to be denoted by $\hat{\mu}_{R_n}(0)$ is asymptotically efficient.

If $d_{i:n} = 1$ for $i = 1,2,\ldots, n$, then $\hat{\mu}_{R_n}$ is median of $\frac{n(n+1)}{2}$ values of $\bar{X}_{jk}$ since the weights are equal. It is called Hodges-Lehmann estimators or W_n estimator. We shall denote it by $\hat{\mu}_{W_n}$.

For logistic distribution with distribution function F, $R(t) = 2t-1$.

$$\therefore \ d_{i:n} = 2\left(\frac{i+1}{2n+1}\right) - 1 - \frac{2i}{2n+1} + 1$$

$$= \frac{2}{2n+1} \quad \text{for all } i .$$

We observe that $d_{i:n}$ is independent of i.

$$W_{jk} = \frac{2}{n(n+1)} ; \qquad j \leq k$$
$$j = 1,2,\ldots, n$$
$$k = 1,2,\ldots, n.$$

This means that to each value of $\bar{X}_{jk}$ equal probability $\frac{2}{n(n+1)}$ is assigned. Thus $\hat{\mu}_{R_n}(0)$ and $\hat{\mu}_{W_n}$ for logistic distribution are the same.

In case of logistic distribution absolute efficiency of $\hat{\mu}_{W_n}$ is one and hence $\hat{\mu}_{W_n}$ is the best possible estimator of μ. Asymptotic efficiency of $\hat{\mu}_{W_n}$ with respect to $\bar{X}$ is 1.10. [See: Lehmann (1983), page 384]. Therefore absolute efficiency of $\bar{X}$ is 0.9090909.

Illustrative Example 3.5.:

Consider example 3.2. We are interested in finding numerical values of $\hat{\mu}_L(0)$ and $\hat{\mu}_{R_n}(0)$ estimators of the location parameter μ of logistic distribution

L - estimator $[\hat{\mu}_L(0)]$

$$\hat{\mu}_L(0) = \sum_{i=1}^{n} W_{in} X_{i:n} \text{ where } W_{in} \propto \frac{i}{n+1}\left(1-\frac{i}{n+1}\right)$$

$\therefore$ $\hat{\mu}_L(0) = k \sum_{i=1}^{n} \frac{i}{n+1}\left(1-\frac{i}{n+1}\right) X_{i:n}$ with k= constant of proportionality.

$$\sum_{i=1}^{n} W_{i:n} = 1 \Rightarrow \frac{k}{n+1}\left[\sum_{i=1}^{n} i - \frac{\sum_{i=1}^{n} i^2}{n+1}\right] = 1$$

$$\Rightarrow k(n+2)n = 6(n+1).$$

$$\therefore \quad k = \frac{6(n+1)}{n(n+2)} = 0.4642857 .$$

i	$W_{in} = (0.0027472)\,i\,(n+1-i)$	i	W_{in}
1	0.0329664	11	0.0604384
2	0.0604384	12	0.0329664
3	0.082416		
4	0.0988992		
5	0.109888		
6	0.1153824		
7	0.1153824		
8	0.109888		
9	0.0988992		
10	0.082416		

$$\hat{\mu}_L(0) = 0.0677535 .$$

R_n and W_n estimators ($\hat{\mu}_{R_n}(0)$ and $\hat{\mu}_{W_n}$):

$\hat{\mu}_{R_n}(0)$ is median of probability distribution of $\bar{X}_{jk}$ with

$$W_{jk} = \frac{2}{n(n+1)} = \frac{1}{78} \cdot$$

Here $\hat{\mu}_{R_n}(0) = \hat{\mu}_{W_n} = 0.0668.$

Comparison of sample median with sample mean :

Let X be a continuous random variable with density f(x) and distribution function $F(x) \equiv \mathcal{F}$. From theorem 3.2. in Lehmann (1983) the efficiency of sample median $\tilde{x}$ with respect to sample mean $\bar{x}$ is given as

$$e_{\tilde{x},\bar{x}}(\mathcal{F}) \doteq 4f^2(0)\ \sigma^2 \text{ where}$$

$$\sigma^2 = var(X).$$

$$\therefore\ e_{\tilde{x},\bar{x}}(\text{standard logistic}) = \frac{\pi^2}{12} \doteq \cdot 82$$

while $e_{\tilde{x},\bar{x}}$ (standard normal) $\doteq 0.637.$

This indicates the fact that the median of a sample drawn from standard logistic distribution is relatively more efficient than that in case of standard normal distribution.

3.3 Tolerance Limits :

An interval $(L(\underline{X}), U(\underline{X}))$ is said to be a γ probability tolerance limit for proportion β if

$$P\left[\int_{L(\underline{X})}^{U(\underline{X})} f(x,\theta)\, dx \geq \beta\right]$$

$$= P\{[F(U(\underline{X})) - F(L(\underline{X}))] \geq \beta]\} = \gamma .$$

One sided lower (γ,β) to tolerance limit $L(\underline{X})$ is defined as

$$P\{[F(L(\underline{X})) \leq 1 - \beta]\} = \gamma .$$

Similarly one sided upper (γ,β) tolerance limit $U(\underline{X})$ is defined as

$$P[F(U(\underline{X})) \geq \beta] = \gamma .$$

One sided tolerance limits for logistic distribution :

Suppose that $L(\underline{X})$ is a lower (γ,β) tolerance limit

i.e.

$$P[F(L(\underline{X}) \leq 1-\beta] = \gamma$$

or $$P[L(\underline{X}) \leq F^{-1}(1-\beta)] = \gamma .$$

$$\therefore \quad P[F^{-1}(1-\beta) \geq L(\underline{X})] = \gamma . \qquad (3.22)$$

The distribution function of logistic distribution is

$$F(x) = \frac{1}{1 + e^{-\frac{\pi}{\sigma\sqrt{3}}(x-\mu)}}$$

$$1-\beta = \frac{1}{1+e^{-\frac{\pi}{\sqrt{3}}(x-\mu)}}$$

$$\Rightarrow \quad x = \mu + \frac{\sqrt{3}}{\pi}\sigma \ln\left(\frac{1-\beta}{\beta}\right) . \qquad (3.23)$$

From (3.23) we have

$$F^{-1}(1-\beta) = \mu + \frac{\sqrt{3}\,\sigma}{\pi} \ln\left(\frac{1-\beta}{\beta}\right).$$

In particular if $\mu = 0, \sigma = 1$

$$F_0^{-1}(1-\beta) = \frac{\sqrt{3}}{\pi}\ln\left(\frac{1-\beta}{\beta}\right).$$

Hence,

$$F^{-1}(1-\beta) = \mu + \sigma F_0^{-1}(1-\beta) . \qquad (3.24)$$

Thus we have to find $L(\underline{X})$ such that

$$P[\mu + \sigma F_0^{-1}(1-\beta) \geq L(\underline{X})] = \gamma \quad . \qquad (3.25)$$

Let $\hat{\mu}$ and $\hat{\sigma}$ be m.l.e.s of μ and σ respectively.

Define

$$T(\underline{X}) = \frac{\hat{\mu} - (\mu + \sigma F_0^{-1}(1-\beta))}{\hat{\sigma}}$$

which is a pivotal quantity and t_γ such that

$$P[T(\underline{X}) \leq t_\gamma] = \gamma$$

$$\therefore P[\mu + \sigma F_0^{-1}(1-\beta) \geq \hat{\mu} - \hat{\sigma}\, t_\gamma] = \gamma \; . \tag{3.26}$$

From (3.22) and (3.26)

$$L(\underline{X}) = \hat{\mu} - t_\gamma \hat{\sigma} . \tag{3.27}$$

The values of tolerance factor t_γ are tabulated for different values of β. Similarly by symmetry an upper (γ,β) tolerance limit $U(\underline{X})$ is

$$U(\underline{X}) = \hat{\mu} + t_\gamma \hat{\sigma} \tag{3.28}$$

3.4. Confidence intervals for the parameters of the logistic distribution :

One of the forms of logistic distribution function is

$$F(x;\mu,\beta) = \frac{1}{1+\exp\{-(\frac{x-\mu}{\beta})\}} \tag{3.29}$$

where

$$\beta = \frac{\sigma\sqrt{3}}{\pi} .$$

Suppose that $\hat{\mu}$ and $\hat{\beta}$ denote the maximum likelihood estimators of μ and β respectively. Antle and Bain (1969) have shown that for any given n,

$$A_n = \frac{(\hat{\mu}-\mu)\sqrt{n}}{\hat{\beta}}, \quad B_n = \frac{\hat{\beta}}{\beta}, \quad A'_n = \frac{(\hat{\mu}-\mu)\sqrt{n}}{\beta}$$

are pivotal quantities.

Antle, Klimko and Harkness (1970) have obtained exact distributions of A_n, B_n and A'_n for n = 2 and estimated by simulation for n=5, 10, 20 and 40. The construction of confidence limits for the parameters is based on these distributions.

Confidence limits for μ when β is known :

Let $\hat{\mu}^*$ be the maximum likelihood estimator of μ when β is known. The distribution of $\hat{\mu}^*$ is symmetric about μ.
The critical values for $\frac{(\hat{\mu}^*-\mu)\sqrt{n}}{\beta}$ are as follows :

	cumulative probability				
n	0.90	0.95	0.975	0.98	0.99
2	2.25	2.96	3.62	3.82	4.43
5	2.20	2.87	3.48	3.66	4.23
10	2.22	2.86	3.47	3.63	4.13
20	2.22	2.85	3.45	3.61	4.03
40	2.22	2.83	3.39	3.53	4.05
∞	2.22	2.85	3.40	3.56	4.03

Suppose $A'_{n,\alpha}$ is such that $P\{A'_n < A'_{n,\alpha}\} = \alpha$.

$$\therefore\ P\left[-A'_{n,1-\frac{\alpha}{2}} < A'_n < A'_{n,1-\frac{\alpha}{2}}\right] = 1-\alpha$$

i.e. $$P\left[-A'_{n,1-\frac{\alpha}{2}} < \left(\frac{\hat{\mu}^* - \mu}{\beta}\right)\sqrt{n} < A'_{n,1-\frac{\alpha}{2}}\right] = 1-\alpha .$$

Hence 100 $(1-\alpha)$ percent confidence interval for μ is given as

$$\left(\hat{\mu}^* - \frac{\beta}{\sqrt{n}} A'_{n,1-\frac{\alpha}{2}} ,\ \hat{\mu}^* + \frac{\beta}{\sqrt{n}} A'_{n,1-\frac{\alpha}{2}}\right) . \tag{3.30}$$

<u>Illustrative Example 3.6</u> :

A random sample of 10 observations was drawn from standard logistic distribution. The observations in the random sample were as follows :

1.3598546,	-0.3736968,	-0.928799,	1.071985
-0.069161 ,	0.4379852,	-0.037858,	0.876197
0.1989784,	-0.6051674 .		

