A STUDY OF

PROCESSES WITH INDEPENDENT INCREMENTS

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CHAPTER 0

Processes with independent increments had been introduced by de Finnett in 1929. Their sample path properties were studied by Levy in 1934.

Most commonly, for inference problem we use $X_1, X_2, \ldots,$ independent and identically distributed random variables. However for a process $\{X(t), t \geq 0\}$, $X(t)$ mutually independent for $t \geq 0$ may not be useful model, is discussed by Kallianpur (page 10) by means of two examples, of which one shows that $X(t)$ does not have realizations in $C[0,1]$ induced by the distribution of $X(t)$ and the other, shows that there is no measurable process equivalent to $X(t)$. A process with stationary and independent increments (SIIP) may be treated as a continuous case analogue of partial sums of independent and identically distributed random variables.

In Chapter 1 we study definitions, relationship with infinite divisibility, representations of characteristic function of SIIP, relationship with martingales. Further we discuss Markov property. We also discuss construction of SIIP. Chapter 2 includes sample path properties, decomposition of SIIP. Finally we discuss strong law of large numbers and central limit theorem for SIIP. Chapter 3 consists of discussion about Kac statistic which is analogous to Kolmogorov-

Smirnov statistic. We also include sequential estimation procedures for the processes which belong to the exponential class of stochastic processes. Further we discuss Cramer-Rao type inequality, a genral form of an efficiently estimable parameter function and a general form of an efficient estimator. Ultimately we study an estimation of canonical measure G which occurs in the representation of a characteristic function of SIIP.

The processes with uncorrelated increments (Doob, page 99), processes with orthogonal increments (Doob, page 99), processes with interchangeable increments (Takacs, page 38), processes with cyclically interchangeable increments (Takacs, page 37) may be treated as a genralization of SIIP. Discussion about the distributions of supremum of SIIP is available in Takacs. Recurrence properties of processes with independent increments are discussed by Kingman (1964). Relationship of SIIP with extremal processes and subordinators is discussed by Kingman (1973). Applications of SIIP to queues, insurance risk, dams are discussed by Prabhu (1980). Inference about Levy processes which can be taken as a special case of SIIP is discussed by Akritas (1981,1982). Inference about gamma and stable processes is studied by Basawa and Brockwell (1978, 1980). Sequential Probability ratio test is discussed in

Gnosh (1970). V-mask for a negative binomial process is discussed by Muddapur (1974). These topics are not discussed in tnis disseration.

CHAPTER 1

1.1 Introduction

This chapter includes definitions and examples of peculiar processes. We also include an example of a process with stationary and independent increments (SIIP) defined on a probability space ($(0,1], \mathbb{B}, P$). Moreover the relationship of SIIP with infinitely divisible characteristic function is studied. Which is helpful in obtaining a general form of the characteristic function of SIIP and its two representations namely, Levy-Khintchine representation and Kolmogorov's representation. Further we discuss construction of SIIP, finite dimensional distributions, mean, variance, covariance function. In the last section relationship of SIIP with martingales and Markov process is studied. Finally we discuss, under certain conditions SIIP holds strong Markov property.

1.2 Definitions and preliminaries

Let $\mathcal{T}$ denote the set of non-negative integers or a finite interval or $[0,\infty)$.

Definition 1 : A stochastic process $\{X(t), t \in \mathcal{T}\}$ defined on a probability space $(\Omega, \mathbb{F}, P)$ with values in $(\mathbb{R}, \mathbb{B})$ is said to be a process with independent increments, if for every positive integer $k \geq 2$ and $\{t_1, t_2, \ldots, t_k\} \in \mathcal{T}$, such that $0 < t_1 < t_2 \ldots < t_k$, the random variables

$$X(t_1) - X(0),\ X(t_2) - X(t_1), \dots, X(t_k) - X(t_{k-1})$$

are independent.

<u>Definition 2</u> : A stochastic process $\{ X(t), t \in \mathcal{T} \}$ defined on a probability space $(\Omega, \mathbb{F}, P)$ with values in $(\mathbb{R}, \mathbb{B})$ is said to be a process with stationary increments, if

$$\mathcal{L}(X(t+h) - X(t)) = \mathcal{L}(X(h))$$

for every h such that $t, t+h \in \mathcal{T}$.

It is clear that if $\{ X(t), t \in \mathcal{T} \}$ is a process with independent increments possessing stationary increments, then for every positive integer $k \geq 2$ and $\{ t_1, t_2, \dots, t_k \} \in \mathcal{T}$

$$\mathcal{L}(X(t_2)-X(t_1), \dots, X(t_k) - X(t_{k-1})) = \mathcal{L}(X(t_2+h)-X(t_1+h), \dots, X(t_k+h)-X(t_{k-1}+h))$$

for every h such that $\{t_1+h, t_2+h, \dots, t_k+h\} \in \mathcal{T}$.

An example of a process with SIIP on probability space $(\mathfrak{X}, \mathbb{B}, P)$ in which $\mathfrak{X}$ is $(0,1]$, $\mathbb{B}$ is the Borel field on $(0,1]$ and P is the Lebesgue measure is given below. To provide such an example, first we prove a lemma which we need.

Lemma 3 : Let $\{ F_n, n \geq 1 \}$ be a sequence of distribution functions. Then there exists a sequence $\{ Z_n, n \geq 1 \}$ of independent random variables on $(\Omega, \mathbb{B}, P)$, such that Z_n has F_n as its distribution function for all $n \geq 1$.

Proof : We discuss the proof in the following three steps.

Step 1 : We generate a sequence $\{X_n, n \geq 1\}$ of independent and identically distributed Bernoulli random variables with $P\{ X_1 = 1 \} = \frac{1}{2}$.

Suppose

$$F_n(x) = 0 \quad \text{if} \quad x < 0$$
$$= \tfrac{1}{2} \quad \text{if} \quad 0 \leq x < 1$$
$$= 1 \quad \text{if} \quad x \geq 1. \qquad (1.2.1)$$

For fixed $n \geq 1$ divide the interval $(0,1]$ into 2^n subintervals of length 2^{-n} each and j-th subinterval will be

$$I_n^{(j)} = ((j-1)2^{-n}, j2^{-n}]$$

for $j = 1,2,\ldots, 2^n$. Define for $\omega \in \Omega$ and $n \geq 1$

$$X_n(\omega) = 1 \quad \text{if } \omega \in \bigcup_{r=1}^{2^{n-1}} I_n^{(2r)}$$
$$= 0 \quad \text{otherwise.}$$

Now we show that $\{X_n, n \geq 1\}$ are independent and identically distributed Bernoulli random variables with F_n given in (1.2.1). Clearly

$$P\{X_n = 0\} = P\{X_n = 1\} = \tfrac{1}{2}$$

for every $n \geq 1$. To show that X_n' s are independent, let us evaluate

$$P\{X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k\}.$$

Clearly

$$P\{X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k\} = 2^{-(k+1)}. \quad (1.2.2)$$

On the other hand

$$P\{X_1 = x_1\} P\{X_2 = x_2\} \ldots P\{X_k = x_k\} = 2^{-(k+1)}. \quad (1.2.3)$$

Therefore

$$P\{X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k\} = P\{X_1 = x_1\} P\{X_2 = x_2\} \ldots P\{X_k = x_k\}. \quad (1.2.4)$$

The relationship (1.2.4) holds for any k and all possible combinations $(x_1, x_2, \ldots, x_k)$. Hence $X_1, X_2, \ldots,$ are independent random variables.

<u>Step 2</u> : We construct a sequence of independent and identically distributed random variables on $(\Omega, \mathbb{B}, P)$ with uniform distribution on (0,1).

Arrange the sequence $\{ X_n, n \geq 1 \}$ defined above in a double array as follows :

$$
\begin{array}{l}
X_{11}, X_{12}, \dots \\
X_{21}, X_{22}, \dots \\
\dots\dots\dots\dots \\
X_{n1}, X_{n2}, \dots \\
\dots\dots\dots\dots
\end{array}
$$

Define for $\omega \in \Omega$

$$U_n(\omega) = \sum_{k=1}^{\infty} 2^{-k} X_{nk}(\omega) . \qquad (1.2.5)$$

Since the series (1.2.5) is dominated above for every ω by a convergent geometric series, U_n converges almost surely. Hence for every $n \geq 1, U_n$ is a limit of measurable functions, so that U_n is a random variable. Since the variables in the different rows are independent and identically distributed random variables, $\{ U_n, n \geq 1 \}$ are independent and identically distributed random variables.

Clearly

$$P\{X_{n1} = x_1, \dots, X_{nk} = x_k\} = 2^{-k} ,$$

for all 2^k possible values of the vector $(x_1, x_2, \dots, x_k)$.

Let

$$S_{nk} = \sum_{i=1}^{k} 2^{-i} X_{ni} .$$

Then S_{nk} assumes the 2^k possible values $j2^{-k}$, $0 \leq j \leq 2^k - 1$ and

$P\{S_{nk} = j2^{-k}\} = 2^{-k}$ for $0 \leq j \leq 2^k - 1$. For any x, $0 \leq x < 1$ there are $[2^k x] + 1$ values of j such that $0 \leq j2^{-k} \leq x$, in which $[2^k x]$ denotes integer part of $2^k x$.

Hence

$$P\{ S_{nk} \leq x \} = \frac{[2^k x] + 1}{2^k} .$$

Clearly, $S_{nk}(\omega)$ series of non-negative terms increases to $U_n(\omega)$ for every $\omega \in \Omega$ as k tends to infinity and hence $\{S_{nk} \leq x\}$ decreases to $\{ U_n \leq x \}$ as k tends to infinity.

Hence

$$\begin{aligned} P\{U_n \leq x\} &= \lim_{k \to \infty} P\{ S_{nk} \leq x \} \\ &= \lim_{k \to \infty} \frac{[2^k x] + 1}{2^k} \\ &= x \end{aligned}$$

Therefore U_n is uniformly distributed over (0,1). Thus $\{ U_n, n \geq 1 \}$ is a sequence of independent and identically distributed random variables, with common distribution which is uniform on(0,1).

1. A countable family $\{\varphi_i\}$ ... in ... is called an orthonormal family in $\mathcal{L}^2[a,b]$ if

$$\int_a^b \varphi_j \varphi_i = 0 \text{ if } i \neq j$$
$$= 1 \text{ if } i = j$$

Ex: $\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots\}$

2. Let Φ be an orthonormal ... $f \in \mathcal{L}^2[a,b]$ and if

$$c_k = \int_a^b f \varphi_k$$

$$f \sim \sum_1^\infty c_k \varphi_k .$$

3. The family Φ is complete if

$$\int_a^b h \varphi_k = 0 \text{ for all } k \Rightarrow h = 0 .$$

4) If Φ ~~...~~ complete has any ... prop ... all

a) Φ is complete

b) set of all ... is dense in $\mathcal{L}^2[a,b]$

c) For every $f \in \mathcal{L}^2[a,b]$ ... functions ... and of $\mathcal{L}^2[a,b]$ i.e.

$$\lim_{n \to \infty} d(s_n, f) = 0$$

where $S_n = \sum_1^n c_k \varphi_k$ ...

and $c_k = \int_a^b f \cdot \varphi_k .$

Step 3 : In order to prove the result for general F_n, define for every $n \geq 1$

$$Y_n(u) = \inf_x [x : F_n(x) \geq u] \text{ for } 0 < u < 1$$
$$= 0 \qquad \text{otherwise .}$$

Let $Z_n(\omega) = Y_n(U_n(\omega))$. Hence $Z_n(\omega)$ is $\mathbb{B}$ measurable. For $0 < u < 1$, $Y_n(u) \leq x$ if and only if $u \leq F_n(x)$, therefore Z_n is a random variable and

$$P\{Z_n \leq x\} = P\{U_n \leq F_n(x)\}$$
$$= F_n(x) \quad . \qquad (1.2.6)$$

Therefore (1.2.6) yields that Z_n has distribution function F_n. Since $U_1, U_2, \ldots$ are independent random variables, lemma follows.

The following theorem gives an example of SIIP defined on $(\Omega, \mathbb{B}, P)$.

Theorem 4 : Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables on $(\Omega, \mathbb{B}, P)$ having common distribution normal with mean zero and unit variance. Let $g_1, g_2, \ldots$ be an arbitrary complete ortnonormal sequence in $L^2[0,T]$ and

$$G_j(t) = \int_0^t g_j(t)\, du \ , \quad j = 1,2,\ldots, .$$

Then for each t, $0 \leq t < T$ the series

$$\sum_{j=1}^{\infty} G_j(t)\ X_j \qquad (1.2.7)$$

converges almost surely to a random variable W(t) which has stationary and independent increments.

Proof : To show that the series (1.2.7) converges almost surely, it is sufficient to show that $\sum_{j=1}^{\infty}$ var $(G_j(t)\ X_j)$ converges. Since X_j' s are standard normal variables

$$\sum_{j=1}^{\infty} \text{var}(G_j(t)\ X_j) = \sum_{j=1}^{\infty} G_j^2\ (t)\ . \qquad (1.2.8)$$

Let us evaluate $\int_0^T I_{(0,t]}^2(u)\ du$ to obtain (1.2.8), where $I_{(0,t]}(u)$ is an indicator function. Since $I_{(0,t]}(u)$ is in $L^2(0,T)$ and $g_1, g_2, \ldots,$ forms a complete orthonormal sequence we get

$$I_{(0,t]}(u) = \sum_{j=1}^{\infty} a_{jt}\ g_j(u)$$

where

$$a_{jt}^2 = \int_0^T g_j(u)\ I_{(0,t]}(u)\ du\ .$$

Hence

$$\int_0^T I_{(0,t]}^2(u)\ du = \int_0^T \left(\sum_{j=1}^{\infty} a_{jt}\ g_j(u)\right)^2 du$$

$$= \int_0^T \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jt}\ a_{kt}\ g_j(u) g_k(u) du$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jt}\, a_{kt} \int_0^T g_j(u) g_k(u)\, du$$

$$= \sum_{j=1}^{\infty} a_{jt}^2$$

$$= \sum_{j=1}^{\infty} \int_0^T g_j(u)\, I_{(0,t]}(u)\, du$$

$$= \sum_{j=1}^{\infty} G_j^2(t) \cdot$$

Since $\int_0^T I^2_{(0,t]}(u)\, du = t$, the series (1.2.8) converges.

. . . .

Denote $\sum_{j=1}^{\infty} G_j(t)\, X_j = W(t)$, then $W(t)$ is normal random variable for every $t \in \mathcal{J}$ follows from the fact that $W(t)$ is a limit of sum of normal variables. Clearly expected value of $W(t)$ is zero. Let us obtain the covariance function, $\mathrm{Cov}\,(W(t), W(s))$ for $0 \leq s,\ t \leq T$.

Now

$$\mathrm{Cov}\,(W(t), W(s))$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} G_j(t)\, G_k(s)\, \mathrm{Cov}\,(X_j, X_k)$$

$$= \sum_{j=1}^{\infty} G_j(t)\, G_j(s) \quad . \qquad (1.2.9)$$

On the other hand

$$\int_0^T I_s(u)\, I_t(u)\, du$$

$$= \int_0^T \Big(\sum_{j=1}^{\infty} a_j\, g_j(u)\Big) \Big(\sum_{k=1}^{\infty} b_k\, g_k\, (u)\Big)\, du$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_j \, b_k \int_0^T g_j(u) \, g_k(u) \, du$$

$$= \sum_{j=1}^{\infty} a_j \, b_j$$

$$= \sum_{j=1}^{\infty} [\int_0^T I_t(u) \, g_j(u) \, du] . [\int_0^T I_s(u) g_j(u) \, du]$$

$$= \sum_{j=1}^{\infty} G_j(t) \quad G_j(s) \quad . \tag{1.2.10}$$

Using (1.2.9) and (1.2.10) we get

$$\text{Cov } (W(t), W(s)) = \int_0^T I_s(u) \, I_t(u) \, du$$

$$= \int_0^{\min(s,t)} du$$

$$= \min (s,t). \tag{1.2.11}$$

From (1.2.11) and the fact that $W(t)$ has normal distribution with mean zero and variance t, it can be seen that the increments are independent. Moreover distribution of $W(t)-W(s)$ for $s < t$ is normal with mean zero and variance $(t-s)$. Similarly the distribution of $W(t+h) - W(s+h)$ with $0 < s+h, t+h < T$ is normal with mean zero and variance $(t-s)$. Therefore the increments are stationary. Hence the theorem.

Now we give an example of a process with independent increments which does not possess stationary increments.