We want to find out 95 percent confidence interval for μ assuming that β is known. The maximum likelihood estimator of μ

$$\hat{\mu}^* \doteq 0.01930318 .$$

Hence 95 percent confidence limits for μ are

$$(-0.5855973 ,\ 0.6242035).$$

Confidence limits for μ if β is unknown :

In this case the confidence interval for μ can be obtained by using the fact that the distribution of $\left(\frac{\hat{\mu}-\mu}{\hat{\beta}}\right)\sqrt{n}$ is symmetric about zero and using critical values given below :

Critical points for the distribution

of $A_n = \frac{(\hat{\mu}-\mu)}{\hat{\beta}}\sqrt{n}$ for logistic distribution

	Cumulative probability				
n	0.90	0.95	0.975	0.98	0.99
2	6.3	12.8	25.7	32.2	64.4
5	2.9	4.0	5.0	5.4	6.7
10	2.50	3.29	4.07	4.30	5.06
20	2.34	3.06	3.67	3.87	4.45
40	2.25	2.93	3.54	3.70	44.19
∞	2.22	2.85	3.40	3.56	4.03

Let $A_{n,\alpha}$ be such that $P[A_n < A_{n,\alpha}] = \alpha$

$\therefore$ A $100(1-\alpha)$ percent confidence interval for μ is

$$\left(\hat{\mu} - \frac{\hat{\beta}}{\sqrt{n}} A_{n,1-\frac{\alpha}{2}},\ \hat{\mu} + \frac{\hat{\beta}}{\sqrt{n}} A_{n,1-\frac{\alpha}{2}}\right) \qquad (3.31)$$

Illustrative example 3.7:

The following are observations in a random sample of size 10 from standard logistic distribution. We are interested in finding 95 percent confidence limits for μ assuming that both μ and β are unknown.

1.4550166, -0.3115278, 1.031834, -0.301224,
0.1063156, 0.6935024, 0.4276079, -1.0350541,
-2.0440766, -0.7087509 .

In section 3.1, we have obtained

$$\hat{\mu} = -0.0318956 \text{ , } \hat{\sigma} = 1.0410532$$

$$\therefore \quad \hat{\beta} = \frac{\hat{\sigma}\sqrt{3}}{\pi} = 0.5764478.$$

Hence, 95 percent confidence limits for μ

are (-0 . 7737284 , 0.7101014) .

Confidence limits for β with μ unknown :

The distribution of $\frac{\hat{\beta}}{\beta}$ is not symmetric. The critical points of $B_n = \frac{\hat{\beta}}{\beta}$ are given below :

Critical points for the distribution of $\frac{\hat{\beta}}{\beta}$ for logistic distribution

n	cumulative probability								
	0.01	0.02	0.025	0.05	0.10	0.90	0.95	0.975	0.98
2	0.01	0.02	0.024	0.049	0.098	1.36	1.66	1.94	2.03
5	0.24	0.29	0.304	0.367	0.454	1.36	1.53	1.66	1.75
10	0.436	0.475	0.492	0.551	0.626	1.28	1.45	1.52	1.55
20	0.588	0.623	0.640	0.689	0.745	1.21	1.29	1.36	1.38

Let us take $B_{n,\alpha}$ as the upper α - percent point of the distribution of B_n i.e.

$$P[B_n < B_{n,\alpha}] = \alpha .$$

Then a $100(1-\alpha)$ percent confidence interval for β is

$$\left(\frac{\hat{\beta}}{B_{n,1-\frac{\alpha}{2}}} , \frac{\hat{\beta}}{B_{n,\frac{\alpha}{2}}} \right) \tag{3.32}$$

Illustrative Example 3.8:

Suppose that following are the observations in a random sample drawn from logistic distribution with mean zero and variance one. We have to find out 95 percent confidence limits for parameter σ with known value of α which is zero.

It is found that maximum likelihood estimate of $\sigma = \hat{\sigma} = 1.0335565$

$$\therefore \ \hat{\beta} = \frac{\hat{\sigma}\sqrt{3}}{\pi} = 0.5697556.$$

Hence 95 percent confidence interval for β is (0.3748392, 1.1580398). This means that 95 percent confidence interval for σ is

$$(0.6799712 \ , \ 2.1007241) \ .$$

3.5. Estimation of parameters of the logistic distribution from symmetrically censored sample and one sided censored sample :

Let $x_{1:n}, x_{2:n}, \ldots, x_{n:n}$ be an ordered random sample of size n from a logistic distribution with distribution function

$$F(x; \mu, \sigma) = \frac{1}{1 + \exp\left\{-\frac{\pi}{\sigma\sqrt{3}}(x-\mu)\right\}} \ .$$

Suppose that q_1 and q_2 are proportions of observations censored from below and above respectively.

Simple estimators for symmetrically censored samples :

When a sample is symmetrically censored, $q_1 = q_2 = \frac{r}{n}$ where r is the number of observations censored at each end. Let $x'_{1:n}, x'_{2:n}, \ldots, x'_{n:n}$ denote the windsorized sample. In windsorization, sample is obtained by replacing the censored extreme observations by the nearest available observation.

Hence,

$$x'_{i:n} = x_{r+1:n} \quad \text{for} \quad 1 \leq i \leq r$$

$$x'_{i:n} = x_{i:n} \quad \text{for} \quad r+1 \leq i \leq n-r$$

$$x'_{i:n} = x_{n-r:n} \quad \text{for} \quad n-r+1 \leq i \leq n .$$

The general form of the simple unbiased estimator of μ having efficiency nearly equal to that of BLUE of μ is proposed by Engelhardt (1975). By trimming we mean the removal of observations at either or both extremes with equal weights assigned to the remaining observations. Thus a simple unbiased estimator of μ is given by

$$\hat{\mu} = \frac{\sum_{i=t+1}^{n-t} x'_{i:n}}{n-2t}, \tag{3.33}$$

where t is the number of observations trimmed from each end of the sample. The number $t = [n\delta]$ is determined in such a way that the asymptotic variance of the estimators of the form (3.33) is minimum. The optimum values of δ for different values of q_1 and q_2 have been tabulated.

If $q_1 = q_2 = q \leq 0.11$, then the optimum value of $\delta = 0.11$. On the other hand when $q_1 = q_2 = q > 0.11$, $\delta < q$.

As $\hat{\mu}$ is linear combination of order statistics, var($\hat{\mu}$) can be computed and compared with that of BLUE. The asymptotic efficiencies of $\hat{\mu}$ with respect to BLUE are obtained by Chernoff, Gastwirth and Johns (1967).

In this case proposed unbiased estimator of σ which has asymptotic efficiency quite close to that of BLUE of σ is

$$\hat{\sigma} = \frac{\sum_{i=1}^{n} | X'_{i:n} - m |}{n\, k_n}, \tag{3.34}$$

where k_n is unbiasing constant and m is median of complete sample.

$$E(\hat{\sigma}) = \sigma$$

$$\Rightarrow E\,[\, \sum_{i=1}^{n} | X'_{i:n} - m | = n\, k_n \sigma.$$

Let $n = 2p$ and n be an even number

$$\therefore \quad \sigma\, n\, k_n = E\,\{\, [\frac{X'_{p:n} + X'_{p+1:n}}{2} - X'_{1:n}] + [\frac{X'_{p:n} + X'_{p+1:n}}{2} - X'_{2:n}] + \cdots$$

$$[\frac{X'_{p:n} + X'_{p+1:n}}{2} - X'_{p:n}] + [X'_{p+1:n} - \frac{X'_{p:n} + X'_{p+1:n}}{2}] + \ldots$$

$$+ [X'_{n:n} - \frac{X'_{p:n} + X'_{p+1:n}}{2}]\}$$

$$= E\{-X'_{1:n} - X'_{2:n} - \ldots - X'_{p:n} + X'_{p+1:n} + \ldots + X'_{n:n}\}$$

$$= -[\mu + \sigma\mu'_1(1,n)] - [\mu + \sigma\mu'_1(2,n)] \ldots$$

$$\ldots [\mu + \sigma\mu'_1(p,n)] + [\mu + \sigma\mu'_1(p+1;n)] \ldots$$

$$\ldots + [\mu + \sigma\mu'_1(n,n)],$$

where

$\mu'_1(i,n)$ = mean of i^{th} order statistic from standard logistic distribution

$i = 1,2,\ldots, n.$

$$\therefore\ k_n = -\frac{2}{n} \sum_{i=1}^{n/2} \mu'_1(i;n) \quad . \qquad (3.35)$$

If n is an odd number and $n = 2p + 1$ then

$$n\,k_n\,\sigma = E\{[m - X'_{1:n}] + [m - X'_{2:n}] + \ldots + [m - X'_{p:n}]$$

$$+ [m - m] + [X'_{p+2:n} - m] + \ldots + [X'_{n:n} - m]\}$$

$$= -[\mu + \sigma\mu'_1(1,n)] - [\mu + \sigma\mu'_1(2,n)] - \ldots [\mu + \sigma\mu'_1(p,n)]$$

$$+ [\mu + \sigma\mu'_1(p+2,n)] \ldots + [\mu + \sigma\mu'_1(n,n)]$$

$$\therefore\; k_n = -\frac{2}{n}\sum_{i=1}^{p}\mu'_1(i,n)$$

$$= -\frac{2}{n}\sum_{i=1}^{[\frac{n}{2}]}\mu'_1(i,n)\;. \tag{3.36}$$

From (3.35) and (3.36)

$$k_n = -\frac{2}{n}\sum_{i=1}^{[\frac{n}{2}]}\mu'_1(i,n)\;, \tag{3.37}$$

where $[\frac{n}{2}]$ = largest integer not exceeding $\frac{n}{2}$. The values of k_n for n = 10, 20, 40, 80 and ∞ are tabulated for different values of q_1 and q_2 by Engelhardt M.(1975).

Illustrative Example 3.9 :

The observations given below are the observations in an ordered random sample in which one observation is censored at each end.

$- 0.7865574, \quad -0.6921597, \quad -0.6451308,$

$- 0.4598864, \quad 0.0573899, \quad 0.3944624,$

$0.6338744, \quad 0.7119401 .$

Here

$$q_1 = q_2 = 0.1 , \quad t = [n\ \delta] = [1.1] = 1$$

$$k_n = 0.5894 , \quad m = 0.2012482 .$$

$$\therefore \hat{\mu} = \frac{\sum_{i=2}^{9} X_{i:10}}{8} = -0.0982584 .$$

$$\hat{\sigma} = \frac{\sum_{i=1}^{10} |X_{i:10} + 0.20124821|}{5.894}$$

$$= 0.99760749915.$$

The above random sample was in fact taken from standard logistic distribution.

Simple estimators in case of one sided censoring :

Suppose that S observations are censored from above. In this case $q_1 = 0$, $q_2 = \frac{s}{n} = q$ (say)

$X'_{1:n}, X'_{2:n}, \ldots, X'_{n:n}$ denote the windsorized sample. The simple unbiased estimator of σ with efficiency nearly equal to B.L.U.E. given by Engelhardt (1975) is

$$\sigma^* = \frac{\sum_{i=1}^{n} |X'_{i:n} - m|}{n\, k_n}, \qquad (3.38)$$

where k_n is unbiasing constant.

In this case the simple unbiased estimator of μ with asymptotic efficiency nearly equal to BLUE of μ is given by

$$\mu^* = \hat{\mu}_b + c_n\, \sigma^* \qquad (3.39)$$

where

$$\hat{\mu}_b = \frac{\sum_{i=t+1}^{n-t} X'_{i:n}}{n-2t},$$

$t = [n\,\delta]$ with δ as trimming fraction and C_n is unbiasing constant.

$$E\{\mu^*\} = \mu$$

$$\Rightarrow E\{\hat{\mu}_b + C_n \sigma^*\} = \mu$$

$$\Rightarrow E\{\hat{\mu}_b\} + \sigma C_n = \mu$$

$$\therefore \quad C_n = -E\left(\frac{\hat{\mu}_b - \mu}{\sigma}\right).$$

Now

$$(n-2t)\ \hat{\mu}_b = \sum_{i=t+1}^{n-t} X'_{i:n}$$

$$= \sum_{i=t+1}^{n-s-1} X_{i:n} + (s-t+1)\, X_{n-s:n}$$

$$\therefore (n-2t)\ E\left(\frac{\hat{\mu}_b - \mu}{\sigma}\right) = \sum_{i=t+1}^{n-s-1} \mu_1'(i:n) + (s-t+1)\ \mu_1'(n-s:n)$$

where $\mu_1'(i:n)$ = mean of i-th order statistic from standard logistic distribution.

$$\therefore\ C_n = -\frac{1}{(n-2t)}\left[\sum_{i=t+1}^{n-s-1} \mu_1'(i:n) + (s-t+1)\ \mu_1'(n-s:n)\right] . \qquad (3.40)$$

The values of C_n are tabulated for n = 10, 20, 40, 80 and ∞ at q = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 by Engelhardt (1975). The values of δ are selected in such a way that the asymptotic variance of (3.39) is minimum. The values of δ are provided in the table for $q_1 = 0$ and $q_2 = 0.0\ (0.1)\ 0.9$. If $q \leq 0.11$, the best value of $\delta = 0.11$ like symmetric case.

Illustrative Example 3.10 :

The ordered sample of size 20 from standard logistic distribution in which 6 observations are censored from above is as follows:

-0.7720175,	-2.0914868,	-1.5531182,	-0.8950532,
-0.6322761,	-0.545905,	-0.3967889,	-0.131908 ,
-0.1245316,	-0.0719511,	0.0164528,	0.0385425,
0.1664982,	0.2744932.	—	—
—	—	—	—

$$q_2 = 0.3 \;,\; k_n = 0.5347 \;,\; C_{20} = 0.0614 \;.$$

$$\sigma^* = \frac{\sum_{i=1}^{20} |X'_{i:20} + 0.0277491|}{10.694}$$

$$= 0.60942048812 \;.$$

$$t = [\,n\delta\,] = [(20)\;(0.14)] = 2 \;.$$

$$\hat{\mu}_b = -\,0.1723481 \;.$$

$$\mu^* = -\,0.13492968203 \;.$$

3.6. Estimation from censored data :

(Best Linear Unbiased Estimators for μ and σ).

Gupta , Qureishi and Shah (1967) have constructed linear unbiased estimators with minimum variance (Best Linear Unbiased Estimators) for the parameters of logistic distribution based on ordered observations in the samples with sizes n = 2, 5, 10, 15, 20, 25 for complete as well as censored sample cases.

Suppose $x_{1:n}, x_{2:n}, \ldots, x_{n:n}$ are n ordered observations from logistic distribution with mean μ and variance σ^2 given by density function

$$f(x;\mu,\sigma) = \frac{\pi}{\sigma\sqrt{3}} \frac{\exp\{-\frac{\pi}{\sigma\sqrt{3}}(x-\mu)\}}{\{1+\exp[-\frac{\pi}{\sigma\sqrt{3}}(x-\mu)]\}^2}$$

with $-\infty < x < \infty$, $-\infty < \mu < \infty$; $\sigma > 0$.

Then the best linear unbiased estimators for μ and σ based on $n - r_1 - r_2$ observations

$$x_{r_1+1:n} < x_{r_1+2:n} < \ldots < x_{n-r_2:n}$$ where r_1

observations are missing in the beginning and r_2 observations are missing at the end are given by

$$\hat{\mu} = \sum_{i=r_1+1}^{n-r_2} a_{i:n}\, x_{i:n} \qquad (3.41)$$

$$\hat{\sigma} = \sum_{i=r_1+1}^{n-r_2} b_{i:n}\, x_{i:n} \qquad (3.42).$$

The coefficients $a_{i:n}$ and $b_{i:n}$ can be obtained in such a way that $\hat{\mu}$ and $\hat{\sigma}$ will have minimum variance in the class of linear unbiased estimators.

$$E\left[\sum_{i=r_1+1}^{n-r_2} a_{i:n}\, x_{i:n}\right] = \mu$$

$$\Rightarrow \sum_{i=r_1+1}^{n-r_2} a_{i:n}\, [\mu + \sigma\,\mu_1'(i:n)] = \mu$$

$$\therefore \quad \sum_{i=r_1+1}^{n-r_2} a_{i:n} = 1 \text{ and } \sum_{i=r_1+1}^{n-r_2} a_{i:n}\, \mu_1'(i:n) = 0 \qquad (3.43)$$

while

$$E\left(\sum_{i=r_2+1}^{n-r_2} b_{i:n}\, x_{i:n}\right) = \sigma$$

$$\Rightarrow \mu \sum_{i=r_1+1}^{n-r_2} b_{i:n} + \sigma \sum_{i=r_1+1}^{n-r_2} b_{i:n}\, \mu_1'(i:n) = \sigma .$$

Thus,

$$\sum_{i=r_1+1}^{n-r_2} b_{i:n} = 0 \quad \text{and} \quad \sum_{i=r_1+1}^{n-r_2} b_{i:n}\, \mu_1'(i:n) = 1. \qquad (3.44)$$

We note that (3.43) and (3.44) are necessary conditions.

The estimators $\hat{\mu}$ and $\hat{\sigma}$ can be written down in the matrix notation using generalized least square theory by Lloyd (1952). Under the set up described above

$$E(\underline{X}) = \underline{P}\ \underline{\theta} \quad \text{where}$$

$$\underline{X}' = (x_{r_1+1:n}, x_{r_1+2:n}, \ldots, x_{n-r_2:n}).$$

$$\underset{\sim}{P} = \begin{bmatrix} 1 & \mu_1'(r_1+1:n) \\ 1 & \mu_1'(r_1+2:n) \\ \cdot & \vdots \\ \cdot & \vdots \\ 1 & \mu_1'(n-r_2:n) \end{bmatrix}_{(n-r_1-r_2)\times 2}.$$

The least square estimates are given by

$$\underset{\sim}{\theta}^* = \begin{pmatrix} \mu^* \\ \sigma^* \end{pmatrix} = \begin{bmatrix} -\underset{\sim}{\alpha}'\,\Gamma\,\underset{\sim}{X} \\ \underset{\sim}{1}'\,\Gamma\,\underset{\sim}{X} \end{bmatrix}, \tag{3.45}$$

where

$$\Gamma = \frac{V^{-1}(\underset{\sim}{1}\,\underset{\sim}{\alpha}' - \underset{\sim}{\alpha}\,\underset{\sim}{1}')\,V^{-1}}{\Delta}$$

$$\begin{aligned}\Delta &= \text{Determinant of } (P'V^{-1}P) \\ &= (\underset{\sim}{1}'\,V^{-1}\,\underset{\sim}{1})\,(\underset{\sim}{\alpha}'\,V^{-1}\,\underset{\sim}{\alpha}) - (\underset{\sim}{1}'V^{-1}\,\underset{\sim}{\alpha})^2,\end{aligned}$$

with V as the variance-covariance matrix of $n-r_1-r_2$ appropriate order statistics.

The estimators of μ and σ given by expressions (3.41),(3.42) and (3.45) are equivalent.

The table of variances and covariances of order statistics in samples of size n = 10,(5) 25 and table of coefficients $a_{i:n}$, $b_{i:n}$ for both complete and censored samples for n=2, 5, 10, 11, 15, 20, 25 are given by Gupta, Qureishi and Shah (1967).

Illustrative Example 3.11 :

The following are observations arranged in an ascending order with smallest observation and two largest observations missing in a random sample drawn from a logistic distribution.

We shall find out the best linear unbiased estimator for μ and σ.

Sr.No.	Ordered Observation $x_{i:n}$	Coefficient for $a_{i:n}$	Coefficient for $b_{i:n}$
1	—	—	—
2	-1.2281979	0.0420	-0.2524
3	-1.0496148	0.0528	-0.1286
4	-0.8041016	0.0697	-0.1069
5	-0.3964635	0.0831	-0.0842
6	-0.2948933	0.0929	-0.0576
7	-0.032913	0.0990	-0.0286
8	0.1876775	0.1014	0.0016
9	0.2690597	0.1000	0.0316
10	0.3521835	0.0949	0.0604
11	0.5588153	0.0860	0.0865
12	0.826016	0.0783	0.1084
13	1.1286019	0.1001	0.3649
14	—	—	—
15	—	—	—

Then best linear unbiased estimators of μ and σ are

$$\hat{\mu} = 0.075064 \; , \quad \hat{\sigma} = 1.1620245 \; .$$

Plackett (1958) has discussed the problem of estimating the μ and σ based on K successive variables

$$Y_u < y_{u+1} < \ldots < y_v \text{ where } v = u+k-1$$

from logistic distribution with mean μ and variance σ^2.

He obtained least square estimator as

$$\theta^* = \begin{bmatrix} \mu^* \\ \sigma^* \end{bmatrix} = (A'V^{-1} A)^{-1} A' V^{-1} y \; , \qquad (3.46)$$

where

$$A' = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ E(\frac{Y_u - \mu}{\sigma}), & E(\frac{Y_{u+1} - \mu}{\sigma}), & \ldots, & E(\frac{Y_v - \mu}{\sigma}) \end{bmatrix}$$

$$y' = [Y_u, Y_{u+1}, \ldots, Y_v] \; .$$

V = Variance-covariance matrix of m

$$\text{with } m' = [\frac{Y_u - \mu}{\sigma} , \frac{Y_{u+1} - \mu}{\sigma} \ldots\ldots \frac{Y_v - \mu}{\sigma}].$$

Further he replaced V by symmetric matrix W as V is very difficult to handle analytically.

$$W=\begin{bmatrix} \frac{p_u q_u}{f_u^2}, & \frac{p_u q_{u+1}}{f_u f_{u+1}}, & \cdots\cdots\cdots & \frac{p_u q_v}{f_u f_v} \\ \frac{p_u q_{u+1}}{f_u f_{u+1}}, & \frac{p_{u+1} q_{u+1}}{f_{u+1}^2} & \cdots & \frac{p_{u+1} q_v}{f_{u+1} f_v} \\ \vdots & \vdots & & \vdots \\ \frac{p_u q_u}{f_u f_v}, & \cdots & \cdots\cdots\cdots & \frac{p_v q_v}{f_v^2} \end{bmatrix}_{k \times k} \tag{3.47}$$

where

$$p_i = F \{ E(\frac{y_i - \mu}{\sigma}) \} ; i = u, u+1 \ldots v.$$

and

$$f_i = f [E (\frac{y_i - \mu}{\sigma})].$$

Define $p_{u-1} = 0$ and $p_{v+1} = 1$.