Let $\{X(t), t \geq 0\}$ be a process with independent increments having distribution of $X(t)$ normal with mean zero and variance t. Define new process $Y(t)$ for $t \geq 0$ as follows :

$$Y(t) = t^2 + X(t) .$$

To show that the increments $Y(s), Y(t)-Y(s)$ for $0 \leq s \leq t$ are independent, let us evaluate

$$P\{Y(s) \leq y_1, Y(t) - Y(s) \leq y_2\}$$

$$= P\{X(s) \leq y_1 - s^2, X(t) - X(s) \leq y_2 - t^2 + s^2\}$$

$$= P\{X(s) \leq y_1 - s^2,\} P\{X(t)-X(s) \leq y_2 - t^2 + s^2\}$$

$$= P\{Y(s) \leq y_1\} \; P\{Y(t) - Y(s) \leq y_2\} .$$

Hence $\{Y(t), t \geq 0\}$ is a process with independent increments. Clearly the distribution of an increment $Y(t) - Y(s)$ is normal with mean $(t^2 - s^2)$ and variance $(t-s)$. On the other hand distribution of $Y(t-s)$ is normal with mean $(t-s)^2$ and variance $(t-s)$, which implies that

$$\mathcal{L}(Y(t-s)) \neq \mathcal{L}(Y(t) - Y(s)).$$

Hence $\{Y(t), t \geq 0\}$ is a process with independent increments but the increments are not stationary.

Next we give an example of a process with stationary but not of independent increments.

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables, with common distribution uniform on (0,1).

Define

$$F_n(t) = \sum_{j=1}^{n} I_{(0,t]}(X_j),$$

$I_{(0,t]}(.)$ denotes indicator function. Then $\{F_n(t), 0 \leq t \leq 1\}$ is a continuous time stochastic process.

Let us consider the distribution of the increments

$$F_n(t) - F_n(s) \text{ and } F_n(t+h) - F_n(s+h)$$

such that $0 \leq s < t < 1$ and $0 < s+h,\ t+h < 1$. By definition

$$F_n(t) - F_n(s) = \sum_{j=1}^{n} I_{(s,t]}(X_j).$$

Hence

$$P\{F_n(t)-F_n(s) = n_1\} = \binom{n}{n_1}(t-s)^{n_1}(1-t+s)^{n-n_1}. \quad (1.2.12)$$

Similarly

$$P\{F_n(t+h)-F_n(s+h) = n_1\} = \binom{n}{n_1}(t-s)^{n_1}(1-t+s)^{n-n_1}. \quad (1.2.13)$$

The relations (1.2.12) and (1.2.13) yields that the process $\{F_n(t),\ 0 \leq t \leq 1\}$ is a process with stationary increments.

To show that the increments are not independent, we obtain the joint distribution of the increments

$$F_n(s) \text{ and } F_n(t) - F_n(s) \text{ for } 0 \leq s < t \leq 1 .$$

Let us obtain

$$P\{F_n(s) = n_1,\ F_n(t)-F_n(s) = n_2\}$$

$$= \frac{n!}{n_1!\,n_2!\,(n-n_1-n_2)!}\ s^{n_1}\ (t-s)^{n_2}\ (1-t)^{n-n_2-n_2} . \quad (1.2.14)$$

On the other hand product of marginal probabilities

$$P\{F_n(s) = n_1\}\quad P\{F_n(t)-F_n(s) = n_2\}$$

$$= \binom{n}{n_1}\ s^{n_1}(1-s)^{n-n_1}\ \binom{n}{n_2}\ (t-s)^{n_2}(1-t+s)^{n-n_2} . \quad (1.2.15)$$

Using (1.2.14) and (1.2.15) clearly the increments are not independent.

1.3 Construction of SIIP .

The theorem discussed below gives a relationship between SIIP and infinitely divisible distribution, by means of this theorem, characteristic function and its representation can be studied.

Definition 5 : A distribution F is infinitely divisible if and only if for each n it can be represented as the distribution of the sum

$$S_n = X_{1.n} + X_{2.n} + \dots + X_{n.n}$$

of n independent random variables with a common distribution F_n. Equivalently the characteristic function ϕ is infinitely divisible if there exists a characteristic function $\Psi_n(u)$ such that

$$\phi(u) = [\ \Psi_n(u)]^n$$

for every $n \geq 1$.

Theorem 6 : If $\{ X(t), t \in \mathcal{T} \}$ is a process with stationary and independent increments, then $X(t)-X(s)$ for $t, s \in \mathcal{T}$ such that $t < s$ has an infinitely divisible distribution for every $t \geq 0$.

Proof : For any positive integer n, $X(t)-X(s)$ can be written as

$$X(t)-X(s) = \sum_{k=1}^{n} Y_{k.n} ,$$

where $Y_{k.n} = X(s+ \frac{k}{n} (t-s)) - X(s+ \frac{k-1}{n} (t-s))$, for $1 \leq k \leq n$.

$\{ Y_{k.n}, k = 1,2,\ldots,n\}$ are independent and identically distributed random variables, which follows from the fact that increments are stationary and independent. Hence the proof.

<u>Theorem 7</u> : Let $\{X(t), t \in T\}$ be a SIIP with $P(X(0)=0)=1$ such that $\phi(u)$ the characteristic function of $X(t)$ is continuous at $t = 0$, then for every u

$$\phi_t(u) = [\ \phi_1(u)\ ]^t .$$

<u>Proof</u> : We can write,

$$X(t+s) = X(t+s)-X(s)+ X(s).$$

Since the increments are independent $X(t+s)-X(s)$ and $X(s)$ are independent. Using the fact that the increments are stationary we get

$$\mathcal{L}\,(X(t+s) - X(s)) = \mathcal{L}\,(X(t)).$$

Hence for u real,

$$\phi_{s+t}(u) = \phi_s(u)\ \phi_t(u). \tag{1.3.1}$$

We obtain,

$$\lim_{s\downarrow 0}\ \phi_{s+t}(u) = \lim_{s\downarrow 0}\ \phi_s(u)\ \ \phi_t(u)$$

$$= \phi_t(u) . \tag{1.3.2}$$

Similarly

$$\lim_{s \downarrow 0} \phi_t(u) = \lim_{s \downarrow 0} \phi_{t-s}(u) \ \phi_s(u)$$

therefore

$$\lim_{s \downarrow 0} \phi_{t-s}(u) = \phi_t(u) \quad . \tag{1.3.3}$$

Hence from (1.3.2) and (1.3.3) it can be seen that $\phi_t(u)$ is continuous for all $t \geq 0$. A measurable solution to (1.3.1) is $\phi_t(u) = [\ \phi_1(u)\ ]^t$ (Breiman, page 304).

Hence the theorem.

Since $\phi_1(u)$ is non vanisning (Chung, page 239) it can be expressed as

$$\phi_t(u) = \exp \{t \log \phi_1(u) \}$$

$$= \exp \{t \Psi (u) \} \quad . \tag{1.3.4}$$

The function $\Psi(u)$ is called the exponent function of the process.

If the assumption that the characteristic function of $X(t)$ is continuous at $t = 0$ is dropped, then the theorem 7 need not hold. This we illustrate in the following example.

Let $\{ X(t), t \geq 0 \}$ be a SIIP having characteristic function $[\ \phi_1^X (u)\ ]^t$. We define a process $Y(t) = X(t)-a(t)$, for non-random function $a(t)$ given by

$$a(t) = 0 \quad \text{if} \quad t = 0$$
$$= 1 \quad \text{if} \quad t > 0.$$

A process $\{Y(t),\ t \geq 0\}$ is a process with independent increments, which we are going to prove in the lemma 13.
The characteristic function of $Y(t)$ will be

$$\phi_t^Y(u) = \exp\{-i\,a(t)\}\phi_t^X(u)$$
$$= \exp\{-i\,a(t)\}[\phi_1^X(u)]^t.$$

Therefore,

$$\phi_t^Y(u) \neq [\phi_1^Y(u)]^t .$$

Without loss of generality, let $\{X(t),\ t \in \mathcal{T}\}$ be a SIIP having $P(X(0) = 0) = 1$. Since the distribution of $X(t)$ is infinitely divisible, the characteristic function $\phi_t(u)$ of $X(t)$ possesses Lévy-Khintchine representation (Doob, page 130). Hence

$$\log \phi_t(u) = i\,u\;a(t) + \int_{-\infty}^{\infty} \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right) \frac{1+x^2}{x^2}\, dG(t,x). \tag{1.3.5}$$

$G(t,.)$ is monotone nondecreasing, right continuous, bounded function in x, with $\lim_{x \to -\infty} G(t,x) = 0$ and $a(t)$ is a constant.
Left hand side of (1.3.5) is continuous, therefore $a(.)$ is continuous.

On the other hand, from the theorem 6 we get

$$\phi_t(u) = [\ \phi_1(u)\ ]^t .$$

Hence

$$\log \ \phi_t(u) = t \log \phi_1(u)$$

$$\log \phi_t(u) = t[iua(1) + \int_{-\infty}^{\infty} (e^{iux}-1-\frac{iux}{1+x^2}) \frac{1+x^2}{x^2} dG(1,x)] \quad (1.3.6)$$

Therefore using (1.3.5) and (1.3.6) one can write

$$a(t) = t.a \quad \text{and}$$

$$G(t,x) = t\ G(x).$$

So (1.3.5) becomes

$$\log \phi_t(u) = iua\ t + t \int_{-\infty}^{\infty} (e^{iux}-1-\frac{iux}{1+x^2}) \frac{1+x^2}{x^2} dG(x). \quad (1.3.7)$$

If Var(X(t)) is finite then Kolmogorov's representation holds for $\phi_t(u)$, and it is given below

$$\log \phi_t(u) = iu\alpha t + t \int_{-\infty}^{\infty} (e^{iux}-1-iux) \frac{1}{x^2} dH(x). \quad (1.3.8)$$

α is a real constant and H(.) is a bounded nondecreasing function, in view of the following lemma.

<u>Lemma 8</u> : If Var (X(t)) is finite,

$$\int_{-\infty}^{\infty} (1+x^2)\ dG(x) \quad \text{is finite.}$$

<u>Proof</u> : Using (1.3.4), $\phi_t(u)$ can be written as

$$\phi_t(u) = e^{t\,\Psi(u)} .$$

It follows from (1.3.7)

$$\Psi(u) = i\,ua + \int_{-\infty}^{\infty} \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right)\frac{1+x^2}{x^2}\, dG(x).$$

It follows from (1.3.7). Note that if Var(X(t)) is finite then $\Psi(u)$ is twice differentiable at 0 and the second derivative $\Psi''(0)$ is of the type

$$0 < -\Psi''(0) = \lim_{h\to 0} -\frac{1}{2h^2}\left[\Psi(2h) - 2\Psi(0) + \Psi(-2h)\right]$$

$$= \lim_{h\to 0} -\frac{1}{2h^2}\int_{-\infty}^{\infty}\left(e^{2ihx} + e^{-2ihx} - 2\right)\frac{1+x^2}{x^2}\, dG(x)$$

$$= \lim_{h\to 0} -\frac{1}{2h^2}\int_{-\infty}^{\infty}\left(\frac{e^{ihx} - e^{-ihx}}{2i}\right)^2 (2i)^2\,\frac{1+x^2}{x^2}\, dG(x).$$

$$= \lim_{h\to 0}\int_{-\infty}^{\infty}\left(\frac{\sin hx}{hx}\right)^2 (1+x^2)\, dG(x) < \infty .$$

Using Fatou's lemma (Loeve, page 125) the result follows.

We give the relationship between the two representations below :

$$H(x) = \int_{-\infty}^{x} (1+y^2)\, dG(y) , \quad x \in \mathbb{R}$$

$$\alpha = a + \int_{-\infty}^{\infty} y\, dG(y) .$$

Conversely

$$G(x) = \int_{-\infty}^{x} \frac{1}{1+y^2} \, dH(y) \qquad x \in \mathbb{R}$$

$$a = \alpha - \int_{-\infty}^{\infty} \frac{y}{1+y^2} \, dH(y) \ .$$

We illustrate below for some processes both the representations of characteristic function.

Example 1 : Let $\{X(t), t \geq 0\}$ be a Brownian motion process (Hoel et. al., page 123) with characteristic function

$$\phi_t(u) = \exp\{t(iu\mu - \tfrac{1}{2} u^2\sigma^2)\}.$$

In the Levy-Khintchine representation $a = \mu$ and

$$G(x) = 0 \qquad \text{if } x < 0$$

$$= \sigma^2 \qquad \text{if } x \geq 0 \ .$$

Similarly in the Kolmogorov's representation

$$\alpha = \mu \quad \text{and}$$

$$H(x) = 0 \quad \text{if } x < 0$$

$$= \sigma^2 \quad \text{if } x \geq 0 \ .$$

Example 2 : Let $\{ X(t), t \geq 0 \}$ be a Poisson process (Hoel et.al., page 96) with characteristic function of $X(t)$ as

$$\phi_t(u) = \exp \{ \lambda t (e^{iu} - 1) \} \ .$$

Therefore

$$\log \phi_t(u) = t\,\lambda(e^{iu}-1)$$

$$= iut\,\frac{\lambda}{2} + \int_{-\infty}^{\infty} \left(e^{iux}-1-\frac{iu\,x}{1+x^2}\right)\frac{1+x^2}{x^2}\,dG(x).$$

Therefore in the Levy-Khintchine representation

$$a = \frac{\lambda}{2} \text{ and } G(x) = 0 \quad \text{if} \quad x < 1$$

$$= \frac{\lambda}{2} \quad \text{if} \quad x \geq 1.$$

Similarly in the Kolmogorov's representation

$$\alpha = \lambda \text{ and } H(x) = 0 \quad \text{if} \quad x < 1$$

$$= \lambda \quad \text{if} \quad x \geq 1.$$

Example 3 : Let $X_1, X_2, \ldots,$ be independent and identically distributed random variables with distribution function $F(x)$ and $N(t)$ be a Poisson process possessing mean λt. Further we assume $N(t)$ and $X_1, X_2, \ldots,$ are independent then compound Poisson process $X(t)$ (Doob, page 419) is given by

$$X(t) = \sum_{j=1}^{N(t)} X_j .$$

Characteristic function of $X(t)$ is given by

$$\phi_t(u) = \exp\{\lambda t\,(h(u)-1)\}$$

where $h(u) = \int_{-\infty}^{\infty} e^{iux}\,dF(x)$.

Hence

$$\log \phi_t(u) = \lambda t\, h(u) - \lambda t$$

$$= iut\,\lambda \int_{-\infty}^{\infty} \frac{x}{1+x^2}\, dF(x) + t\int_{-\infty}^{\infty} \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right) dF(x).$$

Therefore in Levy Khintchine representation

$$\alpha = \lambda \int_{-\infty}^{\infty} \frac{x}{1+x^2}\, dF(x)$$

and $$G(x) = \int_{-\infty}^{x} \frac{\lambda y^2}{(1+y^2)}\, dF(y).$$

Similarly in Kolmogorov's representation

$$\alpha = \lambda \int_{-\infty}^{\infty} x\, dF(x)$$

and $H(x) = \int_{-\infty}^{\infty} \lambda\, y^2\, dF(y)$.

In particular if $F(x) = 0$ for $x < 1$

$= 1$ for $x \geq 1$

we get Poisson process.

Example 4 : A process $\{X(t),\ t \geq 0\}$ with stationary independent increments having $P(X(0)=0)=1$ is said to be gamma process if probability density function of $x(t)$ is

$$f(x,t,\mu) = \frac{\mu^t \exp\{-\mu x\} x^{t-1}}{\Gamma(t)}, \text{if } x > 0, \ \mu > 0$$

$$= 0 \quad \text{otherwise}$$

(Ghosh, page 203).

The characteristic function of $x(t)$ is

$$\phi_t(u) = [1-i\frac{u}{\mu}]^{-t} .$$

Now we write

$$\log \phi_t(u) = -t \log \left(\frac{\mu - iu}{\mu} \right)$$

$$= iut \int_0^\infty \frac{e^{-\mu x}}{1+x^2} dx + t \int_0^\infty \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right) \frac{e^{-\mu x}}{x} dx.$$

So making the proper identification of a and G(,) we get

$$a = \int_0^\infty \frac{e^{-\mu x}}{1+x^2} dx$$

and $G(x) = 0$ for $x < 0$

$$= \int_0^x \frac{y\, e^{-\mu y}}{1+y^2} dy \qquad \text{for } x \geq 0 .$$

Similarly in Kolmogorove's representation

$\alpha = \frac{t}{\mu}$ and $G(x) = 0$ for $x < 0$

$$= \int_0^x y e^{-\theta y} dy \quad \text{for } x \geq 0 .$$

We discuss below a theorem which is a partial converse of the theorem 7.

<u>Theorem 9</u> : Let $\phi_1(u)$ be an infinitely divisible characteristic function then

(1) $\phi_t(u) = [\phi_1(u)]^t$ is a characteristic function for every $t \geq 0$, hence $\phi_t(u)$ is itself an infinitely divisible characteristic function.

(2) There exists a stochastic process $\{X(t), t \geq 0\}$ defined on some probability space satisfying the following conditions,

(i) $x(t)$ has $\phi_t(u)$ as its characteristic function;
and (ii) $\{X(t), t \geq 0\}$ is a SIIP.

Proof : (1) Since $\phi_1(u)$ is an infinitely divisible characteristic function for a positive integer n

$$\phi_1(u) = [\phi(u)]^n,$$

for some characteristic function $\phi(u)$. Using the property that, $\phi_1(u)$ is non vanishing, we can write

$$\phi_1(u) = [\exp\{ \frac{1}{n} \log \phi_1(u)\}]^n.$$

So , $\phi(u) = \exp \{ \frac{1}{n} \log \phi_1(u)\}$.

If m is any positive integer then

$$[\ \phi(u)\ ]^m = \exp \{ \frac{m}{n} \log \phi_1(u) \} \text{ is a}$$

characteristic function. As $\frac{m}{n}$ tends to t,

$\exp \{ \frac{m}{n} \log \phi_1(u)\}$ tends to $[\ \phi_1(u)]^t$ for every u. Clearly $[\phi_1(u)]^t$ is continuous at $u = 0$. Therefore $[\phi_1(u)]^t$ is a characteristic function follows from Levy's continuity theorem (Loeve, page 191). Since for every $n \geq 1$, $[\phi_1(u)]^{t/n}$ is characteristic function. We deduce that $[\phi_1(u)]^t$ is infinitely divisible.

(2) Let for fixed k, $\{t_0,t_1,\ldots,t_k\} \subset [0,\infty)$ such that $0 = t_0 < t_1 < t_2,\ldots, < t_k$. Suppose $Y_1,Y_2,\ldots,Y_k$ are independent random variables, defined for $\{t_1,t_2,\ldots,t_k\}$ with $[\phi_1(u)]^{t_i-t_{i-1}}$ as the characteristic function of Y_i. We denote the joint distribution function of $(Y_1, Y_1+Y_2,\ldots, Y_1+Y_2+\ldots+Y_k)$ by $F_{t_1,t_2,\ldots,t_k}$. In order to establish the existence of a stochastic process in view of Kolmogerov's existence theorem, (Yeh, page 14) we need to show that the family

$$\{F_{t_1,t_2,\ldots,t_k} | \{t_1,t_2,\ldots,t_k\} \subset [0,\infty), k \geq 1\}$$

satisfies the two consistency conditions, symmetry and compatability.

For any permutation $t_{\alpha_1}, t_{\alpha_2},\ldots,t_{\alpha_k}$ of $t_1,t_2,\ldots,t_k$ the ordered arrangement $0 < t_1 < t_2,\ldots, < t_k$ remains the same, hence symmetry follows.

To show compatability let us consider for fixed choice of $\{t_0,t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_k\} \subset [0,\infty)$ such that $0 = t_0 < t_1 < ,\ldots,< t_{i-1} < t_{i+1} <\ldots< t_k$. We can define independent random variables $Y_1, Y_2, \ldots, Y_{i-1}, Y_i+Y_{i+1}, Y_{i+2}, \ldots, Y_k$; such that the characteristic function of Y_m is

$$[\phi_1(u)]^{t_m - t_{m-1}} \text{ for } m=1,2,\ldots,i-1,i+2,\ldots,k;$$

and the characteristic function of Y_i+Y_{i+1} is given by

$$[\phi_1(u)]^{t_{i+1}-t_{i-1}} .$$

Define

$$S_j = Y_1 + Y_2 + \ldots + Y_j, \text{ then}$$

$$\lim_{x_i \to \infty} F_{t_1, t_2, \ldots, t_k}(x_1, x_2, \ldots, x_k)$$

$$= \lim_{x_i \to \infty} P\{S_1 \le x_1, S_2 \le x_2, \ldots, S_k \le x_k\}$$

$$= P\{S_1 \le x_1, S_2 \le x_2, \ldots, S_{i-1} \le x_{i-1}, S_{i+1} \le x_{i+1}, \ldots S_k \le x_k\}$$

Since $S_1, S_2, \ldots, S_{i-1}, S_{i+1}, \ldots, S_k$ can be defined by using independent random variables

$$Y_1, Y_2, \ldots, Y_{i-1}, Y_i+Y_{i+1}, Y_{i+2}, \ldots, Y_k$$

defined as above for $\{t_0, t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_k\} \subset (0,\infty)$.

Hence

$$\lim_{x_i \to \infty} F_{t_1, t_2, \ldots, t_k}(x_1, x_2, \ldots, x_k)$$

$$= F_{t_1, t_2, \ldots, t_{i-1}, t_{i+1}, \ldots, t_k}(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots x_k).$$

and compatability holds.

Therefore there exists a stochastic process $\{X_{(t)}, t \geq 0\}$ with its characteristic function

$$\phi_t(u) = [\phi_1(u)]^t.$$

To show that the increments are independent we use the fact that for $0 < t_1 < t_2$

$$\mathcal{L}(X(t_1), X(t_2)) = \mathcal{L}(Y_1, Y_1+Y_2).$$

For l_1, l_2 real we obtain

$$\mathcal{L}(l_1 X(t_1) + l_2(X(t_2)-X(t_1))$$

$$= \mathcal{L}((l_1-l_2) X(t_1) + l_2 X(t_2))$$

$$= \mathcal{L}((l_1-l_2) Y_1 + l_2 Y_2).$$

Therefore $\mathcal{L}(X(t_1), X(t_2)-X(t_1)) = \mathcal{L}(Y_1, Y_2)$.

Since Y_1 and Y_2 are independent $X(t_1)$ and $X(t_2)-X(t_1)$ are independent.

Hence, in general $X(t_1), X(t_2)-X(t_2), \ldots, X(t_k)-X(t_{k-1})$ are independent and the process $(X(t), t \geq 0\}$ is a process with independent increments. The characteristic function of an increment $X(t)-X(s)$ for $0 < s < t$ and

$$\{s,t\} \subset [0,\infty) \text{ is given by } [\phi_1(u)]^{t-s}.$$

Therefore the increments are stationary. Hence the proof.

In the above theorem if the assumption that $\phi_1(u)$ is infinitely divisibility is dropped, $[\phi_1(u)]^t$ need not be characteristic function for non-integer t, is illustrated in the following example.

Example : Let X be a random variable taking values -1, 0 ,1 with probability $\frac{2}{9}$, $\frac{5}{9}$ and $\frac{2}{9}$ respectively. The characteristic function of X is

$$\phi(u) = \frac{1}{9}[5+4 \cos (u)].$$

Since the distribution has finite support $\phi(u)$ is not infinitely divisible. If possible let $[\phi(u)]^{\frac{1}{3}}$ is a characteristic function, then there exists a characteristic function $\Psi(u)$ such that

$$\phi(u) = [\Psi(u)]^3 .$$

Since, $\phi(u)$ is a characteristic function of three point distribution, $\Psi(u)$ cannot be even the characteristic function of a two point distribution. $\Psi(u)$ can not be the characteristic function of a degenerate random variable Hence $[\phi(u)]^{\frac{1}{3}}$ can not be a characteristic function.

<u>Example (1)</u> Let $\phi(u) = \exp\{-\frac{1}{2} u^2 \sigma^2\}$ be the characteristic function of $N(0, \sigma^2)$, then $[\phi(u)]^t = \exp\{-\frac{1}{2} u^2 \sigma^2 t\}$ is a characteristic function of the increment $X(t+s)-X(s)$ having distribution $N(0, \sigma^2 t)$. We know that $X(t_1), X(t_2)-X(t_1), \ldots,$ are independent normal r.v.s, therefore $(X(t_1), \ldots, X(t_n))$ is a multivariate normal, so $\{X(t), t \geq 0\}$ is a Gaussian process (Doob, page 71). The covariance function will be

$$\mathrm{Cov}(X(t), X(s)) = \sigma_s^2 , \; s \leq t .$$

So $\{X(t), t \geq 0\}$ is a Brownian motion process.

<u>Example (2)</u> Let $\phi(u) = [1-i \frac{\mu}{u}]^{-1}$ so we can obtain $[\phi(u)]^t = [1-i \frac{\mu}{u}]^{-t}$ a characteristic function of $X(t+s)-X(s)$. Distribution of $X(t+s)-X(s)$ is gamma with $P(X(0)=0)=1$ and having probability density function

$$f(x, t, \mu) = \frac{\mu^t e^{-\mu x} x^{t-1}}{\Gamma(t)} \quad \text{if } x > 0, \; t > 0, \; \Gamma > 0 .$$

$$= 0 \qquad \text{otherwise.}$$

Thus $\{X(t), t \geq 0\}$ is a SIIP.

<u>Example (3)</u> Let $\phi(u) = \exp\{\alpha |u|^\beta\}$ then,

$[\phi(u)]^t = \exp\{\alpha t |u|^\beta\}$ will be a characteristic function of stable process (Breiman, page 318). Distribution of the increment $X(t+s)-X(s)$ is stable with parameters α and β. So $\{X(t), t \geq 0\}$ is a stable process with stationary and independent increments.

It is of much importance to know the finite dimensional distributions of a process. The lemma proved below gives the finite dimensional distributions of a SIIP.

Lemma 10 . Finite dimensional distributions of SIIP are determined by the distribution of $X(1)$; if $P(X(0)=0)=1$ and $\phi_t(u)$ is continuous at $t = 0$ for every u real.

Proof : To prove the lemma it suffices to obtain

$$\phi_{t_1 t_2, \ldots, t_n}(u_1, u_2, \ldots, u_n) = E[\exp\{\sum_{i=1}^{n} iu_j \; X(t_j)\}]$$

for $u_1, u_2, \ldots, u_n \in \mathbb{R}$ and $t_1, t_2, \ldots, t_n \in \mathcal{T}$.

$$= E\,[\exp\{i(u_1+u_2+\ldots+u_n)X(t_1)$$

$$+ i(u_2+\ldots+u_n)[X(t_2)-X(t_1)]+\ldots+iu_n[X(t_n)-X(t_{n-1})]\}]$$

$$= \phi_{t_1}(u_1+u_2+\ldots+u_n)\;\phi_{t_2-t_1}(u_2+\ldots+u_n)\ldots\phi_{t_n-t_{n-1}}(u_n).$$

Using theorem 6 we get

$$= [\phi_1(u_1+u_2+\ldots+u_n)]^{t_1}[\phi_1(u_2+\ldots+u_n)]^{t_2-t_1}\ldots[\phi_1(u_n)]^{t_n-t_{n-1}}.$$

Hence the proof.

<u>Lemma 11</u> If $\{X(t), t \in \mathcal{T}\}$ is a process with independent increments and $E|X(t)|^2 < \infty$ then

$$\mathrm{Cov}(X(t), X(s)) = \mathrm{Var}(X(t \wedge s)).$$

$t \wedge s$ stands for min (t,s).

<u>Proof</u> Let $s < t \in \mathcal{T}$, then

$$\begin{aligned}\mathrm{Cov}(X(t),X(s)) &= \mathrm{Cov}(X(t)-X(s)+X(s),X(s))\\ &= \mathrm{Cov}(X(t)-X(s),X(s))+\mathrm{Var}(X(s)),\end{aligned}$$

covariance term vanishes due to independence of increments.
Hence

$$\mathrm{Cov}(X(t),X(s)) = \mathrm{Var}(X(s)) \text{ for } s < t,$$

similarly

$$\mathrm{Cov}(X(t),X(s)) = \mathrm{Var}(X(t)) \text{ for } t < s.$$

Hence the proof.

<u>Lemma 12</u> Let $\{X(t), t \in \mathcal{T}\}$ be a SIIP with finite mean and variance, then

$$E(X(t)) = m_o + m_1 t$$

where $m_o = E(X(0))$ and $m_1 = E(X(1)) - m_o$.

Similarly $\mathrm{Var}(X(t)) = \sigma^2_{X(t)}$

$$= \sigma_o^2 + \sigma_1^2 t$$

where $\sigma_o^2 = E(X(0)-m_o)^2$ and

$\sigma_1^2 = E(X(1)-m_1)^2 - \sigma_o^2$.

Proof : Define $f(t) = E(X(t)-X(o))$ then for every $\{t,s\} \subset \mathcal{J}$,

$$f(t+s) = E(X(t+s)-X(o))$$
$$= E(X(t+s)-X(s))+E(X(s)-X(o)).$$

Since $\mathcal{L}(X(t+s)-X(s)) = \mathcal{L}(X(t))$ due to stationary increments; We get

$$f(t+s) = f(t)+f(s). \qquad (1.3.9)$$

A bounded solution to (1.3.9) is given by

$$f(t) = t\, f(1).$$

Thus we deduce

$$E(X(t)-X(o)) = t\, E(X(1)-X(o))$$

which implies that

$$E(X(t)) = m_o + m_1 t\ .$$

similarly let

$$g(t) = Var(X(t)-X(o)).$$
$$= Var(X(t))-2\, Cov(X(t),X(o))+Var(X(o)).$$

Using lemma 11 we get

$$= \sigma^2_{X(t)} - 2\,\sigma^2_o + \sigma^2_o$$
$$= \sigma^2_{X(t)} - \sigma^2_o\ .$$

We can write

$$g(s+t) = Var\ (X(t+s)-X(o))$$
$$= Var\ (X(t+s)-X(s)+X(s)-X(o)).$$

In view of independence of the increments $X(t+s)-X(s)$ and $X(s)-X(o)$ we get

$$g(s+t) = Var(X(t+s)-X(s)) + Var(X(s)-X(o)).$$

Since $\mathcal{L}(X(t+s)-X(s)) = \mathcal{L}(X(t))$,

$$g(s+t) = g(t) + g(s). \tag{1.3.10}$$

Hence a bounded solution to (1.3.10) will be

$$g(t) = t\, g(1)$$

$$\sigma^2_{X(t)} = \sigma_o^2 + t\,(Var(X(1)-X(o))$$

$$= \sigma_o^2 + t\,(Var(X(1)) - \sigma_o^2)$$

$$= \sigma_o^2 + t\,\sigma_1^2\,.$$

Hence the proof.

If $P(X(o)=o)=1$, then $E(X(t)) = m_1 t$ which is a monotonic function of t and $Var(X(t)) = \sigma_1^2\, t$, which is a nondecreasing function of t. Some properties of process with independent increments, we summarize in the lemma given below.

<u>Lemma 13</u> If $\{X(t),\ t \in \mathcal{T}\}$ is a process with independent increments, then so is

(i) $-X(t)$ $\quad t \in \mathcal{T}$;

(ii) $X(T)-X(T-t)$ $\quad 0 \leq t \leq T$, T fixed;

(iii) $X(t) - a(t)$ $\quad t \in \mathcal{T}$, a(.) is any function of $t \in \mathcal{T}$.

(iv) $X(t+c) - X(c)$ $\qquad t \in \mathcal{T}$, c constant.

Proof (i) Define

$$Y(t) = -X(t).$$

Consider A_1, A_2 subsets of state space of $X(t)$ and $t_1 < t_2$ such that $\{t_1, t_2\} \subset \mathcal{T}$, then

$$P\{Y(t_2) - Y(t_1) \in A_2, Y(t_1) \in A_1\}$$

$$= P\{-[X(t_2)-X(t_1)] \in A_2, -X(t_1) \in A_1\}$$

$$= P\{-[X(t_2)-X(t_1)] \in A_2, -X(t_1) \in A_1\}$$

$$= P\{Y(t_2)-Y(t_1) \in A_2\} P\{Y(t_1) \in A_1\}.$$

Hence $\{Y(t), t \in \mathcal{T}\}$ is a process with independent increments.

(ii) Define

$$Y_T(t) = X(T)-X(T-t).$$

Let A_1, A_2 be as above and $t_1 > t_2$ such that $\{t_1, t_2\} \subset [0,T]$, then

$$P\{Y_T(t_1) \in A_1, Y_T(t_2) - Y_1(t_1) \in A_2\}$$

$$= P\{X(T)-X(T-t_1) \in A_1, X(T)-X(T-t_2)-X(T) + X(T_o-t_1) \in A_2\}$$

$$= P\{X(T)-X(T-t_1) \in A_1, X(T-t_2)-X(T-t_1) \in A_2\}$$

Since $(T-t_1, T)$ and $(T-t_1, T-t_2)$ are nonoverlapping we get

$$= P\{X(T)-X(T-t_1) \varepsilon A_1\} \; P\{X(T-t_2)-X(T-t_1) \varepsilon A_2\}$$

$$= P\{Y_T(t_1) \varepsilon A_1\} \; P\{Y_T(t_2)-Y_1(t_1) \varepsilon A_2\} .$$

Therefore $\{Y_T(t), 0 \leq t \leq T\}$ is a process with independent increments.