Then unbiased estimate of θ is

$$\theta^+ = (A' W^{-1} A)^{-1} A' W^{-1} y \text{ with}$$

$$W^{-1} = \begin{bmatrix} c_u & d_u & 0 & 0 & \vdots & 0 \\ d_u & c_{u+1} & d_{u+1} & 0 & \vdots & 0 \\ 0 & d_{u+1} & c_{u+2} & d_{u+2} & \vdots & 0 \\ \vdots & \vdots & & & \vdots & \\ 0 & 0 & \cdot & \cdot & c_{v-1} & d_{v-1} \\ 0 & 0 & \vdots & \vdots & d_{v-1} & c_v \end{bmatrix}$$

where

$$c_i = \frac{f_i^2\,(p_{i+1} - p_{i-1})}{(p_{i+1} - p_i)\,(p_i - p_{i-1})}$$

and

$$d_i = \frac{-f_i\, f_{i+1}}{p_{i+1} - p_i}\,.$$

Maximum Likelihood Estimators of the Parameters of a Logistic Distribution From Censored Samples :

Let $x_{r_1+1:n}, x_{r_1+2:n}, \ldots, x_{n-r_2:n}$ be an ordered sample of size n from $L(\mu, \sigma)$ distribution from which smallest r_1 and largest r_2 values are censored. The likelihood function of this sample is

$$L_{r_1,r_2} = \frac{n!}{r_1!\, r_2!}\{F(x_{r_1+1:n})\}^{r_1} \{\prod_{i=r_1+1}^{n-r_2} f(x_{i:n})\} \{1-F(x_{n-r_2:n})\}^{r_2}$$

$$\therefore \ln L_{r_1,r_2} = \ln n! - \ln r_1! - \ln r_2! + r_1 \ln F(x_{r_1+1:n})$$

$$+ \sum_{i=r_1+1}^{n-r_2} \ln f(x_{i:n}) + r_2 \ln\{1-F(x_{n-r_2:n})\}$$

$$= \ln n! - \ln r_1! - \ln r_2!$$

$$+ \sum_{r_1+1}^{n-r_2} \ln\left[\frac{\pi}{\sigma\sqrt{3}} \frac{\exp\{\frac{-\pi}{\sigma\sqrt{3}}(x_i-\mu)\}}{[1+\exp\{-\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)\}]^2}\right]$$

$$+ r_1 \ln F(x_{r_1+1:n}) + r_2 \ln\{1-F(x_{n-r_2:n})\}$$

$$= \ln n! - \ln r_1! - \ln r_2! + (n-r_1-r_2)\ln\left(\frac{\pi}{\sigma\sqrt{3}}\right)$$

$$+ 2\sum_{i=r_1+1}^{n-r_2} \ln\{1+\exp[-\frac{\pi}{\sigma\sqrt{3}}(x_i-\mu)]\}^{-1}$$

$$- \sum_{i=r_1+1}^{n-r_2} \frac{\pi}{\sigma\sqrt{3}}(x_{i:n}-\mu) + r_1 \ln F(x_{r_1+1:n})$$

$$+ r_2 \ln\{1-F(x_{n-r_2:n})\} .$$

Put $Z_i = \frac{\pi(x_{i:n}-\mu)}{\sigma\sqrt{3}}$ and $F(Z_i) = \frac{1}{1+\exp(-Z_i)}$.

$$\therefore \ln L_{r_1,r_2} = \ln n! - \ln r_1! - \ln r_2! + (n-r_1-r_2)\ln\left(\frac{\pi}{\sigma\sqrt{3}}\right)$$

$$+ 2\sum_{i=r_1+1}^{n-r_2} \ln F(Z_i) - \sum_{i=r_1+1}^{n-r_2} Z_i + r_1 \ln F(Z_{r_1+1})$$

$$+ r_2 \ln[1 - F(Z_{n-r_2})] . \tag{3.48}$$

The likelihood equations are obtained by differentiating (3.48) with respect to μ and σ and equating those expressions to zero. The maximum likelihood estimators for μ and σ can be obtained by solving these equations iteratively on an electronic computer. Harter and Moore (1967) have obtained asymptotic variances and covariances of the maximum likelihood estimators and tabulated them for censoring proportions $q_1 = 0.0(0.1)\ 0.4$ from below and $q_2 = q_1(0.1)\ (0.9-q_1)$ from above. They have also found out the mean square errors of the maximum likelihood estimates for small samples by a Monte-Carlo study of 1000 samples each of size 10 and compared with the asymptotic variances of the maximum likelihood estimators as well as with the variances of the best linear unbiased estimators.

It was observed that the maximum likelihood estimators and the best linear unbiased estimators are approximately equally efficient for symmetric or nearly symmetric censoring while maximum likelihood estimators are more efficient than best linear unbiased estimator in case of strongly asymmetric censoring.

CHAPTER 4

ESTIMATION OF LOGISTIC FUNCTION IN THE TREATMENT OF BINOMIAL RESPONSE DATA

4.0 Introduction :

We have already remarked that logistic function can be used as a model for probability of response in binomial response data. In this chapter the methods of estimating parameters of logistic function are discussed with the help of illustrative examples.

In Section 4.1 the maximum likelihood estimators are found while minimum logit χ^2 estimator and minimum χ^2 estimators are obtained in Sections 4.2 and 4.3 respectively.

The logistic function as a model for binomial response data corresponding to (1.2) is

$$P_i = \frac{1}{1 + e^{-(\alpha + \beta x_i)}} \tag{4.1}$$

where P_i = true probability of response (death) at dose x_i

$$\alpha = \frac{-\pi \mu}{\sigma \sqrt{3}}, \quad \beta = \frac{\pi}{\sigma\sqrt{3}}.$$

The response (p_i) is measured in terms of observed proportion of deaths of animals P_i.

In the model (4.1), we have assumed that number of animals responded at does x_i is binomially distributed with true probability of response p_i. We are interested in finding out estimates of α and β. Berkson (1955) has discussed maximum likelihood, minimum logit χ^2 and minimum χ^2 estimators of α and β.

4.1. Maximum Likelihood Estimators of α and β :

Let n_i = number of animals exposed at dose x_i ; $i = 1,2,\ldots, k$

P_i = observed proportion of response at doxe x_i

p_i = true probability of response at dose x_i

r_i = number of animals responded at dose x_i.

$= n_i \, P_i$.

The likelihood function of the number of deaths is tnen given by

$$L(r_1,r_2,\ldots,r_k;\alpha,\beta) = \prod_{i=1}^{k} \binom{n_i}{r_i} p_i^{r_i} q_i^{n_i - r_i}$$

$$= \prod_{i=1}^{k} \binom{n_i}{r_i} \prod_{i=1}^{k} [1 + \exp\{-(\alpha + \beta x_i)\}]^{-n_i}$$

$$e^{-\alpha\sum_{i=1}^{k} n_i - \beta\sum_{i=1}^{k} n_i x_i} \; e^{\alpha\sum_{i=1}^{k} r_i} \; e^{\beta\sum_{i=1}^{k} r_i x_i}$$

$$=\{ \prod_{i=1}^{k} \binom{n_i}{r_i} [1 + \exp\{-(\alpha+ \beta x_i)\}]^{-n_i} e^{-\alpha \Sigma n_i} e^{\alpha\Sigma r_i}\}$$

$$\{e^{-\beta\Sigma n_i x_i} e^{\beta\Sigma r_i x_i} \}. \qquad (4.2)$$

Thus $(\sum_{i=1}^{k} r_i = \sum_{i=1}^{k} n_i p_i, \sum_{i=1}^{k} r_i x_i = \sum_{i=1}^{k} n_i p_i x_i)$

is sufficient for (α, β).

Now we proceed to find out likelihood equations for α and β.

Note that

$$\ln p_i = - \ln [1 + \exp\{-(\alpha + \beta x_i)\}]$$

$$\frac{\partial \ln p_i}{\partial \alpha} = \frac{\exp\{-(\alpha+\beta x_i)\}}{[1+\exp\{-(\alpha+\beta x_i)\}]} = q_i$$

$$\frac{\partial \ln}{\partial \alpha} q_i = \frac{\partial}{\partial \alpha} \ln(1-p_i)$$

$$= - \frac{1}{1-p_i} \frac{\partial}{\partial \alpha} p_i$$

$$= - p_i .$$

$$\therefore \frac{\partial}{\partial \alpha} p_i = p_i q_i .$$

$$\ell n\, L(r_1, r_2, \ldots, r_k, \alpha, \beta)$$

$$= \sum_i \{ \ell n \binom{n_i}{r_i} + r_i \ \ell n\ p_i + (n_i - r_i)\ \ell n\ q_i \} \qquad (4.3)$$

$$\frac{\partial}{\partial \alpha} \ell n\ L = \Sigma\ r_i \frac{\partial}{\partial \alpha} \ell n\ p_i + (n_i - r_i) \frac{\partial}{\partial \alpha} \ell n\ q_i$$

$$= \Sigma\ [r_i\ q_i - (n_i - r_i)\ p_i]$$

$$= \Sigma\ r_i - \Sigma\ (r_i + n_i - r_i)\ p_i$$

$$= \Sigma\ r_i - \Sigma\ n_i\ p_i\ . \qquad (4.4)$$

$$\frac{\partial}{\partial \alpha} \ell n\ L = 0 \Rightarrow \Sigma\ n_i\ (P_i - p_i) = 0\ . \qquad (4.5)$$

Similarly $\frac{\partial}{\partial \beta} \ell n\ L = 0 \Rightarrow \Sigma\ n_i x_i (P_i - p_i) = 0.$ (4.6)

Solving (4.5) and (4.6) by iterative procedure, maximum likelihood estimates can be obtained.

Illustrative Example 4.1:

Suppose that in an experiment out of 10 animals exposed, number of animals responded at 3 doses taken as -1,0 and 1 are 2,5 and 7 respectively. The maximum likelihood estimates of α and β are obtained as follows :

Here $n_i = 10$ (const)

Dose (x_i) : - . -1 -, 0 , 1

No.of animals responded (r_i) : 2 , 5 , 7

Observed proportion (P_i): $\frac{2}{10}$, $\frac{5}{10}$, $\frac{7}{10}$.

Let $\hat{p}_1$, $\hat{p}_2$ and $\hat{p}_3$ be estimates of p_1, p_2 and p_3 respectively.

Then

$$\sum_{i=1}^{3} n_i(P_i-\hat{p}_i) = 0 \Rightarrow \sum_{i=1}^{3} (P_i-\hat{p}_i) = 0$$

$$\Rightarrow \hat{p}_1 + \hat{p}_2 + \hat{p}_3 = 1.4 . \tag{4.7}$$

Similarly,

$$\Sigma n_i x_i (P_i - \hat{p}_i) = 0$$

$$\Rightarrow \hat{p}_3 - \hat{p}_1 = 0.5. \tag{4.8}$$

We have

$$\frac{p_i}{1-p_i} = \exp \{ \alpha + \beta x_i \}$$

$$\therefore \frac{p_1}{1-p_1} = \exp \{ \alpha - \beta \}$$

$$\frac{p_2}{1-p_2} = \exp \{ \alpha \} \tag{4.9}$$

$$\frac{p_3}{1-p_3} = \exp \{\alpha + \beta\} .$$

We have to solve (4.7) and (4.8) using (4.9) by iterative procedure. The initial solutions of α and β can be taken as the values corresponding to observed proportions p_1, p_2, p_3. From $\hat{p}_1$ and $\hat{p}_2$ one can find out values of α, β using (4.9) while $\hat{p}_3$ will yield the estimate of $\alpha + \beta$. Thus we can obtain the estimate of $\alpha + \beta$ from $\hat{p}_1$ and $\hat{p}_2$ as well as from $\hat{p}_3$ separately. The desired estimates of α and β will be those for which the difference between the estimates of $\alpha + \beta$ by these two ways is approximately zero.

Sr. No.	I Values of $\hat{p}_1, \hat{p}_2$ and $\hat{p}_3$	II Estimates of α, β and $\alpha + \beta$ from $\hat{p}_1$ ~~and~~ $\hat{p}_2$	III Estimate of $\alpha + \beta$ from $\hat{p}_3$	IV Difference in the estimates of $\alpha+\beta$ obtained in II and III
1	$\hat{p}_1=0.2$ $\hat{p}_2=0.5$ $\hat{p}_3=0.7$	$\hat{\alpha}=0$, $\hat{\beta}=1.38636886$ $\hat{\alpha}+\hat{\beta}=1.3863688$	$\hat{\alpha}+\hat{\beta}=0.8472964$	0.5390724
2	$\hat{p}_1=0.22$ $\hat{p}_2=0.46$ $\hat{p}_3=0.72$	$\hat{\alpha} = -0.1603426$ $\hat{\beta} = 1.053239$ $\hat{\alpha} + \hat{\beta}=0.9449813$	$\hat{\alpha} + \hat{\beta}= 0.9444616$	0.0005197
3	$\hat{p}_1=0.220019$ $\hat{p}_2=0.459962$ $\hat{p}_3=0.720019$	$\hat{\alpha} = -0.1604956$ $\hat{\beta} = 1.1050613$ $\hat{\alpha}+\hat{\beta}= 0.9445657$	$\hat{\alpha}+\hat{\beta} = 0.9445558$ ~~$\alpha+\beta=$~~	0.0000099 $\doteq 0$

Hence the maximum likelihood estimates of α and β are -0.1604956 and 1.1050613 respectively.