(iii) Define

$$Y_a(t) = X(t)-a(t).$$

Then for $t_1 < t_2$ such that $\{t_1,t_2\} \varepsilon \mathcal{J}$ and A_1, A_2 chosen as above, we evaluate

$$P\{Y_a(t_1) \leq x_1, \; Y_a(t_2) \leq x_2\}$$

$$= P\{X(t_1)-a(t_1) \leq x_1, \; X(t_2)-X(t_1)-a(t_2)+ a(t_1) \leq x_2\}$$

$$= P\{X(t_1)-a(t_1) \leq X_1\} \; P\{X(t_2)-X(t_1)-a(t_2)+ a(t_1) \leq x_2\}$$

$$= P\{Y_a(t_1) \leq x_1\} \; P\{Y_a(t_2)-Y_a(t_1) \leq x_2\}.$$

Therefore

$\{Y_a(t_1), t \varepsilon \mathcal{J}\}$ is a process with independent increments.

(iv) $Y^c(t) = X(t+c)-X(c)$ can be shown to have independent increments for any c constant; using similar argument used in (ii). Hence the proof.

It is proved in (iv) for c constant $Y^c(t)$ is a process with independent increments, however with some assumptions about continuity of sample paths (iv) holds if c is , , replaced by stopping times. This is established in theorem 21.

1.4. Relationship with martingales and Markov process.

Definition 14 : A process $\{X(t), 0 \leq t < \infty\}$ is called a martingale if $E(|X(t)|) < \infty$ for all $t \geq 0$ and if for all $0 \leq s \leq t$,

$$E(X(t)|X(\tau), \tau \leq s) = X(s) \text{ almost surely (Breiman, page 300).}$$

Theorem 15 : If $\{X(t), \quad 0 \leq t < \infty\}$ is a process with independent increments with finite expectation for all $t \geq 0$ then, $\{X(t) - E(X(t)), 0 \leq t < \infty\}$ is a martingale.

Proof. Without loss of generality let us assume $E(X(t)) = 0$. If $E(X(t))$ is non-zero, we subtract $E(X(t))$ from $X(t)$. $\{X(t)-E(X(t)), 0 \leq t < \infty\}$ is a process with independent increments follows from lemma 13 (iii). For $0 \leq s \leq t$

$$\begin{aligned} &E(X(t)|X(\tau), \tau \leq s) \\ &= E(X(t)-X(s) + X(s)|X(\tau), \tau \leq s) \\ &= E(X(t)-X(s)|X(\tau), \tau \leq s) + E(X(s)|X(\tau), \tau \leq s). \end{aligned}$$

Since $X(t)-X(s)$ is independent of $X(\tau)$, $\tau \leq s$ we deduce

$$\begin{aligned} E(X(t)|X(\tau), \tau \leq s) &= E(X(t)-X(s)) + X(s) \\ &= X(s) \quad \text{with probability one.} \end{aligned}$$

Next theorem relates a process having independent increments with Markov process.

Definition 16 : A process $\{X(t), t \geq 0\}$ is called Markov with state space $S \in \mathbb{B}$ if $X(t) \in S$, $t \geq 0$, and for any $B \subset S$, t, $\tau \geq 0$.

$$P(X(t+\tau) \in B \mid X(s), s \leq t) = P(X(t+\tau) \in B \mid X(t))$$

with probability one (Breiman, page 319).

Theorem 17. A process $\{X(t), t \geq 0\}$ having independent increments is a Markov process.

Proof. For any $t \geq 0$, $X(t)$ can be expressed as a sum of independent random variables as follows. We consider $0 = t_0 \leq t_1, \ldots \leq t_n = t$ and define

$$Y_k = X(t_k) - X(t_{k-1}), \; k=1,2,\ldots,n.$$

Then $X(t) = \sum_{k=1}^{n} Y_k$. The random variables $Y_1, Y_2, \ldots, Y_n$ are independent follows from the conditions of the theorem. In order to verify that the process is Markov it is enough to show for any B subset of state space

$$P\{X(t_n) \in B \mid X(t_1), X(t_2), \ldots, X(t_{n-1})\}$$

$$= P\{X(t_n) \in B \mid X(t_{n-1})\}$$

with probability one (Breiman, page 319). For any Borel sets C,D let us obtain

$$P\{X(t_{n-1}) \varepsilon C, Y_n \varepsilon D | Y_1, Y_2, \ldots, Y_{n-1}\}$$

$$= E\{I_C(X(t_{n-1}))\ I_D(Y_n) | Y_1, Y_2, \ldots, Y_{n-1}\}$$

$$= I_C(X(t_{n-1}))\ E(I_D(Y_n)). \qquad (1.4.1)$$

Similarly we evaluate

$$P\{X(t_{n-1}) \varepsilon C, Y_n \varepsilon D | X(t_{n-1})\}$$

$$= E\{I_C(X(t_{n-1}))\ I_D(Y_n) | X(t_{n-1})\}$$

$$= I_C(X(t_{n-1}))\ E(I_D(Y_n)) . \qquad (1.4.2)$$

Using (1.4.1) and (1.4.2) we deduce that

$$P\{X(t_{n-1}) \varepsilon C, Y_n \varepsilon D | Y_1, Y_2, \ldots, Y_{n-1}\}$$

$$= P\{X(t_{n-1}) \varepsilon C, Y_n \varepsilon D | X(t_{n-1})\}.$$

Particularly for $A \varepsilon \mathbb{B} X \mathbb{B}$

$$P\{(X(t_{n-1}), Y_n) \varepsilon A | Y_1, Y_2, \ldots, Y_{n-1}\}$$

$$= P\{(X(t_{n-1}), Y_n) \varepsilon A | X(t_{n-1})\}.$$

Therefore for $B \varepsilon \mathbb{B}$

$$P\{X(t_{n-1}) + Y_n \varepsilon B | Y_1, Y_2, \ldots, Y_{n-1}\}$$

$$= P\{X(t_{n-1}) + Y_n \varepsilon B | X(t_{n-1})\} .$$

Hence

$$P\{X(t_n) \varepsilon B | Y_1, Y_2, \ldots, Y_{n-1}\} = P\{X(t_n) \varepsilon B | X(t_{n-1})\}. \quad (1.4.3)$$

Since the σ-field generated by $\{Y_1, Y_2, \ldots, Y_{n-1}\}$ is same as that of $\{X(t_1), X(t_2), \ldots, X(t_{n-1})\}$, (1.4.3) becomes

$$P\{X(t_n) \varepsilon B | X(t_1), X(t_2), \ldots, X(t_{n-1})\} = P\{X(t_n) \varepsilon B \mid X(t_{n-1})\}.$$

Hence the theorem.

An example illustrating that the converse of the theorem 17 does not hold is given below.

Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed uniform random variables over (0,1).

Define

$$F_n(t) = \sum_{j=1}^{n} I_{[0,t]}(X_j)$$

where $I_{[0,t]}(X_j) = 1$ if $X_j \varepsilon [0,t]$
$= 0$ otherwise.

Then $\{F_n(t), 0 \leq t \leq 1\}$ is a continuous time stochastic process. Moreover it satisfies Markov property. It is verified as follows.

Choose $0 < t_1 < t_2, \ldots < t_k < 1$, and denote

$F_n(t_1) = r_1$, $F_n(t_j) - F_n(t_{j-1}) = r_j$, $j=2,\ldots,k$.

Now we obtain

$$P\{F_n(t_k)=m \mid F_n(t_j)\ j=1,2,\ldots k-1\}$$

$$= \frac{\dfrac{n!}{r_1!\,r_2!\ldots r_k!\,(n-m)!}\, t_1^{r_1}(t_2-t_1)^{r_2}\ldots(t_k-t_{k-1})^{r_k}(1-t_k)^{n-m}}{\dfrac{n!}{r_1!\,r_2!\ldots r_{k-1}!\,(n-m+r_k)!}\, t_1^{r_1}(t_2-t_1)^{r_2}\ldots(t_{k-1}-t_{k-2})^{r_{k-1}}(1-t_{k-1})^{n-m+r_k}}$$

$$= \frac{(n-m+r_k)!}{r_k!\ (n-m)!} \cdot \frac{(t_k-t_{k-1})^{r_k}\ (1-t_k)^{n-m}}{(n-t_{k-1})^{n-m+r_k}}, \quad r_1, r_2, \ldots, r_k \geq 0, \sum_{i=1}^{k} r_i = m \leq n$$

$$= P\{F_n(t_k) = m \mid F_n(t_{k-1})\}.$$

Thus we deduce $\{F_n(t),\ 0 \leq t \leq 1\}$ is a Markov process. The process $\{F_n(t),\ 0 \leq t \leq 1\}$ does not possess an independent increments is shown in section 1.2, example 2.

A SMP with additional assumptions regarding sample paths obeys strong Markov property. In order to prove this, the necessary results are included below.

Definition 18 : For any process $\{X(t), t \in \mathcal{T}\}$, a random variable $t^* \geq 0$ will be called a stopping time if for every $t \geq 0$,

$$\{t^* \leq t\} \in \mathbb{F}(X(\tau),\ \tau \leq t),$$

$\mathbb{F}(X(\tau), \tau \leq t)$ is the σ field generated by random variables $(X(\tau), \tau \leq t)$ (Breiman, page 268).

Lemma 19 : If t^* is a stopping time corresponding to the process $\{X(t), t \in J\}$ then

$$t_n^* = \frac{k}{n} \quad \text{if} \quad \frac{k-1}{n} < t^* \leq \frac{k}{n}, k > 1; \qquad (1.4.4)$$

is a stopping time and for $B \in \mathbb{F}\{X(s) | s \leq t\}$,

$$B \cap \{t_n^* \leq t\} \in \mathbb{F}(X(s), s \leq t).$$

Proof. For $\frac{k}{n} < t \leq \frac{k+1}{n}$, from definition of t_n^*,

$$\{t_n^* \leq t\} = \{t^* \leq \frac{k}{n}\} \in \mathbb{F}(X(s), s \leq \frac{k}{n}). \qquad (1.4.5)$$

Since $\mathbb{F}(X(s), s \leq \frac{k}{n}) \subset \mathbb{F}(X(s), s \leq t)$.

$$\{t_n^* \leq t\} \in \mathbb{F}(X(s), s < t).$$

Thus we can say t_n^* is a stopping time. Now, from the fact that $t^* \leq t_n^*$ we get $B \in \mathbb{F}(X(t), t \leq t^*)$, implies that $B \in \mathbb{F}(X(t), t \leq t_n^*)$. We have $B \cap \{t_n^* \leq t\} \subset \mathbb{F}(X(s), s \leq t)$ therefore,

$$B \cap \{t_n^* \leq t\} = B \cap \{t^* \leq \frac{k}{n}\} \in \mathbb{F}(X(s), s \leq t).$$ Hence the proof.

Definition 20 : $D([a,b])$ is the class of all functions $X(t), a \leq t \leq b$, which have only jump discontinuities and which are right continuous (Breiman, page 299).

Next theorem proves that $\{X(t), t \geq 0\}$ posseses strong Markov property, under certain conditions.

Theorem 21. For $\{X(t), t \geq 0\}$ a SIIP with sample paths in $D([0,\infty))$, if t^* is any stopping time, then $\{X(t+t^*)-X(t^*), t \geq 0\}$ has the same distribution as that of $\{X(t), t \geq 0\}$ and is independent of $\mathbb{F}(X(t), t \leq t^*)$.

Proof. We discuss the proof in two setps as follows :

Step 1 : Here we consider t^* to be discrete, taking countable number of values $\{\tau_k\}$. Considering $A_1, A_2, \ldots, A_j \in \mathbb{B}_1$; $t_1, t_2, \ldots t_j \geq 0$ and $B \in \mathbb{F}(X(t), t \leq t^*)$ we obtain

$$P\{Y(t_1) \in A_1, Y(t_2) \in A_2, \ldots, Y(t_j) \in A_j, B)\},$$

$$= \sum_{k=1}^{\infty} P\{Y(t_1) \in A_1, \ldots, Y(t_j) \in A_j, t^* = \tau_k, B\} , \quad (1.4.6)$$

where

$$Y(t_j) = X(t_j + t^*) - X(t^*) .$$

Furthermore, we note that

$$\{t^* = \tau_k\} \cap \{t^* \leq \tau_k\} \cap B \subset \{ t^* = \tau_k\} \cap B.$$

Since $w \in \{t^* = \tau_k\} \cap B$ implies that $w \in \{t^* \leq \tau_k\} \cap \{t^* = \tau_k\} \cap B$,

we have

$$\{t^* = \tau_k\} \cap B \subset \{t^* \leq \tau_k\} \cap \{t^* = \tau_k\} \cap B.$$

Thus

$$\{t^* = \tau_k, B\} \in \mathbb{F}\{X(t),\ t \leq \tau_k\}.$$

Therefore (1.4.6) becomes

$$\sum_{k=1}^{\infty} P\{Y(t_1) \in A_1, \ldots, Y(t_j) \in A_j, t^* = \tau_k, B\}$$

$$= \sum_{k=1}^{\infty} P\{X(t_1+\tau_k)-X(\tau_k) \in A_1, \ldots, X(t_j+\tau_k)-X(\tau_k) \in A_j\}$$

$$\times P\{t^* = \tau_k, B\}.$$

$$= \quad P\{X(t_1) \in A_1, \ \ldots, X(t_j) \in A_j\} \ .P(B). \qquad (1.4.7)$$

Choosing $B = \Omega$ (1.4.7) becomes

$$P\{Y(t_1) \in A_1, \ldots, Y(t_j) \in A_j\} = P\{X(t_1) \in A_1, \ldots, X(t_j) \in A_j\}.$$

<u>Step 2</u> : If t^* is not discrete then we use t_n^* defined by (1.4.4). Define

$$Y_n(t) = X(t + t_n^*) - X(t_n^*).$$

Referring to lemma 19 and taking $B \in \mathbb{F}(X(t), t \leq t_n^*)$, by the similar arguments used in step 1 we get

$$P\{Y_n(t_1) < x_1, \ldots, Y_n(t_j) < x_j, B\}$$

$$= P\{X(t_1) < x_1, \ldots, X(t_j) < x_j\} \ P\{B\}.$$

The sample paths are in $D([0,\infty))$ yield, for every w and t

$X(t+t_n^*) - X(t_n^*) \to X(t+t^*) - X(t^*)$. This implies

$Y_n(t) \to Y(t)$ as $n \to \infty$ for every w and t,. Thus at every continuity point $(x_1, x_2, \ldots, x_j)$ of distribution function of $Y(t_1), Y(t_2), \ldots, Y(t_j)$, we conclude

$$P\{Y(t_1) < x_1, \ldots, Y(t_j) < x_j, B\}$$

$$= P\{X(t_1) < x_1, \ldots, X(t_j) < x_j\} P(B). \qquad (1.4.8)$$

The relation (1.4.8) holds for any $B \in \mathbb{F}(X(t), t \leq t^*)$, therefore, $\{X(t), t \geq 0\}$ is independent of $\mathbb{F}(X(t), t \leq t^*)$. Hence the proof.

The conditions for sample paths of SIIP to be in $D([0,\infty))$ are given in the next chapter.

CHAPTER 2

2.1 Introduction

In this chapter we discuss centering of SIIP and its sample patn properties. We also discuss the decomposition of SIIP into continuous and discrete components which are independent. The components are independent and the continuous component is a Gaussian process. Finally we dicuss characterization of Wiener process and Poisson process.

2.2 Sample patn properties

Below we include results which make possible to choose a non-random function $a(\cdot)$ such that the sample function of $\{X(t)-a(t),\ t \geq 0\}$ possess some continuity properties.

Any function $a(\cdot)$ of such type is called a centering function and $\{X(t)-a(t), t \geq 0\}$ is called a centered process. Moreover $\{X(t)-a(t),\ t \geq 0\}$ is a process with independent increments (lemma 13, chapter 1).

Lemma 1: If $\{X(t),\ t \geq 0\}$ is a process with independent increments, then a non-random function $a(t)$ can be chosen in such a manner that $X(t)-a(t)$ has no discontinuities of second kind (or jump discontinuities).