These estimates of α and β can be used to find out the estimate of quantity of dose at desired percentage of response. For example, we may be interested in finding out estimate of dose x_i for which 95 percent response is observed, i.e.

$$0.95 = \frac{1}{1+\exp\{-(-0.1604956 + 1.1050613\, x_i)\}} \, .$$

$$\therefore \quad \exp\{\, 0.1604956 + 1.1050613\, x_i\} = \frac{0.95}{0.05} \, .$$

$$\therefore \quad \hat{x}_i = 2.8097396 \, .$$

There are some situations for which the iterative method works but we do not get permissible solutions. For instance let us consider the following example.

Dose x_i :	-1	0	1
No. of animals responded : (r_i)	7	5	2
Observed proportion: (P_i)	0.7	0.5	0.2

$$\sum_{i=1}^{3} n_i(P_i-\hat{p}_i) = 0$$

$$\Rightarrow \hat{p}_1 + \hat{p}_2 + \hat{p}_3 = 1.4 \; . \qquad (4.10)$$

$$\sum_i n_i x_i \; (P_i-\hat{p}_i) = 0$$

$$\Rightarrow \hat{p}_3-\hat{p}_1 = -0.5 \; . \qquad (4.11)$$

Now,

$$\frac{p_i}{1-p_i} = \exp \{ \alpha + \beta x_i \} \quad i = 1,2,3. \qquad (4.12)$$

The equations (4.10) and (4.11) are to be solved iteratively subject to the condition (4.12). It is found that for $\alpha = -0.1604956$ and $\beta = -1.1050613$, these equations are satisfied.

But as the value of β is negative, the solution of likelihood equations are not consistent. In this case $P_1 > P_2 > P_3$ for $x_1 < x_2 < x_3$ which does not satisfy assumption in the model. However probability of getting such a sample is positive.

4.2. Minimum logit χ^2 estimators:

Logit χ^2 is defined as

$$\chi^2(\text{logit}) = \sum_i n_i P_i \, Q_i(\tilde{\ell}_i - \hat{\ell}_i)^2 \quad , \qquad (4.13)$$

where $\ell_i = \ell n(\frac{P_i}{Q_i})$ and $\hat{\ell}_i = \hat{\alpha} + \hat{\beta} x_i$.

The estimate obtained by minimizing the quantity logit χ^2 is called minimum logit χ^2 estimate.

Now,

$$\frac{\partial\chi^2(\text{logit})}{\partial\alpha} = 2\sum_i n_i P_i Q_i\ (\ell_i - \hat{\alpha} - \hat{\beta}x_i)$$

$$\therefore \quad \frac{\partial\chi^2(\text{logit})}{\partial\alpha} = 0 \Rightarrow \sum_i n_i P_i Q_i(\ell_i - \hat{\alpha} - \hat{\beta}x_i) = 0\ . \qquad (4.14)$$

Similarly

$$\frac{\partial\chi^2(\text{logit})}{\partial\beta} = -2\sum_i n_i P_i Q_i\ (\ell_i - \hat{\alpha} - \hat{\beta}x_i)x_i$$

$$\frac{\partial\chi^2(\text{logit})}{\partial\beta} = 0 \Rightarrow \sum_i n_i P_i Q_i(\ell_i - \hat{\alpha} - \hat{\beta}x_i)x_i = 0. \qquad (4.15)$$

The normal equations (4.14) and (4.15) can solved by an iterative procedure or by a procedure of obtaining least square solution of the straight line

$$\hat{\ell}_i = \alpha + \beta x_i$$

using $n_i P_i Q_i$ as the weight of observation ℓ_i. Then the estimates for α and β given by Berkson (1953) are

$$\hat{\alpha} = \frac{\sum_i n_i P_i Q_i \ell_i - \hat{\beta}\sum_i n_i P_i Q_i x_i}{\sum_i n_i P_i Q_i} \qquad (4.16)$$

$$\hat{\beta} = \frac{\sum_i n_i p_i Q_i \ell_i x_i - \frac{(\sum_i n_i p_i Q_i \ell_i)(\sum_i n_i p_i Q_i x_i)}{\sum_i n_i p_i Q_i}}{\sum_i n_i p_i Q_i x_i^2 - \frac{(\sum_i n_i p_i Q_i x_i)^2}{\sum n_i p_i Q_i}} . \quad (4.17)$$

The minimum logit χ^2 estimates are not sufficient for (α, β) as an observed p_i of 0 can be replaced by observed p_i of 1 in equations (4.14) and (4.15) without changing the solution.

putting $\bar{x} = \frac{\sum_i n_i p_i Q_i x_i}{\sum_i n_i p_i Q_i}$ and $\bar{\ell} = \frac{\sum_i n_i p_i Q_i \ell_i}{\sum_i n_i p_i Q_i}$ in (4.16)

and (4.17)

$$\hat{\alpha} = \bar{\ell} - \hat{\beta}\,\bar{x} \quad (4.18)$$

$$\hat{\beta} = \frac{\sum_i n_i p_i Q_i (\ell_i - \bar{\ell})(x_i - \bar{x})}{\sum_i n_i p_i Q_i (x_i - \bar{x})^2} . \quad (4.19)$$

Illustrative Example 4.2 :

Consider the example 4.1.

$$\ell_1 = \ell n \left(\frac{p_1}{Q_1}\right) = -1.3862944$$

$$\ell_2 = \ell n \left(\frac{p_2}{Q_2}\right) = 0$$

$$\ell_3 = \ell n \left(\frac{p_3}{Q_3}\right) = 0.8472964 .$$

x_i	P_i	$n_iP_iQ_i$	$n_iP_iQ_ix_i$	$n_iP_iQ_ix_i^2$	$n_iP_iQ_i\ell_i$	$n_iP_iQ_i\ell_ix_i$
1	0.2	1.6	-1.6	1.6	-2.218	2.218
0	0.5	2.5	0	0	0	0
1	0.7	2.1	2.1	2.1	1.779	1.779
Total	—	6.2	0.5	3.7	-0.439	3.997

$$\bar{x} = \frac{\sum_i n_iP_iQ_ix_i}{\sum_i n_iP_iQ_i} = 0.08064516$$

$$\bar{\ell} = \frac{\sum_i n_iP_iQ_i\ell_i}{\sum_i n_iP_iQ_i} = -0.0708064$$

and

$$\left.\begin{aligned} &\sum n_iP_iQ_ix_i^2 = 3.7 \\ &\frac{(\sum_i n_iP_iQ_ix_i)^2}{\sum_i n_iP_iQ_i} = 0.0403226 \end{aligned}\right\} \Rightarrow \sum_i n_iP_iQ_i(x_i-\bar{x})^2 = 3.6596774$$

$$\left.\begin{aligned} &\sum_i n_iP_iQ_i\ell_ix_i = 3.997 \\ &\frac{(\sum_i n_iP_iQ_i\ell_i)(\sum n_iP_iQ_ix_i)}{\sum n_iP_iQ_i} = -0.0354 \end{aligned}\right\} \Rightarrow \sum_i n_iP_iQ_i(\ell_i-\bar{\ell})(x_i-\bar{x}) = 4.03240322$$

$$\therefore \quad \hat{\beta} = \frac{\Sigma n_i P_i Q_i(\ell_i - \bar{\ell})(x_i - \bar{x})}{\Sigma \; n_i P_i Q_i (x_i - \bar{x})^2} = 1.1018466 \cdot$$

and

$$\hat{\alpha} = \bar{\ell} - \hat{\beta}\,\bar{x} = -0.15966504338.$$

4.3. Minimum χ^2 estimators:

Let n_i = number of animals exposed at dose x_i;

$i = 1,2,\ldots, k$.

r_i = number of animals responded at dose $x_i = n_i p_i$

P_i = probability of response at dose x_i

$q_i = 1 - P_i$; $i = 1,2,\ldots,k$.

p_i = observed proportion of animals responded at dose x_i

$Q_i = 1 - P_i$

e_i = expected number of animals responded at dose $x_i = n_i P_i$

$n_i - e_i$ = expected number of animals not responded at x_i

s_i = number of animals not affected at dose $x_i = n_i Q_i$.

Then Pearson χ^2 is given by

$$\chi^2 = \sum_{i=1}^{k} \frac{(r_i - e_i)^2}{e_i} + \sum_{i=1}^{k} \frac{(s_i - n_i + e_i)^2}{n_i - e_i}$$

$$= \sum_{i=1}^{k} \frac{(n_i P_i - n_i p_i)^2}{n p_i} + \sum_{i=1}^{k} \frac{(n_i Q_i - n_i q_i)^2}{n_i q_i}$$

$$= \sum_{i=1}^{k} \frac{n_i (P_i - p_i)^2}{p_i} + \sum_{i=1}^{k} \frac{n_i (p_i - P_i)^2}{q_i}$$

$$\therefore \chi^2 = \sum_{i=1}^{k} \frac{n_i (P_i - p_i)^2}{p_i \; q_i} , \qquad (4.20)$$

which has χ^2 distribution with $(k-1) \times 1 = k-1$ d.f.

Now,

$$\frac{\partial \chi^2}{\partial \alpha} = \frac{\partial \chi^2}{\partial p_i} \frac{\partial p_i}{\partial \alpha}$$

$$= \sum_i n_i \left\{ \frac{2 p_i q_i (P_i - p_i) + (P_i - p_i)^2 (1 - 2p_i)}{(p_i q_i)^2} \right\} \frac{\partial p_i}{\partial \alpha}$$

$$= \sum_i \frac{n_i (P_i - p_i)}{(p_i q_i)^2} \{ 2p_i \; q_i + (1 - 2p_i) \; (P_i - p_i) \} \frac{\partial p_i}{\partial \alpha}$$

$$= \sum_i \frac{n_i (P_i - p_i)}{(p_i q_i)^2} \{ p_i + P_i - 2p_i \; P_i \} \frac{\partial p_i}{\partial \alpha}$$

$$= \sum_i \frac{n_i (P_i - p_i)}{(p_i q_i)^2} \{ p_i (1 - P_i) + P_i (1 - p_i) \} \frac{\partial p_i}{\partial \alpha}$$

$$\therefore \frac{\partial \chi^2}{\partial \alpha} = \sum_{i=1}^{k} \frac{n_i (P_i - p_i)}{(p_i q_i)^2} \{ p_i Q_i + P_i q_i \} \frac{\partial p_i}{\partial \alpha} .$$

But $\frac{\partial p_i}{\partial \alpha} = p_i\, q_i$; $i = 1,2,\ldots, k.$

Hence,

$$\frac{\partial \chi^2}{\partial \alpha} = 0 \Rightarrow \sum_{i=1}^{k} \frac{n_i(P_i - p_i)}{p_i q_i} \{ P_i q_i + p_i Q_i \} = 0. \tag{4.21}$$

Similarly

$$\frac{\partial \chi^2}{\partial \alpha} = \frac{\partial \chi^2}{\partial p_i} \quad \frac{\partial p_i}{\partial \beta}$$

and

$$\frac{\partial p_i}{\partial \beta} = p_i q_i\, x_i \,.$$

$$\therefore \frac{\partial \chi^2}{\partial \beta} = 0 \Rightarrow \sum_{i=1}^{k} \frac{n_i(P_i - p_i)x_i}{p_i q_i} \{ P_i q_i + p_i Q_i \} = 0 \,. \tag{4.22}$$

Minimum χ^2 estimators for α and β are obtained by solving (4.21) and (4.22) simultaneously. An iterative procedure for solving these equations is as follows :

One can take initial values of $\hat{p}_1$ and $\hat{p}_2$ as P_1 and P_2 respectively. $\hat{p}_1$ and $\hat{p}_2$ will specify the values of $\hat{\alpha}$ and $\hat{\beta}$. For these values of $\hat{\alpha}$ and $\hat{\beta}$, $\hat{p}_3$ can be determined using the relation

$$\hat{p}_3 = \frac{\exp\{\hat{\alpha} + \hat{\beta}\}}{1 + \exp\{\hat{\alpha} + \hat{\beta}\}} \,.$$

Substituting values of $\hat{p}_1, \hat{p}_2$ and $\hat{p}_3$, the left hand sides of (4.21) and (4.22) can be found. The iterative procedure is terminated at those values of $\hat{p}_1$, $\hat{p}_2$ and $\hat{p}_3$ for which left hand sides of (4.21) and (4.22) become approximately equal to zero. Corresponding values of $\hat{\alpha}$ and $\hat{\beta}$ are the required minimum χ^2 estimates of α and β.

Illustrative Example 4.3:

Consider the illustrative example 4.1. We shall find minimum χ^2 estimates of α and β using iterative procedure.

Initial value of $\hat{p}_1 = p_1 = 0.2$

and Initial value of $\hat{p}_2 = p_2 = 0.5$.

Values of $\hat{p}_1$ and $\hat{p}_2$ (1).	$\hat{\alpha}$ and $\hat{\beta}$ for (1). (2)	$\hat{p}_3$ correspo-nding to (2).	Left hand side of (4.21).	Left hand side of (4.22).
$\hat{p}_1 = 0.2$ $\hat{p}_2 = 0.5$	$\hat{\alpha} = 0$ $\hat{\beta} = 1.38637886$	$\hat{p}_3 = 0.8$	-0.2375	-0.2375
$\hat{p}_1 = 0.22$ $\hat{p}_2 = 0.48$	$\hat{\alpha} = -0.08116044$ $\hat{\beta} = 1.1345378$	$\hat{p}_3 = 0.7508913$	-0.0672962	-0.070035
$\hat{p}_1 = 0.22$ $\hat{p}_2 = 0.46$	$\hat{\alpha} = -0.16016875$ $\hat{\beta} = 1.1056795$	$\hat{p}_3 = 0.7202114$	0.00229	-0.0026204
$\hat{p}_1 = 0.221$ $\hat{p}_2 = 0.46$	$\hat{\alpha} = -0.16016875$ $\hat{\beta} = 1.0985935$	$\hat{p}_3 = 0.7187813$	0.001423	0.0020491
$\hat{p}_1 = 0.22063$ $\hat{p}_2 = 0.04601$	$\hat{\alpha} = -0.160012$ $\hat{\beta} = 1.102011$	$\hat{p}_3 = 0.7195034$	$0.0006309 \doteq 0$	$0.0004251 \doteq 0$

Thus minimum X^2 estimates for (α, β) are

$$(-0.160012,\ 1.102011).$$

Name of the estimate	α	β
Maximum likelihood estimate	-0.1604956	1.1050613
Minimum logit χ^2 estimate	-0.15966505	1.1018466
Minimum χ^2 estimate	-0.160012	1.102011

However there are samples which do not yield finite estimates for α or β or both α and β. For example suppose that $[p_1, p_2, p_3]$ are observed proportions of killed animals at doses $[x_1, x_2, x_3]$.

In this case $[p_1, 0, 0]$, $[0, 0, p_3]$, $[0, p_2, 1]$

$[p_1, 1, 1]$, $[1, 1, p_3]$, $[1, p_2, 0]$

are the samples which lead to infinite estimates.

Working Rule :

The problem in estimating the parameters mentioned above can be resolved by one rule suggested by Berkson (1953) which is as follows :

As observation zero is replaced by $\frac{1}{2n}$ and for an observation one take value $1 - \frac{1}{2n}$. Illustration of this rule is given in Berkson (1955).

CHAPTER 5

BIVARIATE LOGISTIC DISTRIBUTION

5.0. Introduction :

There is no natural extension of univariate logistic distribution to multivariate logistic distribution unlike the normal distribution. In this chapter we give some bivariate logistic distributions and study their properties.

Gumbel's bivariate logistic distribution and its properties are discussed in section 5.1.

Section 5.2. deals with the bivariate logistic distribution which is generalization of Gumbel's bivariate distribution and the expressions for its cumulant generating function, mode, correlation coefficient, conditional expectations etc.

Section 5.3 consists of another form of bivariate logistic distribution and its properties.

5.1. Gumbel's Bivariate Logistic Distribution :

Gumbel's bivariate distribution is a bivariate distribution with distribution function as

$$F(x,y) = \frac{1}{[1+e^{-x}+e^{-y}]} \; ; \quad \begin{array}{l} -\infty < x < \infty \\ -\infty < y < \infty \end{array} \qquad (5.1)$$

The properties of this distribution have been studied by Gumbel (1961).

The probability density function of this distribution is

$$f(x,y) = \frac{2\, e^{-x-y}}{(1+e^{-x}+e^{-y})^3} \quad . \tag{5.2}$$

The marginal distributions are

$$f_1(x) = \frac{e^{-x}}{(1+e^{-x})^2} \quad \text{and} \quad f_2(y) = \frac{e^{-y}}{(1+e^{-y})^2} \quad .$$

$$\therefore \; f(x,y) \neq f_1(x) \; f_2(y) \; .$$

Hence the variables X and Y are not independent.

This distribution is asymmetric about (0,0) as $f(-x, -x) > f(x,x)$, $x > 0$.

For $x > 0$, we have

$$f(-x, -x) = \frac{2\, e^{2x}}{(1+2e^{x})^3}$$

$$f(x,x) = \frac{2\, e^{-2x}}{(1+2e^{-x})^3} \quad .$$

It is enough to show that :

$$e^{2x}(1+2e^{-x})^3 > e^{-2x}(1+2e^{x})^3$$

i.e. $e^{2x} - e^{-2x} > 2(e^{x} - e^{-x})$.

Let $g(x) = e^{2x} - e^{-2x} - 2(e^{x} - e^{-x})$.

$$g(0) = 0.$$

$$g'(x) = 2[(e^{2x} + e^{-2x}) - (e^{x} + e^{-x})]$$

$$= 2[h(2x) - h(x)] \text{ where}$$

$$h(x) = e^{x} + e^{-x}.$$

$$h'(x) > 0 \Rightarrow g'(x) > 0.$$

$$\therefore \quad f(-x, -x) > f(x,x).$$

It is interesting to note that the Gumbel's bivariate logistic distribution is not symmetric about (0,0) but marginal distributions are symmetric about zero.

The m.g.f. of Gumbel's bivariate logistic distribution is given by

$$G(t_1,t_2) = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-(1-t_1)x}\, e^{-(1-t_2)y}}{(1+e^{-x}+e^{-y})^3}\,dx\,dy$$

$$= 2 \int_{-\infty}^{\infty} e^{-(1-t_2)y} dy \int_{-\infty}^{\infty} \frac{e^{-(1-t_1)x}}{(1+e^{-x}+e^{-y})^3} dx ,$$

putting $\frac{1+e^{-y}}{1+e^{-x}+e^{-y}} = Z$ in the second integral

$$G(t_1,t_2) = 2 \int_{-\infty}^{\infty} e^{-(1-t_2)y} dy$$

$$[(1+e^{-y})^{-(2+t_1)} \int_0^1 Z^{1+t_1}(1-Z)^{-t_1} dZ]$$

$$= 2 \frac{\Gamma(2+t_1)\Gamma(1-t_1)}{\Gamma(3)} \int_{-\infty}^{\infty} e^{-(1-t_2)y} (1+e^{-y})^{-(2+t_1)} dy$$

$$= \Gamma(1-t_1)\, \Gamma(1-t_1+t_2)\, \Gamma(1-t_2) .$$

As the marginal expectations of X and Y are zero and standard deviations $\frac{\pi}{\sqrt{3}}$, the correlation coefficient is $\frac{3E(XY)}{\pi^2}$.

Now,

$$E(XY) = \left. \frac{\partial^2 \ln G(t_1,t_2)}{\partial t_1 \partial t_2} \right]_{t_1=t_2=0}$$

$$= \psi'(1) = \pi^2/6 .$$

$\therefore$ Correlation coefficient between X and Y $= \rho_{XY} = \frac{1}{2}$ (5.3).

Hence the use of this distribution is very limited.

Regression of Y on X :

The conditional moment generating function of Y given X=x is

$$G(t_2|X=x) = [F(x)]^{t_2} \Gamma(2+t_2)\,\Gamma(1-t_2)$$

$$\therefore \ \ln [G(t_2|X=x)] = t_2 \ln F(x) + \ln\Gamma(1+t_2)$$

$$+ \ln \Gamma(1+t_2) + \ln \Gamma(1-t_2).$$

$$\frac{\partial \ln G(t_2|X=x)}{\partial t_2} = \ln F(x) + \frac{1}{1+t_2} + \frac{d}{dt_2} \Gamma(1+t_2)$$

$$- \frac{d}{dt_2} \ln \Gamma(1-t_2). \quad (5.4)$$

The conditional expectations can be obtained by putting $t_2 = 0$ in (5.4) ,

$$E[Y|X=x] = 1 + \ln F(x) .$$

Similarly $E[X|Y=y] = 1 + \ln F(y)$.

$$\lim_{x\to-\infty} E[Y|X=x] = -\infty, \quad E[Y|X=0] = 0.30685$$

$$\lim_{x\to\infty} E[Y|X=x] = 1 \quad \text{and}$$

$$\lim_{y\to-\infty} E[X|Y=y] = -\infty, \quad E[X|Y=0] = 0.30685$$

$$\lim_{y\to\infty} E[X|Y=y] = 1.$$

Thus conditional expectation of one variable is an increasing function of the other and it becomes asymptotically parallel to the axis at distance unity. Differentiating (5.4) w.r.t. t_2 and putting $t_2 = 0$, we get conditional variance of Y given X=x as

$$\sigma^2[\,Y \mid X = x\,]$$

$$= -\frac{1}{(1+t_2)^2} + \psi'(1+t_2) + \psi'(1-t_2)]_{t_2=0}$$

$$= 2\psi'(1) - 1$$

$$= \frac{\pi^2}{3} - 1 .$$

Similarly,

$$\sigma^2[X \mid Y = y] = \frac{\pi^2}{3} - 1.$$

Hence the conditional variances of X and Y are equal.