Proof : Let $X(t)$ and $\tilde{X}(t)$ be two identical copies taken on the same probability space such that $X(t)$ and $\tilde{X}(t)$ are independent of each other. Define

$$X^*(t) = X(t) - \tilde{X}(t) \qquad \text{for all } t \geq 0.$$

Let $\phi_t(u)$ and $\phi_{s,t}(u)$ be the characteristic functions of $X(t)$ and $X(t)-X(s)$ respectively. Hence $h_t(u) = |\phi_t(u)|^2$ and $h_{s,t}(u) = |\phi_{s,t}(u)|^2$ will be the characteristic functions of $X^*(t)$ and $X^*(t)-X^*(s)$ respectively. Clearly $0 < h_t(u) \leq 1$ and for $0 < s < t$

$$h_t(u) = h_s(u) \quad h_{s,t}(u) \tag{2.2.1}$$

implies that $h_t(u)$ is a monotonically non-increasing and bounded function of t. Therefore $h_{t-0}(u)$ and $h_{t+0}(u)$ exist for $t \geq 0$. To complete the proof we need to show that $\lim_{h \downarrow 0} X(t+h) = X(t+0)$ and $\lim_{h \downarrow 0} X(t-h) = X(t-0)$ in probability.

Using the relation

$$h_{t+0}(u) = \lim_{s \downarrow t} h_s(u),$$

we show the existence of $X(t+0)$. $h_{t+0}(u)$ being a limit of infinitely divisible characteristic function cont. at t=0, it is also infinitely divisible characteristic function (Chung, page 244). So it is nonvanishing and we can choose $\delta > 0$ such that $h_{t+0}(u) > 0$ for $|u| < \delta$, which implies that for all $|u| < \delta$ there is $s_0 > t$ such that $h_s(u) > 0$ for $t < s < s_0$. If $t < s_1 < s_2$ from (2.2.1) we get

$$h_{s_1 s_2}(u) = \frac{h_{s_1}(u)}{h_{s_2}(u)} .$$

Therefore

$$\lim_{s_2 \downarrow t} h_{s_1 s_2}(u) = \lim_{s_1 \downarrow t\ s_2 \downarrow t} \frac{h_{s_1}(u)}{h_{s_2}(u)} = 1.$$

Thus for every $\varepsilon > 0$,

$$\lim_{s_1 \downarrow t, s_2 \downarrow t} P\{|X^*(s_2) - X^*(s_1)| > \varepsilon\} = 0 .$$

We deduce

$$X^*(s) \to X^*(t+0) \text{ in probability as } s \downarrow t .$$

Similarly existence of $X^*(t-0)$ can be proved. Using lemma 3 of Gihman [2] et.al. (page 384) for a symmetric process

$$P\{ \sup_{t < s < t+\delta} |X^*(s) - X^*(t+0)| > \varepsilon \}$$

$$\leq 2P\{ | X^*(t+\delta) - X^*(t+0) | > \varepsilon \} . \qquad (2.2.2)$$

Allowing $\delta \to 0$ in (2.2.2) we conclude $X^*(t)$ has only jump discontinuities with probability one. Thus for almost all fixed ω_o , $X(t,\omega) - \tilde{X}(t, \omega_o)$ has no discontinuities of second kind. So $a(t) = \tilde{X}(t, \omega_o)$ can be considered as centering function.

We consider for further discussion the process to be in $D([0,\infty))$ that is the class of all functions $\{X(t),\ t \geq 0\}$ which have only jump discontinuties and which are right continuous.

Definition 2 : A process $\{X(t),\ t \geq 0\}$ with stationary and independent increments satisfying the following conditions (a) and (b) is called a Levy process.

(a) X(t) is continuous in probability that is, for every $\varepsilon > 0$

$$P\{|X(t)| > \varepsilon\} \to 0 \quad \text{as} \quad t \to 0 .$$

(b) There exist left and right limits $X(t^-)$ and $X(t^+)$, further X(t), is right continuous.

Next lemma gives the conditions for continuity of X(t) in probability.

Lemma 3 Let $\{X(t), t \geq 0\}$ be a SIIP such that $\phi_t(u)$ the characteristic function of X(t), is continuous at $t = 0$ for every u, then X(t) is continuous in probability.

Proof Let us assume that $P(X(0)=0)=1$, hence

$$X(t+s) = X(t+s)-X(s)+X(s)-X(0) .$$

Using stationary independent increments property, we get

$$\mathcal{L}(X(t+s)) = \mathcal{L}(X(t) + X(s))$$

and

$$\phi_{t+s}(u) = \phi_t(u) \ \ \phi_s(u).$$

Therefore

$$\lim_{s\downarrow 0} \ \phi_{t+s}(u) = \lim_{s\downarrow 0} \ \phi_t(u) \ \ \phi_s(u).$$

Since $\phi_0(u) = 1$ we can write

$$\phi_t(u) = \lim_{s\downarrow 0} \ \phi_{t+s}(u) \ . \tag{2.2.3}$$

Similarly

$$\lim_{s\downarrow 0} \phi_t(u) = \lim_{s\downarrow 0} \phi_{t-s}(u) \ \phi_s(u)$$

$$\phi_t(u) = \lim_{s\downarrow 0} \phi_{t-s}(u) \ . \tag{2.2.4}$$

From (2.2.3) and (2.2.4) it follows that $\phi_t(u)$ is continuous in t. Therefore

$$\lim_{s\downarrow 0} \phi_s(u) = 1$$

which implies that $X(s) \to 0$ in distribution as $s \to 0$ hence $X(s) \to 0$ in probability as $s \to 0$.

Since

$$\mathcal{L}(X(s)) = \mathcal{L}(X(t)-X(t-s)) = \mathcal{L}(X(t+s)-X(s)).$$

It can be seen that

$X(t+s) \to X(t)$ in probability as $s \to 0$ and $X(t-s) \to X(t)$ in probability as $s \to 0$.

Lemma 4 If $\{X(t), t \geq 0\}$ is a Levy process then

$$P\{|X(t)-X(t-0)| > 0\} = 0$$

for all $t \geq 0$.

Proof Since $X(t)$ is continuous in probability for $\varepsilon > 0$,

$$P\{|X(t) - X(t-0)| > \varepsilon\}$$

$$= \lim_{h \to 0} P\{|X(t)-X(t-h)| > \varepsilon\} = 0.$$

Therefore for every $n > 0$

$$P\{|X(t)-X(t-0)| > \frac{1}{n}\} = 0 .$$

But

$$P\{ \bigcup_{n=1}^{\infty} \{|X(t)-X(t-0)| > \frac{1}{n}\}$$

$$= P\{|X(t)-X(t-0)| > 0\}$$

$$= 0 .$$

Hence the proof.

Definition 5 A process $\{X(t), t \geq 0\}$ is said to have no fixed discontinuity at $t_o \geq 0$, if for $\varepsilon > 0$,

$$\lim_{t \to t_o} P(|X(t)-X(t_o)| > \varepsilon) = 0 .$$

Next, theorem proves that a process can be centered so that, the number of points of fixed discontinuity is at most countable. First we prove a required lemma.

Lemma 6 Let $X_1, X_2, \ldots$ be independent and identically distributed random variables.

If $d(X, X_n) = \inf_{\varepsilon} \{ \varepsilon \mid P\{|X - X_n| \geq \varepsilon\} \leq \varepsilon \}$

for random variable X, then X_n converges to X in probability as n tends to infinity, if and only if $d(X, X_n)$ converges to zero as n tends to infinity.

Proof If $d(X, X_n)$ converges to zero, then

$d(X, X_n) < \delta \quad \forall \quad n \geq n_0(\delta)$. which means

$a_n = \inf_{\varepsilon} \{ \varepsilon \mid P\{|X-X_n| \geq \varepsilon \} \leq \varepsilon \} < \delta$.

Therefore there exists ε_n such that $\varepsilon_n < a_n + \eta$, where $\eta > 0$. Suppose $a_n < \delta$, then η can be taken $\frac{\delta - a_n}{2}$,

hence

$$\begin{aligned} \varepsilon_n < a_n + \eta &= a_n + \frac{\delta - a_n}{2} \\ &= \frac{\delta - a_n}{2} \\ &< \delta \end{aligned}$$

Therefore for every $n \geq n_0$

$$P\{|X-X_n| \geq \delta\} \leq P\{|X-X_n| \geq \varepsilon_n\} \leq \varepsilon_n < \delta ,$$

which implies that, for every $n \geq n_0$

$$P\{|X-X_n| \geq \delta\} \leq \delta .$$

Thus $X_n \to X$ in probability as $n \to \infty$.

Conversely if $X_n \to X$ in probability we get

$$P\{|X-X_n| \geq \varepsilon\} \leq \delta \text{ for every } n > n_0(\delta) .$$

Therefore, for $\varepsilon \geq \delta$

$$P\{|X-X_n| \geq \varepsilon\} \leq \delta \leq \varepsilon \tag{2.2.5}$$

and for $\varepsilon < \delta$ we get

$$P\{|X-X_n| \geq \delta\} \leq P\{|X-X_n| \geq \varepsilon\} \leq \delta . \tag{2.2.6}$$

Thus from (2.2.5) and (2.2.6)

$$P\{|X-X_n| \geq \delta\} \leq \delta .$$

Hence $d(X, X_n) \to 0$, if $X_n \to X$ in probability as $n \to \infty$.

<u>Theorem 7</u> Let $\{X(t), t \geq 0\}$ be a process with independent increments. Then there is a function $a(\cdot)$ defined for $t \geq 0$, such that $Z(t)=X(t)-a(t)$ has at most countably many points of fixed discontinuity.

<u>Proof</u> Suppose $0 \leq s_1 \leq s_2 \ldots$ are such that, $s_n < t$ and $s_n \to t$. Since the increments are independent,

$$\sum_{j=1}^{\infty} (X(s_{j+1}) - X(s_j))$$

is a series of independent random variables. Therefore

$$X(t)-X(s_1) = \sum_{j=1}^{n}(X(s_{j+1})-X(s_j)) + (X(t)-X(s_{n+1})).$$

Using theorem 2.8 (Doob, page 119) we conclude

$$\lim_{n\to\infty} (X(s_n) - a(s_n))$$

exists and finite with probability one for some $\{a(s_n);n \geq 1\}$ as centering constants. Therefore if $Z(s_n)=X(s_n)- a(s_n)$ then

$$\lim_{n\to\infty} Z(s_n)$$

exists and is finite with probability one.

Hence

$Z(s_n) \to Z(t - 0)$ in probability as $s_n \uparrow t$ and

$Z(s_n) \to Z(t + 0)$ in probability as $s_n \downarrow t$.

Define, $d(X ,X_n) = \inf_{\varepsilon} \{ \varepsilon \mid P\{|X-X_n| \geq \varepsilon\} \leq \varepsilon\}$, then from lemma 6, for each $t \geq 0$, random variable $Z(t)$ is a point of a complete metric space, so that the random variables of the $Z(t)$ process define a function $f(t)$, for $t \geq 0$ with the values in this metric space. Since $Z(t+0)$ and $Z(t-0)$ exist, therefore $f(t+0)$ and $f(t-0)$ exist for $t \geq 0$.

Let

$$T_n = \{t \mid |f(t+0) - f(t-0)| \geq \frac{1}{n}\}.$$

If $t \in T(n)$ and $f(t-0)$ exists then an interval can be obtained such that t is the right end point of an interval containing t but no other point of $T(n)$. Thus a set of intervals each containing a single point of $T(n)$ and all points of this set are contained in the intervals and the set of non-overlapping intervals constituting $[0,\infty)$ can be obtained. Since the set of disjoint intervals is at most enumerable, $T(n)$ is atmost enumerable. Hence the proof.

We discuss below a theorem regarding the boundedness of sample functions of $X(t)$.

Theorem 8 Let $\{X(t), t \geq 0\}$ be a centeredseparable process with independent increments. Then almost all sample functions of the process are bounded for $c \leq t \leq d$.

Proof Let $m(t)$ be a median of $X(d)-X(t)$. Consider $c \leq s_n \leq d$ and $s_n \uparrow t$, then every limiting value of the sequence $\{m(s_n)\}$ is a median of $X(d)-X(t-0)$. For $c \leq t \leq d$ $m(t)$

Doob Thm. 2.2 pp 12[illegible]

Let $y_1, y_2, \ldots y_n$ be mutually independent r.v. & let $x_j = y_1 + \cdots + y_j$. Then, if $x_n - x_1, x_n - x_2, \ldots$ have [illegible] dist.

(2.4) $$2P\{x_n(\omega) \geq \lambda + 2 [illegible] - 2\sum_{j=1}^{n} [illegible] \leq P\{ [illegible] \}$$

$$\leq 2P\{ [illegible] \}$$

for any $\lambda > 0$, [illegible] the right hand half of (2.4) remains valid [illegible] [illegible] [illegible] [illegible] [illegible] [illegible] is bounded in t therefore $|m(t)| \leq k$. Let

$Z(t) = X(t) - X(c) + m(t) \quad c \leq t \leq d$, clearly $\{Z(t), t \geq 0\}$ is process with independent increments and median of $Z(d)-Z(t)$ is zero. If $Y_1, Y_2, \ldots$ are independent random variables and $X_j = Y_1+Y_2+\ldots+Y_j$ and if $X_n - X_k$ have zero median then using theorem 2.2 (Doob,106) we write,

$$P\{\max_{1\leq j\leq n} X_j(\omega) \geq \lambda\} \leq 2P\{X_n(\omega) \geq \lambda\}.$$

Hence, for $\lambda > 0$ and $c = t_0 < t_1, \ldots, < t_n < d$,

$$P\{\max_{1\leq j\leq n} Z(t_j, \omega) \geq \lambda\} \leq 2P\{Z(d, \omega) \geq \lambda\}.$$

This implies

$$P\{\max_{1\leq j\leq n} X(t_j, \omega) - X(c, \omega) \geq \lambda + k\}$$
$$\leq 2P\{X(d, \omega) - X(c, \omega) \geq \lambda\} \qquad (2.2.7)$$

Relation (2.2.7) is true for all finite subsets $\{t_j\}$ of $[c, d]$. Since $\{X(t), t > 0\}$ is separable, we write

$$P\{\sup_{c\leq t\leq d} X(t, \omega) - X(c, \omega) \geq \lambda + k\}$$
$$\leq 2P\{X(d, \omega) - X(c, \omega) \geq \lambda\}. \qquad (2.2.8)$$

Similarly for $-X(t)$ one can write

$$P\{\sup_{c\le t\le d} X(c,\omega)-X(t,\omega)\ge\lambda+k\}$$
$$\le 2P\{X(c,\omega)-X(d,\omega)\ge\lambda\} \quad . \qquad (2.1.9)$$

Hence from (2.1.8) and (2.1.9) it follows

$$P\{\sup_{c\le t\le d} |X(c,\omega)-X(t,\omega)|\ge\lambda+k\}\le 2P\{|X(c,\omega)-X(d,\omega)|\ge\lambda\} \qquad (2.1.10)$$

But

$$P\{|X(c,\omega)-X(d,\omega)|\ge\lambda\}\le \frac{Var(X(c,\omega)-X(d,\omega))}{\lambda^2}$$
$$= \frac{\sigma^2|c-d|}{\lambda^2} .$$

Therefore for sufficiently large λ

$P\{|X(c,\omega)-X(d,\omega)|\ge\lambda\}$ can be made arbitrarily small. Hence the proof.

2.3 Characterization of Wiener and Poisson processes

This section is devoted to characterization of Wiener process and Poisson process.

Theorem 9 An independent increment process $\{X(t), t\ge 0\}$ with $P(X(0)=0)=1$ is Wiener process having continuous mean $a(t)$ and continuous covariance function $\sigma^2(\min(s,t))$, $\sigma^2(0)=0$ if $X(t)$ is continuous for almost all ω.

<u>Proof</u> Let t_{n_k}, $k = 1,2,\ldots,m_n$ be a subdivision of the interval (s,t) into subintervals of equal length; such that

$$\sum_{k=1}^{m_n} P\{|X(t_{n_k}) - X(t_{n_{k-1}})| > \frac{1}{n}\} < \frac{1}{n} \qquad (2.3.1)$$

For $X(t)$ continuous and possesing independent increments, choice of such subintervals is possible due to theorem 4, of Gihman [2] et. al.,(page 188).

Define

$$X_{n_k} = X(t_{n_k}) - X(t_{n_{k-1}}) \text{ and}$$

$$X'_{n_k} = X_{n_k} \quad \text{if} \quad |X_{n_k}| \leq \frac{1}{n}$$

$$= 0 \qquad \text{otherwise.}$$

Let $X'_n = \sum_{k=1}^{m_n} X'_{n_k}$, then

$$P\{X'_n \neq X(t) - X(s)\} = \sum_{k=1}^{m_n} P\{|X(t_{n_k}) - X(t_{n_{k-1}})| > \frac{1}{n}\}$$
$$\leq \frac{1}{n} .$$

Hence X'_n converges in probability to $X(t)-X(s)$ as n tends to infinity. (2.3.2)

Denote $a'_{n_k} = E(X'_{n_k})$, $\sigma^2_{n_k} = Var(X'_{n_k})$

$$a'_n = \sum_{k=1}^{m_n} a'_{n_k}, \quad \sigma^2_n = \sum_{k=1}^{m_n} \sigma^2_{n_k} .$$

We discuss the proof in two cases as follows

Case (i) $\lim\limits_{n \to \infty} \sigma_n^2 < \infty$ (2.3.3)

Using (2.3.3) it follows that, there exists a subsequence n_j such that $\lim\limits_{n \to \infty} \sigma_{n_j}^2 = \sigma^2 < \infty$. For further discussion we express X'_{n_j} as

$$X'_{n_j} = a'_{n_j} + \sum_{k=1}^{m_{n_j}} (X'_{(n_j)_k} - a'_{(n_j)_k}) \qquad (2.3.4)$$

By central limit theorem $(X'_{n_j} - a'_{n_j})$ converges in distribution to normal random variable X with mean zero and variance σ^2. From (2.3.2) it can be seen that X'_{n_j} converges in probability. Hence a'_{n_j} converges to a limit say a. Therefore

$$X(t)-X(s) = a + X \; . \qquad (2.3.5)$$

Hence the proof.

Case (ii) $\lim\limits_{n \to \infty} \sigma_n^2 = \infty$.

For any $c > 0$, q_n can be chosen such that

$$\sum_{k=1}^{q_n} \sigma_{n_k}^2 \to c \; .$$

To exhibit the choice of q_n, let

$$q'_n = \left[\frac{c}{\min\limits_{1 \le k \le m_n} \sigma_{n_k}^2} \right] \text{ and } q''_n = \left[\frac{c}{\max\limits_{1 \le k \le m_n} \sigma_{n_k}^2} \right] .$$

[x] denotes integral part of x.

Then

$$\sum_{k=1}^{q_n''} \sigma_{n_k}^2 < c < \sum_{k=1}^{q_n'} \sigma_{n_k}^2 .$$

Define

$$Q_n = \min \{ j > q_n'' \mid \sum_{k=1}^{j} \sigma_{n_k}^2 > c \}.$$

Then

$$\sum_{k=1}^{Q_n} \sigma_{n_k}^2 \geq c \text{ and } \sum_{k=1}^{Q_n - 1} \sigma_{n_k}^2 < c .$$

Hence

$$c \leq \sum_{k=1}^{Q_n} \sigma_{n_k}^2 < c + \sigma_{n_{Q_n}}^2 .$$

Therefore $\sum_{k=1}^{Q_n} \sigma_{n_k}^2 \to c$ as $n \to \infty$.

Using the central limit theorem we get $\sum_{1}^{q_n} (X_{n_k}' - a_{n_k}')$ converges in distribution to a normal random variable.

Hence

$$\overline{\lim_{n\to\infty}} |E\{\exp(iuX_n')\}| \leq \lim_{n\to\infty} | \prod_{k=1}^{m_n} \exp\{iu(X_{n_k}' - a_{n_k}')\}|$$

$$= \exp\{\frac{-u^2c^2}{2}\} .$$

Since c is an arbitrary

$$\lim_{n\to\infty} E\{\exp(iu X_n')\} = 0,$$

which contracticts (2.3.2) . Hence case (ii) cannot hold. Therefore X(t)-X(s) has a normal distribution and

$$a(t) = E(X(t)),\ \sigma^2(t) = Var(X(t)).$$

Theorem 10 Let $\{X(t), t \geq 0\}$ be a Levy process with $P(X(0)=0) = 1$. If almost all sample functions are step functions with jump 1, then X(t) is a Poisson process.

Proof Suppose for each n

$$s \leq t_{n_0} < t_{n_1} \cdots < t_{n_{m_n}} \leq t$$

be subdivision of an interval [s,t].

We denote

$$X_{n_k} = X(t_{n_k}) - X(t_{n_{k-1}}).$$

Define

$$X'_{n_k} = X_{n_k} \quad \text{if } X_{n_k} = 0 \text{ or } 1$$

$$= 1 \quad \text{if } X_{n_k} > 1.$$

Therefore,

$$X'_n = \sum_{k=1}^{m_n} X'_{n_k}.$$

$P\{\sup_{1\leq k\leq m_n} X_{n_k} > 1\} = 0$ follows from the fact that the jumps of X(t) are of size 1. We need to evaluate

$$P\{X'_n \neq X(t)-X(s)\} = \sum_{k=0}^{m_n-1} P(X_{n_k} > 1)$$

to show that X_n converges to X(t) -X(s) in probability.

Clearly

$$P\{\sup_{1\le k\le m_n} X_{n_k} > 1\} = \sum_{k=0}^{m_n-1} \prod_{j=0}^{k-1} P\{X_{n_j} \le 1\} P\{X_{n_k} > 1\}$$

$$\ge \sum_{k=0}^{m_n-1} P\{X_{n_k} > 1\} \prod_{j=0}^{m_n-1} P\{X_{n_j} \le 1\}$$

$$= \sum_{k=0}^{m_n-1} P\{X_{n_k} > 1\} [1-P\{\sup_{1\le k\le m_n} X_{n_k} > 1\}].$$

Thus

$$\sum_{k=0}^{m_n-1} P\{X_{n_k} > 1\} \le P\{\sup_{1\le k\le m_n} X_{n_k} > 1\}[1-P\{\sup_{1\le k\le m_n} X_{n_k} > 1\}]^{-1}$$

Hence

$$X_n \to X(t) - X(s) \text{ as } n \to \infty \cdot \qquad (2.3.6)$$

$$E \exp\{-\alpha[X(t)-X(s)]\} = \lim_{n\to\infty} E \exp\{-\alpha X'_n\}$$

$$= \lim_{n\to\infty} \prod_{k=1}^{m_n} E \exp\{-\alpha X'_{n_k}\}$$

$$= \lim_{n\to\infty} \prod_{k=1}^{m_n} [(1-P_{n_k}) + P_{n_k} \exp(-\alpha)]$$

where $P_{n_k} = P\{X_{n_k}=1\} = P\{X_{n_k} \ge 1\}$.

$$= \lim_{n\to\infty} \prod_{k=1}^{m_n} [1- P_{n_k}(1- \exp(-\alpha))]$$

$$\le \lim_{n\to\infty} \prod_{k=1}^{n} \exp\{-P_{n_k}(1-\exp(-\alpha))\}$$

$$= \lim_{n\to\infty} \exp\{-P_n(1-\exp(-\alpha)\} \qquad (2.3.7)$$

Where $P_n = \sum_{k=1}^{m_n} P_{n_k}$.

Hence (2.3.7) reduces to

$$E\{-\alpha[X(t)-X(s)]\} = \exp\{-(1-\exp(-\alpha)) \overline{\lim_{n\to\infty}} P_n\}.$$

Since X(t) is continuous in probability a $\delta > 0$ exists such that $|t_1 - t_2| < \delta$

$$P\{|X(t_2)-X(t_1)| = 0\} = P\{|X(t_2)-X(t_1)| \le \tfrac{1}{2}\} > 0. \qquad (2.3.8)$$

Therefore

$$P\{X(t)-X(s) = 0\} > 0$$

and $\max_{1\le k\le m_n} P_{n_k} \to 0$ as $n \to \infty$ due to stochastic continuity of X(t). Note that

$$P\{X(t)-X(s) = 0\} = P\{\sum_{k=1}^{m_n} X_{n_k} = 0\}$$

$$= \prod_{k=1}^{m_n} (1 - P_{n_k}).$$

Therefore

$$-\log P\{X(t)-X(s) = 0\} = \sum_{k=1}^{m_n} -\log(1-P_{n_k}).$$

Clearly

$$\sum_{k=1}^{m_n} P_{n_k} < -\log P\{X(t)-X(s) = 0\} \le \sum_{k=1}^{m_n} P_{n_k} + \frac{\max_{1\le k\le m_n} P_{n_k}}{1-\max_{1\le k\le m_n} P_{n_k}}. \qquad (2.3.9)$$

Taking limit as $n \to \infty$ in the relation (2.3.9) we get

$$\lim_{n\to\infty} \sum_{k=1}^{m_n} p_{nk} = -\log P\{X(t)-X(s) = 0\}.$$

Thus (2.3.7) reduces to

$$E(\exp \{-\alpha (X(t)-X(s))\})= \exp \{P(1-\exp(-\alpha))\}.$$

Hence the proof.

2.4. Decomposition of a process with independent increments.

In this section we study the decomposition of a separable and stochastically continuous process with independent increments into a continuous and discrete components. Moreover, a continuous component is independent of each of the remaining components. Further we show that the continuous component is a Gaussian process.

Let $\{X(t),\ t \geq 0\}$ be separable and stochastically continuous process with independent increments. We also assume that S is the range of X(t). Denote $A_\varepsilon = \{x: |x| > \varepsilon\}$ and $\mathcal{A}_\varepsilon$ σ-field of Borel sets contained in A. Then

$$X(t,A_\varepsilon) = \sum_{s\leq t} (X(s+0) -X(s-0)) I_{A_\varepsilon} (X(s+0)-X(s-0)) \qquad (2.4.1)$$

$I_{A_\varepsilon}(.)$ is an indicator function. Thus

$$X_\varepsilon(t) = X(t)-X(t,A_\varepsilon) \qquad (2.4.2)$$

will be a process obtained from $X(t)$, after discarding the jumps of size exceeding ε. We prove a lemma which is helpful in obtaining decomposition of $X(t)$.

Lemma 11 : If $\{X_\varepsilon(t), t \geq 0\}$ is a process defined in (2.4.2) then for every $\varepsilon > 0$,

$$E(|X_\varepsilon(t)|^2) < \infty.$$

Proof. Let for each n

$$0 = t_{n_0} < t_{n1} \dots < t_{nn} = t \quad \text{and}$$

$$\lim_{n\to\infty} \max_{0 \leq k \leq n} (t_{nk} - t_{n(k-1)}) = 0.$$

Define

$$X_{nk} = X_\varepsilon(t_{nk}) - X_\varepsilon(t_{n(k-1)})$$

$$\text{if } |X_\varepsilon(t_{n_k}) - X_\varepsilon(t_{n(k-1)})| \leq 2\varepsilon$$

$$= 0 \quad \text{otherwise.}$$

In order to show that $\sum_{k=1}^{n} X_{nk}$ converges in probability, let us evaluate

$$P\{X_\varepsilon(t) \neq \sum_{k=1}^{n} X_{nk}\} = \sum_{k=1}^{n} P\{X_{nk} = 0\} .$$

Clearly

$$P\{\sup_{1\le k\le n} X_{nk} = 0\} = P\{\sup_{1\le k\le n} |X_\varepsilon(t_{nk})-X_\varepsilon(t_{n(k-1)})| > \varepsilon\}$$

$$= \sum_{k=1}^{n} \prod_{j=1}^{k-1} P\{|X_\varepsilon(t_{nj})-X_\varepsilon(t_{n(j-1)})| \le 2\varepsilon\} \times P\{|X_\varepsilon(t_{nk})-X_\varepsilon(t_{n(k-1)})| > 2\varepsilon\}$$

$$\ge \sum_{k=1}^{n} P\{|X_\varepsilon(t_{nk})-X_\varepsilon(t_{n(k-1)})| > 2\varepsilon\} \times \prod_{j=1}^{n-1} P\{|X_\varepsilon(t_{nj})-X_\varepsilon(t_{n(j-1)})| \le 2\varepsilon\}$$

$$= \sum_{k=1}^{n} P\{|X_\varepsilon(t_{nk})-X_\varepsilon(t_{n(k-1)})| > 2\varepsilon\} \left[1-P\{\sup_{1\le j\le n-1} |X_\varepsilon(t_{nj})-X_\varepsilon(t_{n(j-1)})| > 2\varepsilon\}\right].$$

Thus,

$$\sum_{k=1}^{n} P\{|X_\varepsilon(t_{nk})-X_\varepsilon(t_{n(k-1)})| > 2\varepsilon\} \le P\{\sup_{1\le k\le n} |X_\varepsilon(t_{nk})-X_\varepsilon(t_{n(k-1)})| > 2\varepsilon\} \times \left[1-P\{\sup_{1\le k\le n} |X_\varepsilon(t_{nk})-X_\varepsilon(t_{n(k-1)})| > 2\varepsilon\}\right]^{-1} \quad (2.4.3)$$

Since $X(t)$ is stochastically continuous, $X_\varepsilon(t)$ is stochastically continuous (Gikhman [1], et.al., page 257) which implies that

$$P\{\sup_{1\le k\le n} |X_\varepsilon(t_{nk})-X_\varepsilon(t_{n(k-1)})| > 2\varepsilon\},$$

tends zero as n tends to infinity. Therefore allowing $n \to \infty$, right hand side of (2.4.3) reduces to zero. Thus

$$X_\varepsilon(t) = \lim_{n\to\infty} \sum_{k=1}^{n} X_{nk} . \tag{2.4.4}$$

Clearly every term in the left hand side of (2.4.4) is less than or equal to 2ε in absolute value. Further we show that $\sum_{k=1}^{n} Var(X_{nk})$ is convergent. Suppose it possible $\sum_{k=1}^{n} Var(X_{nk})$ does not converge. Define

$$Y_{nk} = \frac{X_{nk} - E(X_{nk})}{\sqrt{\sum_{k=1}^{n} Var(X_{nk})}} .$$

Then $\sum_{k=1}^{n} Y_{nk}$ will converge in distribution to a normal variate with mean 0 and variance 1. Therefore

$$\lim_{n\to\infty} P\{\sum_{k=1}^{n} X_{nk} > \alpha \sqrt{\sum_{k=1}^{n} Var(X_{nk})} + \sum_{k=1}^{n} E(X_{nk}) \}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} \exp \{ - \frac{1}{2} u^2\} \, du \tag{2.4.5}$$

and

$$\lim_{n\to\infty} P\{\sum_{k=1}^{n} X_{nk} < -\alpha \sqrt{\sum_{k=1}^{n} Var(X_{nk})} + \sum_{k=1}^{n} E(X_{nk})\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\alpha} \exp \{ - \frac{1}{2} u^2\} \, du \tag{2.4.6}$$

The relations (2.4.5) and (2.4.6) contradict the boundedness of $\sum_{k=1}^{n} X_{nk}$ which follows from the relation (2.4.4). Hence

$\sum_{k=1}^{\infty} \mathrm{Var}(X_{nk})$ is convergent. Using Chebyshev's inequality we write

$$P\{|\sum_{k=1}^{n} X_{nk} - \sum_{k=1}^{n} E(X_{nk})| > \delta\} \leq \frac{\mathrm{Var}\,(\sum_{k=1}^{n} X_{nk})}{\delta^2}.$$

Therefore $\sum_{k=1}^{n} X_{nk} - \sum_{k=1}^{n} E(X_{nk})$ is bounded in probability which implies that $E(\sum_{k=1}^{n} X_{nk})$ is bounded. Note that

$$E(|\sum_{k=1}^{n} X_{nk}|^2) = E\,(\sum_{k=1}^{n} X_{nk})^2$$

$$= \mathrm{Var}\,(\sum_{k=1}^{n} X_{nk}) + (E\,(\sum_{k=1}^{n} X_{nk}))^2.$$

Hence $E(|\sum_{k=1}^{n} X_{nk}|^2)$ is bounded. Since

$$E(|X_\varepsilon(t)|^2) \leq \overline{\lim_{n\to\infty}}\; E(|\sum_{k=1}^{n} X_{nk}|^2) \quad .$$

We get $E(|X_\varepsilon(t)|^2) < \infty$.

Let $\{\varepsilon_n\}$ denote a sequence which decreases to 0. Further we denote the set of x such that $\varepsilon_k < |x| \leq \varepsilon_{k-1}$ by Δ_k for k=2,3,... and Δ_1 be the set of all x , such that $|x| > \varepsilon_1$. The processes $X(t,\Delta_1),\ldots,X(t,\Delta_k)$, and $X(t) - \sum_{j=1}^{k} X(t,\Delta_j)$ are mutually independent (Gikhman [1], et al., page 260). Moreover $X(t,\Delta_j)$ is stochastically continuous as X(t) is stochastically continuous (Gikhman [1] et,al., page 257). Now we prove the theorem regarding decomposition of X(t).