5.2. Generalization Of Gumbel's Bivariate Logistic Distribution :

The distribution function of a bivariate logistic distribution having two marginal distributions also logistic can be written in the form

$$F(x,y) = [1+ e^{-x} + e^{-y}]^{-v}; \quad -\infty < x < \infty$$
$$-\infty < y < \infty$$
$$v > 0. \qquad (5.5)$$

This bivariate logistic distribution is nothing but generalization of Gumbel's bivariate logistic distribution. The distribution given by (5.5) may arise from two independent variates having extreme value distribution with

$$P[X < x, Y < y] = \exp \{- \lambda(e^{-x} + e^{-y})\}$$

where λ has a gamma distribution with shape parameter v.
The expression (5.5) may be obtained as :

$$P[X < x, Y < y]$$
$$= \int_0^\infty \exp \{- \lambda(e^{-x} + e^{-y})\} \frac{\lambda^{v-1} e^{-\lambda}}{\Gamma(v)} d\lambda$$
$$= \int_0^\infty \frac{\exp\{- \lambda(e^{-x} + e^{-y} +1)\}\lambda^{v-1}}{\Gamma(v)} d\lambda$$
$$= \frac{1}{(1-e^{-x}+ e^{-y})^v} ;$$ [illegible] .

The p.d.f. corresponding to (5.5) is

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \, \partial y}$$

$$= -\frac{\partial}{\partial x} \left[\frac{v e^{-y}}{(1+e^{-x}+ e^{-y})^{v+1}}\right]$$

$$= \frac{v(v+1)\, e^{-x-y}}{(1+e^{-x}+e^{-y})^{v+2}}; \qquad (5.6)$$

Properties of the distribution :

Satterthwaite and Hutchinson (1978) have studied the properties of above type of bivariate logistic distribution.

The bivariate logistic distribution with p.d.f. given by (5.6) is not symmetric about (0,0). For this result it is enough to show that $f(-x,-x) \neq f(x,x)$ for at least one value of x.

$$f(-x,-x) = \frac{v(v+1)\, e^{2x}}{(1+2e^{x})^{v+2}}$$

and $$f(x,x) = \frac{v(v+1)\, e^{-2x}}{(1+2e^{-x})^{v+2}}.$$

In particular let us take $x = +1$.

$$f(-1,-1) = (0.0043)\, v(v+1).$$

$$f(1, 1) = (0.015)\, v(v+1).$$

$\therefore$ $f(-1, -1) \neq f(1,1)$ for any value of $v > 0$.

Marginal distributions of X and Y.

The probability density function of the marginal distribution of X is

$$f_1(x) = \int_{-\infty}^{\infty} \frac{v(v+1)\, e^{-x-y}}{(1+e^{-x}+ e^{-y})^{v+2}}\, dy$$

putting $t = \dfrac{1}{1+e^{-x} + e^{-y}}$, we have

$$f_1(x) = \int_{0}^{\frac{1}{1+e^{-x}}} v(v+1)\, e^{-x}\, t^{v}\, dt$$

$$= \frac{v e^{-x}}{(1+e^{-x})^{v+1}} \quad . \qquad (5.7)$$

Similarly the p.d.f. of marginal distribution of Y is

$$f_2(y) = \frac{v\, e^{-y}}{(1+e^{-y})^{v+1}} \quad . \qquad (5.8)$$

We note that X and Y are not independent.

Mode of the distribution :

In order to find the mode of the distribution, consider

$$\ln f(x,y) = \ln v + \ln (v+1) - x - y-(v+2) \ln(1+e^{-x}+ e^{-y})$$

$$\frac{\partial \ln f(x,y)}{\partial x} = 0$$

$$\Rightarrow \frac{(v+2)\ e^{-x}}{1+e^{-x}+ e^{-y}} = 1 . \qquad (5.9)$$

and

$$\frac{\partial \ln f(x,y)}{\partial y} = 0$$

$$\Rightarrow \frac{(v+2)\ e^{-y}}{1+ e^{-x}+ e^{-y}} = 1 . \qquad (5.10)$$

Solving (5.9) and (5.10) for x and y we see that the mode of the distribution lies on the line $x = y$. (5.11)

Now

$$f(x,x) = \frac{v(v+1)\ e^{-2x}}{(1+2e^{-x})^{v+2}}$$

$$= h(x).\ (\text{say})$$

$h'(x) = 0$ gives

$$1 + 2\ e^{-x} = (v+2)\ e^{-x}$$

i.e.

$$x = \ln v \quad . \qquad (5.12)$$

(5.11) and (5.12) indicate that the mode of the bivariate logistic distribution is $(\ln v, \ln v)$.

Cumulant Generating Function :

The m.g.f. of the bivariate logistic distribution is given by

$$G(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{t_1 x}\, e^{t_2 y}\, dx\, dy$$

$$= v(v+1) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-(1-t_1)x}\, e^{-(1-t_2)y}}{(1+e^{-x}+e^{-y})^{v+2}}\, dx\, dy$$

$$= v(v+1) \int_{-\infty}^{\infty} e^{-(1-t_2)y} dy \int_{-\infty}^{\infty} \frac{e^{-(1-t_1)x}}{(1+e^{-x}+e^{-y})^{v+2}} dx. \quad (5.13)$$

Putting $\frac{1+e^{-y}}{1+e^{-x}+e^{-y}} = Z$ in the second integral of (5.13) we have

$$G(t_1,t_2) = v(v+1) \int_{-\infty}^{\infty} e^{-(1-t_2)y}\, dy$$

$$[(1+e^{-y})^{-(v+t_1+1)} . \int_0^1 Z^{v+t_1}(1-Z)^{-t_1}\, dz\ ]$$

$$= v(v+1) \int_{-\infty}^{\infty} e^{-(1-t_2)y} dy\ [\frac{(1+e^{-y})^{-(v+t_1+1)}\, \Gamma(v+t_1+1)\ \Gamma(1-t_1)}{(v+1)\ \ \Gamma(v+1)}]$$

$$= \frac{\Gamma(v+t_1+1)\ \Gamma(1-t_1)}{\Gamma(v)} \int_{-\infty}^{\infty} e^{-(1-t_2)y}(1+e^{-y})^{-(v+t_1+1)}\, dy$$

$$= \frac{\Gamma(1-t_1)\ \Gamma(v+t_1+t_2)\ \Gamma(1-t_2)}{\Gamma(v)} \tag{5.14}$$

Taking logrithms of both sides of (5.14) we get

$$\ln G(t_1,t_2) = \ln \Gamma(1-t_1) + \ln \Gamma(v+t_1+t_2) + \ln \Gamma(1-t_2) - \ln \Gamma(v).$$

Differentiating the above expression with respect to t_1 and putting $t_1 = t_2 = 0$

$$E(X) = \frac{\partial}{\partial t_1} \ln G(t_1,t_2)\Big]_{t_1=t_2=0}$$

$$= \frac{-\Gamma'(1-t_1)}{\Gamma(1-t_1)} + \frac{\Gamma'(v+t_1+t_2)}{\Gamma(v+t_1+t_2)}\Bigg]_{t_1=t_2=0}$$

$$= \frac{-\Gamma'(1)}{\Gamma(1)} + \frac{\Gamma'(v)}{\Gamma(v)}$$

$$= -\psi(1) + \psi(v)$$

$$= \gamma + \psi(v)$$

where γ = Euler's constant and $\psi(v)$ is digamma function.

Similarly $E(Y) = \gamma + \psi(v)$.

Now $$\mathrm{Var}(X) = \frac{\partial^2 \ln G(t_1,t_2)}{\partial t_1^2}\Bigg]_{t_1=t_2=0}$$

$$= \psi'(1-t_1) + \psi'(v+ t_1+t_2)]_{t_1 = t_2 = 0}$$

$$= \psi'(1) + \psi'(v)$$

$$= \frac{\pi^2}{6} + \xi(2, v),$$

where $\xi(2, v) = \sum_{m=0}^{\infty} (m+ v)^{-2}$; the Riemann Zeta function, Similarly $Var(Y) = \frac{\pi^2}{6} + \xi(2, v)$.

$$Cov(X,Y) = \frac{\partial^2 \ln G(t_1,t_2)}{\partial t_1 \partial t_2}]_{t_1=t_2 = 0}$$

$$= \psi'(v)$$

$$= \xi(2, v).$$

Hence the correlation coefficient comes out to be

$$\rho = \frac{\xi(2 , v)}{\xi(2,v) + \frac{\pi^2}{6}} .$$

Regression of Y on X :

The conditional probability density function of Y given X=x is

$$f[y|X=x] = \frac{(v+1) e^{-y} (1+e^{-x})^{v+1}}{(1+e^{-x} + e^{-y})^{v+2}} . \qquad (5.15)$$

The conditional moment generating function of Y given X=x is

$$G(t_2|X=x) = \int_{-\infty}^{\infty} f(y|X=x)\, e^{t_2 y}\, dy$$

$$= \int_{-\infty}^{\infty} \frac{(v+1)\, e^{-y}(1+e^{-x})^{v+1}\, e^{t_2 y}}{(1+e^{-x}+e^{-y})^{v+2}}\, dy$$

putting $\frac{1+e^{-x}}{1+e^{-x}+e^{-y}} = Z$, we get

$$G[t_2|X=x] = \int_0^1 (v+1)\, Z^{v+t_2}\, (1-Z)^{-t_2}\, dz$$

$$= (v+1)\, (1+e^{-x})^{t_2}\, \frac{\Gamma(v+1+t_2)\, \Gamma(1-t_2)}{\Gamma(v+2)}$$

$\therefore$ $G(t_2|X=x)$

$$= [F(x)]^{t_2|v}\, \frac{\Gamma(v+1+t_2)\, \Gamma(1-t_2)}{\Gamma(v+1)}.$$

Thus the regression of Y on X is

$$E[Y|X=x] = \frac{1}{v}\, \ln F(x) + \psi(v+1) + C$$

$$= -\ln(1+e^{-x}) + \psi(v+1) + C. \qquad (5.16)$$

Similarly

$$E[X|Y=y] = -\ln(1+e^{-y}) + \psi(v+1) + C. \qquad (5.17)$$

Now

$$\text{Var}\,[Y|X = x]$$

$$= \frac{\partial^2}{\partial t_2^2}\,\ell n\, G\,(t_2|X{=}x)\,]_{t_2 = 0}$$

$$= \frac{\pi^2}{6} + \xi(\,2,\ v{+}1)\ . \tag{5.18}$$

Also,

$$\text{Var}\,(X|Y{=}y) = \frac{\pi^2}{6} + \xi\,(2,\ v+1)\ . \tag{5.19}$$

Hence the conditional variances of X and Y are equal.

5.3. Another family of bivariate logistic distribution :

The distribution function of another bivariate logistic distribution given by Gumbel (1961) is

$$F(x,y) = \frac{[1+\alpha\ e^{-x-y}(1+e^{-x})^{-1}(1+e^{-y})^{-1}]}{(1+e^{-x})\ (1+e^{-y})}\ ;\ -1 \le \alpha \le 1. \tag{5.20}$$

The expression (5.20) represents a parametric family of distributions. The density function corresponding to (5.20) can be written as

$$f(x,y) = \frac{e^{-x-y}}{(1+e^{-x})^2(1+e^{-y})^2}\,[1+\alpha(\frac{1-e^{-x}}{1+e^{-x}})(\frac{1-e^{-y}}{1+e^{-y}})] \tag{5.21}$$

$$f(-x,-y) = \frac{e^{x+y}}{(1+e^x)^2\,(1+e^y)^2}\,[1+\alpha\,\frac{(1-e^x)\,(1-e^y)}{(1+e^x)\,(1+e^y)}\,].$$

Since

$$\frac{e^{-x-y}}{(1+e^{-x})^2(1+e^{-y})^2} = \frac{e^{x+y}}{(1+e^x)^2\,(1+e^y)^2}$$

and

$$\frac{(1-e^{-x})(1-e^{-y})}{(1+e^{-x})(1-e^{y})} = \frac{(1-e^x)\,(1-e^y)}{(1+e^x)\,(1+e^y)}\,.$$

$$f(-x,-y) = f(x,y) \quad . \tag{5.22}$$

$$f(-x,y) = \frac{e^{x-y}}{(1+e^x)^2(1+e^{-y})^2}\,[\,1+\alpha\,(\,\frac{1-e^x}{1-e^x})\,(\frac{1-e^{-y}}{1+e^{-y}})]$$

$$= \frac{e^{y-x}}{(1+e^{-x})^2(1+e^y)^2}\,[\,1+\alpha\,\frac{(1-e^{-x})\,(1-e^y)}{(1+e^{-x})\,(1+e^y)}\,]$$

$$= f(x,-y) \quad . \tag{5.23}$$

From (5.22) and (5.23), we observe that the bivariate logistic distribution with density function given by (5.21) is symmetric about (0,0).