Theorem 12 : If $\{X(t),\ t \geq 0\}$ is a separable and stochastically continuous process with independent increments, then there is a continuous process $X_0(t)$ such that

$$X(t) = X_0(t) + X(t,\Delta_1) + \sum_{j=2}^{\infty}[X(t,\Delta_j) - E(X(t,\Delta_j)]$$

Proof. We can express $X_{\varepsilon_1}(t)$ as

$$X_{\varepsilon_1}(t) = \sum_{k=2}^{m} X(t,\Delta_k) + X_{\varepsilon_{m+1}}(t) \qquad (2.4.7)$$

The terms on right hand side of (2.4.7) are independent which yields

$$\sum_{k=2}^{n} Var\ (X(t,\Delta_k)) \leq Var\ (X_{\varepsilon_1}(t)).$$

Since $Var(X_{\varepsilon_1}(t))$ is finite $\sum_{k=2}^{n} Var\ (X(t,\Delta_k))$ converges as $n \to \infty$. Then a subsequence $\{n_k\}$ with $n_1 = 1$ can be chosen such that

$$\sum_{j=n_k}^{\infty} Var\ (X(T,\Delta_j)) \leq \frac{1}{k^6}\ .$$

In order to show that $\sum_{j=2}^{n_k} [X(t,\Delta_j) - E(X(t,\Delta_j))]$

converges uniformly with probability one as $k \to \infty$, we obtain for $T < \infty$,

$$P\{\sup_{0\leq t\leq T} |\sum_{j=n_k+1}^{n_{k+1}} [X(t,\Delta_j)-E(X(t,\Delta_j))]-$$

$$\sum_{j=2}^{n_k} [X(t,\Delta_j)- E(X(t,\Delta_j))]| > \frac{1}{k^2} \}$$

$$\leq P\{\sup_{0\leq t\leq T} |\sum_{j=n_k+1}^{n_{k+1}}[X(t,\Delta_j)-E(X(t,\Delta_j))]| > \frac{1}{k^2} \}$$

$$\leq \overline{\lim_{m\to\infty}} P\{ \sup_{\ell\leq mT} |\sum_{j=n_k+1}^{n_{k+1}}[X(\frac{\ell}{m}, \Delta_j)-E(X(\frac{\ell}{m},\Delta_j))] \geq \frac{1}{k^2} \}$$

$$\leq \overline{\lim_{m\to\infty}} k^4 E(|\sum_{j=n_k+1}^{n_{k+1}}[X(T,\Delta_j)- E(X(T,\Delta_j))]|^2)\leq \frac{1}{k^2}. \quad (2.4.8)$$

The relation (2.4.8) follows due to Kolmogorov's inequality (Giknman [1] et.al., page 119). Now we obtain

$$\lim_{T\to\infty} P\{\sup_{0\leq t\leq T} |\sum_{j=n_k+1}^{n_{k+1}}[X(t,\Delta_j) - E(X(t,\Delta_j))]| > \frac{1}{k^2}\}$$

$$\leq \frac{1}{k^2} .$$

Therefore,

$$P\{ \sup_{t\geq 0} |\sum_{j=n_k+1}^{n_{k+1}}[X(t,\Delta_j) - E (X(t,\Delta_j))]| > \frac{1}{k^2} \}$$

$$\leq \frac{1}{k^2} .$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent, in view of Borel-Cantelli theorem (Giknman [1] et.al. page 112) it follows that

$$P\{\sum_{j=n_k+1}^{n_{k+1}} |X(t,\Delta_j) - E(X(t,\Delta_j))| \geq \frac{1}{k^2}\} \leq \frac{1}{k^2} .$$

Hence

$$\sum_{j=2}^{n_k} [X(t,\Delta_j) - E(X(t,\Delta_j))]$$

converges uniformly with probability one. Further we show that

$$X_{\varepsilon_1}(t) - \sum_{j=2}^{n_k} [X(t,\Delta_j) - E(X(t,\Delta_j))]$$

converges uniformly with probability one to $X_0(t)$. Note that $X(t,\Delta_j)$ is a stochastically continuous and

$$\sup_{0 \leq t \leq \infty} E(| X(t,\Delta_j) |^2) < \infty .$$

Therefore in view of theorem 6 (Gikhman [1] et.al.,page 72) we get

$$\lim_{t \to s} E(X(t,\Delta_j)) = E(X(s,\Delta_j)).$$

The process $X_{\varepsilon_1}(t) - \sum_{j=2}^{n_k} [X(t,\Delta_j) - E(X(t,\Delta_j))]$

does not have jumps of size exceeding ε_{n_k} in absolute value. Therefore

$$\lim_{k \to \infty} X_{\varepsilon_1}(t) - \sum_{j=2}^{n_k} [X(t,\Delta_j) - E(X(t,\Delta_j))]$$

$$= X_0(t)$$

uniformly with probability one and $X_0(t)$ is continuous with probability one. Thus we get

$$X(t) = X_o(t) + X(t,\Delta_1) + \sum_{j=2}^{\infty}[X(t,\Delta_j)-E(X(t,\Delta_j)].$$

Since $X_{\varepsilon_1}(t) - \sum_{j=1}^{n}[X(t,\Delta_j)-E(X(t,\Delta_j)]$

is independent of each of the processes $X(t,\Delta_j)$ for $j=1,2,\ldots,$ and $X_o(t)$ being a limit of

$$X_{\varepsilon_1}(t) - \sum_{j=2}^{n}[X(t,\Delta_j) - E(X(t,\Delta_j))]$$

is independent of each of $X(t, \Delta_j)$. Now we show that $E(|X_o(t)|^2)$ is finite. Note that

$$X_{\varepsilon_1}(t) = X_o(t) + \sum_{j=2}^{\infty} [X(t, \Delta_j)- E(X,\Delta_j))] .$$

Since the terms on the left hand side are independent and $E(|X_{\varepsilon_1}(t)|^2) < \infty$ we get

$$E(|X_o(t)|^2) < \infty .$$

$X_o(t)$ being a limit of process with independent increments, it is a process with independent increments. If $P(X_o(0)=0) =1$ then in view of lemma 9 the process $\{X(t),t \geq 0\}$ is a Gaussian process.

2.5. Strong law of large numbers and Central limit theorem.

In this section we study the limiting behaviour of SIIP. .. theorem discussed below is a strong law of large numbers for SIIP.

<u>Theorem 13</u> : Let $\{X(t),\ t \geq 0\}$ be a separable SIIP such that $E(X(t)-X(0)) = 0$. Then

$$P\{ \lim_{t\to\infty} \frac{X(t)}{t} = 0 \} = 1.$$

<u>Proof</u>. Let $[t]$ denote the integer part of t. Then clearly

$$X([t])-X(0) = \sum_{j=1}^{[t]} [X(j)-X(j-1)]$$
$$= \sum_{j=1}^{[t]} Y_j$$

where $Y_j = X(j)-X(j-1)$; $j=1,2,\ldots,[t]$. Since $X(t)$ is a SIIP, $\{Y_j, j \geq 1\}$ is a sequence of independent and identically distributed random variables with mean $E(Y_1)$. In view of 'strong law of large numbers' for independent and identically distributed random variables, we get

$$\lim_{t\to\infty} \frac{\sum_{j=1}^{[t]} Y_j}{[t]} = E(Y_j) = 0 \qquad (2.5.1)$$

with probability one. Note that $X(t)$ is SIIP hence

$$\mathcal{L}\left(\sup_{j \leq t \leq j+1} |X(t)-X(j)|\right) = \mathcal{L}\left(\sup_{0 \leq t \leq 1} |X(t)-X(0)|\right).$$

If $Z(j) = \sup_{j \leq t \leq j+1} |X(t)-X(j)|$ then $Z_1, Z_2, \ldots$ are independent and identically distributed random variables.

Gihman [1]

Lemma 2 page [illegible] ([illegible])

Let $X(t)$ be a separable [illegible] indep. increments for which there exist [illegible]

$$P\{|X(T)-X(s)|>c\} \leq \alpha \quad \text{for [illegible]}$$

Then for [illegible] $\alpha > 0$

$$P\{\sup_{0\leq s\leq T} |X(s)| > [illegible]\} \leq \frac{1}{1-\alpha} [illegible]$$

[illegible]

Let

$$\sum_{n=1}^{\infty} P\{\sup_{n\leq t\leq n+1} |X(t)-X(n)| > \varepsilon [illegible]\} [illegible]$$

Let [illegible] $C > 0$ [illegible]

$$P\{\sup_{[illegible]} |X(t)| > [illegible]\} \leq [illegible]$$

$$\sum_{n=1}^{\infty} P\{\sup \frac{|X(t)| [illegible]}{\varepsilon} > n\} \leq [illegible] \frac{[illegible]}{\varepsilon}.$$

$$\sum_{n=1}^{\infty} P [illegible]$$

Therefore

$$\lim_{t\to\infty} \frac{1}{[t]} \sum_{j=1}^{[t]} Z(j) = E(Z(1)) \tag{2.5.2}$$

with probability one. Similarly

$$\lim_{t\to\infty} \frac{1}{[t]-1} \sum_{j=1}^{[t]-1} Z(j) = E(Z(1)) \tag{2.5.3}$$

with probability one. Subtracting (2.5.3) from (2.5.2) we get

$$\lim_{t\to\infty} \frac{1}{[t]} Z([t]) \to 0$$

with probability one. We express

$$\frac{X(t)-X(0)}{t} = \frac{X(t)-X([t])}{[t]} \times \frac{[t]}{t} + \frac{X([t])-X(0)}{[t]} \times \frac{[t]}{t}. \tag{2.5.4}$$

Using (2.5.1) and (2.5.3) we get form (2.5.4)

$$\lim_{t\to\infty} \frac{X(t)-X(0)}{t} = \lim_{t\to\infty} \frac{X(t)}{t} = 0 \text{ with probability one.}$$

The following is a version of central limit theorem for SIIP.

<u>Theorem 14</u> : If $\{X(t), t \geq 0\}$ is a SIIP with $E(X(t)) = \rho\, t$ and $Var(X(t)) = \sigma^2 t$; where ρ and σ are real finite constants. Then

$$P\left\{ \frac{X(t)-\rho t}{\sigma\sqrt{t}} \leq x \right\} \to \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} u^2\right\} du \text{ as } t \to \infty.$$

Proof. Let us assume $P(X(0)=0) = 1$. Define

$$Y_j = X(j)-X(j-1) \qquad j=1,2,\dots, .$$

Then we can write

$$X([t]) = \sum_{j=1}^{[t]} Y_j , \qquad (2.5.5)$$

where $[t]$ is a integer part of t. Since $\{X(t), t \geq 0\}$ is a SIIP, $\{Y_j, j \geq 1\}$ is a sequence of independent and identically distributed random variables. Clearly

$$E(Y_j) = \rho \quad \text{and } Var(Y_j) = \sigma^2 .$$

Using central limit theorem we can say

$$\frac{X([t]) - \rho [t]}{\sigma \sqrt{[t]}} \qquad (2.5.6)$$

converges in distribution to a normal variable with mean 0 and variance 1. Note that $\{X(t), t \geq 0\}$ is a SIIP so we have

$$\mathcal{L}(X(t)- X([t]) = \mathcal{L}(X(t-[t])).$$

Using Chebyshev's inequality we get

$$P \left\{ \frac{|X(t-[t]) - \rho(t-[t])|}{\sigma \sqrt{t}} \geq \varepsilon \right\} \leq \frac{\sigma^2 (t-[t])}{\varepsilon^2 \sigma^2 t} . \qquad (2.5.7)$$

Allowing $t \to \infty$ in (2.5.7), it can be seen that

$$\lim_{t\to\infty} \frac{X(t-[t]) - \rho(t-[t])}{\sigma\sqrt{t}} = 0 \qquad (2.5.8)$$

in probability. Therefore $\frac{X(t-[t]) - \rho(t-[t])}{\sigma\sqrt{t}}$

converges to 0 in distribution, as $t \to \infty$.

We express

$$\frac{X(t) - \rho t}{\sigma\sqrt{t}} = \frac{X([t]) - \rho[t]}{\sigma\sqrt{[t]}} \times \frac{\sigma\sqrt{[t]}}{\sigma\sqrt{t}} + \frac{X(t-[t]) - \rho(t-[t])}{\sigma\sqrt{t}}$$

Since $\sqrt{\frac{[t]}{t}}$ increases to 1 as $t \to \infty$, we can say

$$\frac{X(t) - \rho t}{\sigma\sqrt{t}}$$

converges in distribution to a normal variable with mean 0 and variance 1 as $t \to \infty$.

CHAPTER 3

3.1. Introduction

This chapter is devoted to inference regarding SIIP. In some situations discussed in section (3.2), sample size is a random variable. In order to test the goodness of fit for the model with the help of a sample of random size, a statistic analogous to Kolmogorov-Smirnov statistic is suggested by M.Kac. We treat this as an application of a SIIP. Further we obtain an asymptotic distribution of a Kac statistic. Next we include sequential estimation procedure for multidimensional stochastic processes which belong to the exponential class. We obtain maximum likelihood equation, sufficient estimator, efficient estimator, Cramer-Rao type inequality. Ultimately a general form of an efficiently estimable parameter function and the corresponding estimator is determined. We also include illustrations time to time. Further we estimate the canonical measure G which occurs in the canonical representation of the characteristic function of SIIP.

3.2. Kolmogorov Smirnov Statistic and Kac Statistic.

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a continuous distribution having distribution functions F(.). The fundamental problem of statistics is to test the hypothesis $F = F_0$ against $F \neq F_0$. Let without loss of generality F_0 corresponds to a uniform distribution on (0,1). One statistic that is commonly used is Kolmogorov-Smirnov statistic D_n, which is given by

$$D_n = \sup_{0 \leq t \leq 1} |F_n(t) - t|.$$

$F_n(t)$ represents an empirical distribution function defined as follows :

$$F_n(t) = \frac{1}{n}\sum_{j=1}^{n} I_{[X_j \leq t]} \quad \text{if} \quad n > 0$$

$$= 0 \qquad\qquad \text{if} \quad n = 0 \qquad\qquad (3.2.1)$$

$I_{[X_j \leq t]}$ is an indicator function. If the alternative hypothesis is one sided, then either D_n^+ or D_n^- is used, where

$$D_n^+ = \sup_{0 \leq t \leq 1} \{ t - F_n(t) \}$$

and

$$D_n^- = \sup_{0 \leq t \leq 1} \{ F_n(t) - t \} \qquad (3.2.2)$$

In order to study the asymptotic properties of D_n, D_n^+, D_n^-, let us consider the family

$$\{Z_n(t),\ 0 \leq t \leq 1\}$$

in which,

$$Z_n(t) = F_n(t)-t \quad (3.2.3)$$

Clearly $\{Z_n(t), 0 \leq t \leq 1\}$ is a stochastic process. The process $\{Z_n(t), 0 \leq t \leq 1\}$ can be transformed to a process with uncorrelated increments as defined in Doob (page, 99) which can be treated as a generalisation of the process with independent increments. Let us consider the transformation

$$Y_n(t) = (1+t)\, Z_n\left(\frac{1}{1+t}\right) \quad (3.2.4)$$

To show that $\{Y_n(t), 0 \leq t \leq 1\}$ is a process with uncorrelated increments, we need to obtain covariance function of the process. Let $0 \leq s \leq t \leq 1$, $j = 1,2,\ldots,N$ and $k=1,2,\ldots,N$ then

$$\begin{aligned} E(I_{[X_j \leq t]}) &= P\{X_j \leq t\} \\ &= t \end{aligned} \quad (3.2.5)$$

which implies that $E(Z_n(t)) = 0$ and $E(Y_n(t)) = 0$.

Note that

$$E(I_{[X_j \leq t]}\, I_{[X_k \leq s]}) = \begin{cases} P\{X_j \leq t\}\, P\{X_k \leq s\} & \text{if } j \neq k \\ P\{X_j \leq \min(s,t)\} & \text{if } j = k. \end{cases}$$

$$= \begin{cases} st & \text{if } j \neq k \\ \min(s,t) & \text{if } j = k \end{cases}$$

Therefore,

$$\text{Cov}\,(I_{[X_j \le t]}, I_{[X_k \le s]}) = \begin{cases} 0 & \text{if } j \ne k \\ \min(s,t) - st & \text{if } j = k. \end{cases}$$

Hence, (3.2.6)

$$\text{Cov}\,(Z_n(s),\, Z_n(t)) = \min\,(s,t) - st. \qquad (3.2.7)$$

Using (3.2.5) Cov $(Y_n(s), Y_n(t))$ can be obtained as follows

$$\text{Cov}(Y_n(s), Y_n(t)) = (1+t)(1+s)\ \text{Cov}(Z_n(s), Z_n(t))$$

$$= (1+t)(1+s)\left[\min\left(\frac{1}{1+s}, \frac{1}{1+t}\right) - \frac{1}{(1+t)(1+s)}\right].$$

Since, $s,t, \ge 0$ we can write

$$= \min\,(1+t,\ 1+s) - 1$$

$$= \min\,(s,t).$$

Thus we can see for $s < 1$.

$$\text{cov}\,(Y_n(s), Y_n(t) - Y_n(s)) = 0$$

which implies that the process $\{Y_n(t), 0 \le t \le 1\}$ is a process with uncorrelated increments.