The marginal distribution of X is given by

$$f_1(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_{-\infty}^{\infty} \frac{e^{-x-y}}{(1+e^{-x})^2(1+e^{-y})^2}\, dy + \frac{\alpha(1-e^{-x})e^{-x}}{(1+e^{-x})(1-e^{-x})^2} \int_{-\infty}^{\infty} \frac{e^{-y}(1-e^{-y})}{(1+e^{-y})^3}\, dy$$

putting $t = \frac{1}{1+e^{-y}}$ in the second integral

$$f_1(x) = \frac{e^{-x}}{(1+e^{-x})^2} + \frac{\alpha(1-e^{-x})e^{-x}}{(1+e^{-x})(1+e^{-x})^2} \left[\int_0^1 t\, dt - \int_0^1 (1-t)dt \right]$$

$$= \frac{e^{-x}}{(1+e^{-x})^2} .$$

Thus the marginal distribution of X is logistic with mean zero and variance $\frac{\pi^2}{3}$. Similarly marginal distribution of Y is also logistic with mean zero and variance $\frac{\pi^2}{3}$. Conditional distribution of X given Y=y is

$$f(x|Y=y) = \frac{e^{-x}}{(1+e^{-x})^2} \left[1 + \alpha \left(\frac{1-e^{-x}}{1+e^{-x}}\right) \left(\frac{1-e^{-y}}{1+e^{-y}}\right) \right] .$$

Hence the conditional expectation of X given Y = y is

$$E[\, X|Y = y\,] = \int_{-\infty}^{\infty} \frac{x\, e^{-x}}{(1+e^{-x})^2}\, dx + \alpha \int_{-\infty}^{\infty} \frac{x\, e^{-x}(1-e^{-x})(1-e^{-y})}{(1+e^{-x})^3(1+e^{-y})}\, dy$$

$$= \alpha \left(\frac{1-e^{-y}}{1+e^{-y}}\right) \int_{-\infty}^{\infty} \frac{x\, e^{-x}(1-e^{-x})}{(1+e^{-x})^3}\, dx$$

putting $t = \frac{1}{1+e^{-x}}$

$$E[X|Y=y] = \frac{\alpha(1-e^{-y})}{(1+e^{-y})}\Big[\,2\int_0^1 t\,\ell n\,t\,dt - \int_0^1 \ell n\,t\,dt$$

$$-2\int_0^1 t\,\ell n\,(1-t)\,dt + \int_0^1 \ell n\,(1-t)dt\Big]$$

$$= \frac{\alpha(1-e^{-y})}{1+e^{-y}}$$

$$= \alpha[\,2F(y) - 1]\,. \qquad (5.24)$$

Now,

$$E(XY) = E_Y\,[\,E(XY|Y=y)]$$

$$= \int_{-\infty}^{\infty} \alpha\,y\,(2F(y)-1)\,f(y)\,dy\,.$$

$$\therefore\quad E(XY) = 2\alpha\int_{-\infty}^{\infty} yF(y)\,f(y)\,dy - \alpha\int_{-\infty}^{\infty} y\,f(y)\,dy$$

$$= 2\alpha\int_{-\infty}^{\infty} \frac{y\,e^{-y}}{(1+e^{-y})^3}\,dy - 0\,.$$

putting $t = \frac{1}{1+e^{-y}}$

$$E(XY) = 2\alpha\int_0^1 [\,\ell n\,t - \ell n\,(1-t)]\,t\,dt$$

$$= 2\alpha\Big[\int_0^1 t\,\ell n t\,dt - \int_0^1 t\,\ell n\,(1-t)\,dt\Big]$$

$$= 2\alpha \{ [t^2(-\frac{\ln t}{2} - \frac{1}{4})]_0^1 - \int_0^1 \ln t \, dt + \int_0^1 t \ln t dt \}$$

$$= \alpha \quad . \tag{5.25}$$

Using (5.25) the correlation coefficient between X and Y is

$$\rho_{XY} = \frac{3\alpha}{\pi^2} ,$$

which is a function of α .

As $-1 \leq \alpha \leq 1$, $|\rho_{XY}| < 0.3038847$.

Thus this type of bivariate distribution is more useful is practice as compared to the bivariate distribution given by (5.1).

Mode of the distribution :

Let

$$\frac{e^{-x}}{1+e^{-x}} = p, \ \frac{1}{1+e^{-x}} = q, \ \frac{e^{-y}}{1+e^{-y}} = p^*, \ \frac{1}{1+e^{-y}} = q^* .$$

$0 < p < 1$, $0 < q < 1$ similarly $0 < p^* < 1$, $0 < q^* < 1$.

The the density function (5.21) can be written down as

$$\phi\ (p,p^*) = (pq)\ (p^*\ q^*)\ \{\ 1 + \alpha(q-p)\ (q^* - p^*)\ \}$$

$$= p(1-p)\ p^*(1-p^*)\ \{1+(1-2p)\alpha(1-2p^*)\} \tag{5.26}$$

put $p = \frac{1}{2} + \delta_1$ and $p^* = \frac{1}{2} + \delta_2$;

$$-\frac{1}{2} < \delta_1 < \frac{1}{2} \quad \text{and} \quad |\delta_2| < \frac{1}{2} .$$

Then (5.26) is given by

$$\begin{aligned} g(\delta_1, \delta_2) &= \left(\frac{1}{4} - \delta_1^2\right)\left(\frac{1}{4} - \delta_2^2\right)\{1 + 4\delta_1 \alpha \delta_2\} \\ &= \left\{\frac{1}{16} - \frac{\delta_1^2}{4} - \frac{\delta_2^2}{4} + \delta_1^2 \delta_2^2\right\}\{1 + 4\delta_1 \alpha \delta_2\} \end{aligned} \quad (5.27)$$

$$\begin{aligned} \frac{\partial g(\delta_1 \delta_2)}{\partial \delta_1} &= \left(-\frac{2\delta_1}{4} + 2\delta_1 \delta_2^2\right)\{1 + 4\delta_1 \alpha \delta_2\} + \\ &\quad + 4\delta_2 \alpha \left\{\frac{1}{16} - \frac{\delta_1^2}{4} - \frac{\delta_2^2}{4} + \delta_1^2 \delta_2^2\right\} \end{aligned}$$

$$\frac{\partial g(\delta_1, \delta_2)}{\partial \delta_1} = 0$$

$$\Rightarrow \frac{4\delta_2 \alpha \left\{\frac{1}{16} - \frac{\delta_1^2}{4} - \frac{\delta_2^2}{4} + \delta_1^2 \delta_2^2\right\}}{\left(\frac{\delta_1}{2} - 2\delta_1 \delta_2^2\right)} = 1 + 4\delta_1 \alpha \delta_2 . \quad (5.28)$$

Similarly,

$$\frac{g(\delta_1 \delta_2)}{\partial \delta_2} = 0$$

$$\Rightarrow \frac{4\delta_1 \alpha \left\{\frac{1}{16} - \frac{\delta_1^2}{4} - \frac{\delta_2^2}{4} + \delta_1^2 \delta_2^2\right\}}{\left(\frac{\delta_2}{2} - 2\delta_2 \delta_1^2\right)} = 1 + 4\delta_1 \alpha \delta_2 . \quad (5.29)$$

From (5.28) and (5.29) we have,

$$\frac{4\delta_1\{\frac{1}{16}-\frac{\delta_1^2}{4}-\frac{\delta_2^2}{4}+\delta_1^2\,\delta_2^2\}}{\{\frac{\delta_2^2}{2}-2\delta_1^2\,\delta_2^2\}} = \frac{4\delta_2\{\frac{1}{16}-\frac{\delta_1^2}{4}-\frac{\delta_2^2}{4}+\delta_1^2\delta_2^2\}}{\{\frac{\delta_1}{2}-2\delta_1\delta_2^2\}}$$

$$\Rightarrow \delta_1\{\frac{\delta_1}{2}-2\delta_1\,\delta_2^2\} = \delta_2\{\frac{\delta_2}{2}-2\delta_1^2\,\delta_2\}$$

$$\Rightarrow \delta_1^2 = \delta_2^2 .$$

Taking $\delta_1^2 = \delta_2^2$ in (5.27) we get

$$h(\delta_1) = (\frac{1}{4}-\delta_1^2)^2\{1+4\delta_1^2\}$$

$$\therefore \quad h'(\delta_1) = (\frac{1}{4}-\delta_1^2)\,\{-4{}_1(1+4\delta_1^2)+8\delta_1(\frac{1}{4}-\delta_1^2)\}$$

$$h'(\delta_1) = 0$$

$$\Rightarrow 4\delta_1\,(1+4\delta_1^2) = 2\delta_1(1-4\delta_1^2) \because \delta_1^2 \neq \frac{1}{4}.$$

If $\delta_1 \neq 0$ then $2(1+4\delta_1^2) = 1-4\delta_1^2$.

$$\text{i.e. } \delta_1^2 = -\frac{1}{12}.$$

But this value of δ_1^2 is not admissible. Hence $\delta_1 = 0$, $\delta_2 = 0$. Thus the function $f(x,y)$ is maximum at $p = \frac{1}{2}$ and $p^* = \frac{1}{2}$ which means that mode is (0,0).

We observe that the correlation coefficient is a function of α while the mode is independent of α. Ali, Mikhail and Haq (1978) also have obtained the same type of bivariate logistic distribution given by (5.20) with $F(x,y)$ as the probability of the joint failure of both the components of a system before (x,y). They have studied some properties of the distribution.

Since $F(x,y) \leq F_1(x)\ F_2(y)$ for $-1 \leq \alpha \leq 0$; $\forall (x,y)$, $F(x,y)$ is negatively quadrant dependent for negative values of α. $F(x,y) \geq F_1(x)\ F_2(y)$ for $0 \leq \alpha \leq 1\ \forall (x,y)$. Hence $F(x,y)$ is positively quadrant dependent for positive values of α. If the value of $\alpha = 0$, we get stochastic independence of X and Y.

Let $F_1(x) = U$ and $F_1(y) = V$

$$\therefore F(x,y) \equiv F(u,v) = \frac{uv}{[1+\alpha(1-u)(1-v)]}.$$

Now

$$\frac{F(u,v)}{F_1(u)} = \frac{v}{1+\alpha(1-u)(1-v)}$$

$$\therefore \frac{\partial}{\partial u}\left(\frac{F(u,v)}{F_1(u)}\right) = \frac{\alpha(1-v)}{[1+\alpha(1-u)(1-v)]^2} > 0 \text{ if } \alpha > 0.$$

Thus $F(u,v)$ is left tail increasing [decreasing] for positive [negative] values of α.

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