In some situations like the number of insurance claims during the next year, number of telephone calls for one week, number of insects trapped in three hours, sample size will not be fixed.

Therefore let $N, X_1, X_2, \ldots,$ be independent random variables; N having poisson distribution with parameter λ; $X_1, X_2, \ldots,$ are considered in particular uniform random variables on (0,1). A modified empirical distribution function $F_\lambda^*(t)$ analogous to $F(t)$, is given by M. Kac in 1949 (Beekman, page 142) which is defined as

$$F_\lambda^*(t) = \frac{1}{\lambda} \sum_{j=1}^{N} I_{[X_j \leq t]} \quad \text{if } N > 0 \qquad 0 \leq t \leq 1. \qquad (3.2.8)$$

The one sided Kac statistic analogous to D_n^- is

$$K_\lambda^-(t) = \sup_{0 \leq t \leq 1} \{t - F_\lambda^*(t)\}. \qquad (3.2.9)$$

For convenience we define

$$X_\lambda(t) = \sqrt{\lambda}\ \{t - F_\lambda^*(t)\}.$$

Clearly $\{X_\lambda(t),\ 0 \leq t \leq 1\}$ is also a stochastic process. We obtain below mean and covariance function of $X_\lambda(t)$.

Let $0 \leq s \leq t \leq 1$; $j=1,2,\ldots N$ and $k=1,2,\ldots,N$, then

$$E(X_\lambda(t)) = E\{E(X_\lambda(t))\ |N=r\}$$

Using (3.2.5) we get

$$= E\{\sqrt{\lambda}(t - \frac{r}{\lambda} t)\} \qquad (3.2.10)$$

$$= 0$$

Now,

$$\mathrm{Cov}\ (X_\lambda(s),\ X_\lambda(t))$$

$$=\mathrm{Cov}(\sqrt{\lambda}\,[s-F_\lambda^*(s)],\ \sqrt{\lambda}\,[t-F_\lambda^*\ (t)])$$

$$=\ \lambda\ \mathrm{Cov}\ (F_\lambda^*\ (s),\ F_\lambda^*(t))$$

$$=\lambda\ \{\mathrm{Cov}(\ E(F_\lambda^*(s)\,|N=r),\ E(F_\lambda^*\ (t)\,|N=r))\ +$$

$$E(\mathrm{Cov}(F_\lambda^*(s),F_\lambda^*(t))\,|\ N=r)\}. \qquad (3.2.11)$$

Let us evaluate

$$\mathrm{Cov}\ (E(F_\lambda^*\ (s)\,|N=r),\ E(F_\lambda^*(t)\,|N=r))$$

$$=\mathrm{Cov}\ (\ \frac{r}{\lambda}\ t\ ,\ \ \frac{r}{\lambda}\ s)$$

$$=\ \frac{st}{\lambda}\quad .$$

$$\mathrm{Cov}\ (F_\lambda^*\ (t),\ F_\lambda^*(s)\ \ |N=r)$$

$$=\mathrm{Cov}(\ \frac{1}{\lambda}\ \sum_1^r I_{[X_j\ \leq t]},\ \frac{1}{\lambda}\ \sum_1^r\ I_{[X_k\ \leq\ s]}). \qquad (3.2.12)$$

Using (3.2.6) the relation (3.2.12) samplifies to

$$\mathrm{Cov}(\ F_\lambda^*(t),\ F_\lambda^*(s)\,|N=r)$$

$$=\frac{r}{\lambda^2}[\ \min(s,t)-\ st].$$

Hence,

$$E\{\ \mathrm{Cov}[(F_\lambda^*(t),\ F_\lambda^*(s)\,|N=r]\}$$

$$=\ \frac{1}{\lambda}\ [\min\ (s,t)-\ st].$$

Then (3.2.11) becomes

$$Cov(X_\lambda(s), X_\lambda(t)) = \lambda[\frac{st}{\lambda} + \frac{\min(s,t)}{\lambda} - \frac{st}{\lambda}]$$
$$= \min(s,t).$$

In order to show that the process $\{X_\lambda(t), 0 \leq t \leq 1\}$ posseses independent increments; we include the following lemma.

Lemma 1 : The random variables $Y(\Delta_1, N)$ and $Y(\Delta_2, N)$ are independent if Δ_1 and Δ_2 are non-overlapping subintervals of (0,1).

Where,

$$Y(\Delta, N) = \sum_{j=1}^{N} I_\Delta(X_j)$$

and

$$I_\Delta(x) = 1 \quad \text{if } x \in \Delta$$
$$= 0 \quad \text{otherwise.}$$

Proof. In order to prove the lemma let us obtain for

$$0 \leq k_1, k_2 \leq n \text{ and } k_1 + k_2 \leq n$$

$$P\{Y(\Delta_1, N) = k_1, Y(\Delta_2, N) = k_2\}$$

$$= \sum_{n=0}^{\infty} P\{Y(\Delta_1, N) = k_1, Y(\Delta_2, N) = k_2 | N=n\} \times P[N=n]$$

$$= \sum_{n=k_1+k_2}^{\infty} \frac{n!}{k_1!k_2!(n-k_1-k_2)!} \Delta_1^{k_1} \Delta_2^{k_2} (1-\Delta_1-\Delta_2)^{n-k_1-k_2} \times \frac{e^{-\lambda} \lambda^n}{n!}$$

$$= \sum_{n=k_1+k_2}^{\infty} e^{-\lambda} \frac{(\lambda\Delta_1)^{k_1}}{k_1!} \frac{(\lambda\Delta_2)^{k_2}}{k_2!} \frac{(\lambda(1-\Delta_1-\Delta_2))^{n-k_1-k_2}}{(n-k_1-k_2)!}$$

$$= \frac{(\lambda\Delta_1)^{k_1}}{k_1!} \frac{(\lambda\Delta_2)^{k_2}}{k_2!} e^{-\lambda} \sum_{n=k_1+k_2}^{\infty} \frac{(\lambda(1-\Delta_1-\Delta_2))^{n-k_1-k_2}}{(n-k_1-k_2)!}$$

$$= e^{-\lambda(}\frac{(\lambda\Delta_1)^{k_1}}{k_1!} \times e^{-\lambda\Delta_2} \frac{(\lambda\Delta_2)^{k_2}}{k_2!}$$

$$= P\{Y(\Delta_1,N) = k_1\} \; P\{Y(\Delta_2,N) = k_2\} .$$

Hence the proof.

Since $X_\lambda(s) = \sqrt{\lambda}\ \{ t - \frac{Y((o,s),N)}{\lambda} \}$

and for $s < t$

$$X_\lambda(t) - X_\lambda(s) = \sqrt{\lambda}\{(t-s) - \frac{Y((s,t),N)}{\lambda}\}$$

the increments $X_\lambda(s)$ and $X_\lambda(t)-X_\lambda(s)$ are independent.

Hence, N is a Poisson random variable, is a sufficient condition for $\{X_\lambda(t),\ 0 \le t \le 1\}$ being a process with independent increments.

Next lemma shows that N is a Poisson random variable is a necessary condition for $\{X_\lambda(t),\ 0 \le t \le 1\}$ being a process with independent increments.

<u>Lemma 2</u> : Let $X_1, X_2, \ldots$ be independent and identically distributed random variables, each having probability density function $f(.)$, N be a non-negative integer valued random variable independent of X_i's. If $Y(\Delta,N)$ represents the number of those of X_i's among the first N, which fall within the interval Δ and for nonoverlapping intervals Δ_1 and Δ_2 the random variables $Y(\Delta_1,N)$ and $Y(\Delta_2,N)$ are independent then N follows Poisson distribution.

<u>Proof</u>. Define,

$$I_\Delta(x) = 1 \qquad \text{if } x \in \Delta$$
$$= 0 \qquad \text{otherwise}$$

Then $Y(\Delta_1,N) = \sum_{j=1}^{N} I_{\Delta_1}(X_j)$ and

$$Y(\Delta_2,N) = \sum_{j=2}^{N} I_{\Delta_2}(X_j) \ .$$

Since $Y(\Delta_1,N)$ and $Y(\Delta_2,N)$ are independent, for $u_1, u_2 \in \mathbb{R}$ we get

$$E[\exp\{i(u_1 \sum_{j=1}^{N} I_{\Delta_1}(X_j) + u_2 \sum_{j=1}^{N} I_{\Delta_2}(X_j))\}]$$

$$= E[\exp\{iu_1 \sum_{j=1}^{N} I_{\Delta_1}(X_j)\}] \, E[\exp\{iu_2 \sum_{j=1}^{N} I_{\Delta_2}(X_j)\}]. \qquad (3.2.13)$$

Let $P\{N=r\} = P(r)$ then in view of independence of N and X_i's and (3.2.13) we write

$$\sum_{r=0}^{\infty} P(r)(E[\exp\{i(u_1 I_{\Delta_1}(x)+u_2\, I_{\Delta_2}(x))\}])^r$$

$$=\sum_{r=0}^{\infty} P(r)(E[\exp\{iu_1\, I_{\Delta_1}(x)\}])^r x \sum_{r=0}^{\infty} P(r)(E[\exp\{iu_2 I_{\Delta_2}(x)\} \quad (3.2.14)$$

We can see that

$$E[\exp\{iu_1\, I_{\Delta_1}(x)\}]$$

$$= P(I_{\Delta_1}(x)=0) + \exp\{iu_1\}P(I_{\Delta_1}(x)=1)$$

$$= 1 + (\exp\{iu_1\}-1)\int_{\Delta_1} f(x)\, dx. \quad (3.2.15)$$

Similarly,

$$E(\exp\{iu_2\, I_{\Delta_2}(x)\} = 1+(\exp\{iu_2\}-1)\int_{\Delta_2} f(x)dx. \quad (3.2.16)$$

Now,

$$E[\ \exp\{iu_1\, I_{\Delta_1}(x) + iu_2\, I_{\Delta_2}(x)\}]$$

$$=P(I_{\Delta_1}(x)=0, I_{\Delta_2}(x)=0)+\exp\{iu_1\}P(I_{\Delta_1}(x)=1, I_{\Delta_2}(x)=0)$$

$$+ \exp\{iu_2\}\ P(I_{\Delta_1}(x)=0, I_{\Delta_2}(x)=1)+ \exp\{iu_1 + iu_2\}\, x$$

$$P(I_{\Delta_1}(x)=1, I_{\Delta_2}(x)=1).$$

Since Δ_1 and Δ_2 are non-overlapping we get

$$E[\exp\{iu_1 I_{\Delta_1}(x) + iu_2 I_{\Delta_2}(x)\}$$

$$= 1+(\exp\{iu_1\}-1)\int_{\Delta_1} f(x)dx + (\exp\{iu_2\}-1)\int_{\Delta_2} f(x)dx. \quad (3.2.17)$$

Taking $u_1=u_2 = \pi$ in (3.2.15), (3.2.16) and (3.2.17) we write (3.2.14) as

$$\sum_{r=0}^{\infty} P(r)(1-2\int_{\Delta_1} f(x)dx -2\int_{\Delta_2} f(x)\ dx)^r$$

$$= \sum_{r=0}^{\infty} P(r)(1-2\int_{\Delta_1} f(x)dx)^r \sum_{r=0}^{\infty} P(x)(1-2\int_{\Delta_2} f(x)\ dx)^r. \quad (3.2.18)$$

Setting

$$a = 1-2\int_{\Delta_1} f(x)\ dx\ ,\ b = 1-2\int_{\Delta_2} f(x)\ dx$$

and $g(a) = \sum_{r=0}^{\infty} a^r P(r)$ relation (3.2.18) becomes

$$g(a+b-1) = g(a).g(b). \quad (3.2.19)$$

The relation (3.2.19) holds for real numbers a,b between -1 and 1 if non-overlapping intervals Δ_1 and Δ_2 can be found such that $a = 1-2\int_{\Delta_1} f(x)\ dx$ and $b = 1-2\int_{\Delta_2} f(x)\ dx$. Since g is analytic in unit circle we write

$$\frac{\partial}{\partial a} g(a+b-1) = \frac{\partial}{\partial a} g(a)\ g(b)$$

$$\frac{\partial^2}{\partial a\, \partial b} g(a+b-1) = \frac{\partial}{\partial a} g(a)\ \ \frac{\partial}{\partial b} g(b)$$

Hence we write

$$g'(a+b-1) = g'(a)\, g'(b).$$

If $b = 1$ then we get $g''(a) = c\, g'(a)$ where $c = g'(1)$.

Therefore

$$g(a) = \alpha \exp\{c\, a\} + \beta.$$

Clearly

$$g'(1) = \alpha\, C \exp\{C\} = C\,.$$

This implies $\alpha = \exp\{-C\}$.

Since

$$g(1) = \sum_{r=0}^{\infty} P(r) = 1$$

and

$$g(1) = \alpha \exp\{C\} + \beta$$

$$= 1 + \beta\,.$$

We get $\beta = 0.$

Hence

$$g(a) = \exp\{C(a-1)\}.$$

Thus $P\{N=r\} = \exp\{-C\} \frac{C^r}{r!}$; $\quad r = 0,1,2,\ldots,.$

In order to study the limiting distribution of Kac statistic $K_\lambda^-(t)$ defined in (3.2.9) we study the limiting distribution of the process $\{X_\lambda(t), 0 \le t \le 1\}$ as $\lambda \to \infty$.

Let $\{X(t),\ 0 \leq t \leq 1\}$ be a Gaussian process (Doob, page 71) with $P(X(0)=0)=1$, $E(X(t))=0$ and

$$\mathrm{Cov}\big(X(t),X(s)\big) = \min(s,t).$$

Denote

$$K^{-}(t) = \sup_{0 \leq t \leq 1} X(t).$$

Since $X_\lambda(t)$ for each t and fixed N say $N=r$, is a sum of independent and identically distributed random variables with finite variance we get

$$\lim_{\lambda \to \infty} \phi_{X_\lambda(t_1),\ldots,X_\lambda(t_k)}(u_1,u_2,\ldots,u_k)$$

$$=\phi_{X(t_1),\ldots,X(t_k)}(u_1,u_2,\ldots,u_k)$$

$$=\exp\{-\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} u_i u_j \min(t_i,t_j)\}. \qquad (3.2.20)$$

Where $u_1,u_2,\ldots,u_k$ are real numbers and

$\phi_{X_\lambda(t_1),\ldots,X_\lambda(t_k)}(u_1,u_2,\ldots,u_k)$ is a characteristic function of $X_\lambda(t_1),\ldots,X_\lambda(t_k)$ for any

$$\{t_1,\ldots,t_k\} \in [0,1].$$

Theorem 3 : For any real number α

$$\lim_{\lambda\to\infty} P\{\sqrt{\lambda}\ K_\lambda^-(t) \le \alpha\} = P\{K^-(t) \le \alpha\}. \qquad (3.2.21)$$

Proof. To prove the theorem we need to obtain

$$\lim_{\lambda\to\infty} P\{\sqrt{\lambda}\ K_\lambda^-(t) \le \alpha\} = \lim_{\lambda\to\infty} P\{\sup_{0\le t\le 1} X_\lambda(t) \le \alpha\}.$$

Using the separable version of $X_\lambda(t)$ we write

$$P\{\sup_{0\le t\le 1} X_\lambda(t) \le \alpha\} = P\{\lim_{r\to\infty} \sup_{1\le k\le 2^r} X_\lambda(\tfrac{k}{2^r}) \le \alpha\}.$$

Since the sets are monotonic we write

$$P\{\lim_{r\to\infty} \sup_{1\le k\le 2^r} X_\lambda(\tfrac{k}{2^r}) \le \alpha\} = \lim_{r\to\infty} P\{\sup_{1\le k\le 2^r} X_\lambda(\tfrac{k}{2^r}) \le \alpha\}.$$

Therefore

$$\lim_{\lambda\to\infty} P\{\sqrt{\lambda}K_\lambda^-(t) \le \alpha\} = \lim_{\lambda\to\infty} \lim_{r\to\infty} P\{\sup_{1\le k\le 2^r} X_\lambda(\tfrac{k}{2^r}) \le \alpha\}. \qquad (3.2.22)$$

Define

$$S_k = \sum_{j=1}^{k} 2^{\frac{-r}{2}} [X_\lambda(\tfrac{j}{2^r}) - X_\lambda(\tfrac{j-1}{2^r})],$$

$$R_k = \sum_{j=1}^{k} 2^{-\frac{r}{2}} [X(\tfrac{j}{2^r}) - X(\tfrac{j-1}{2^r})]$$

and

$$P_n(\alpha) = P\{\max (S_1, S_2, \ldots, S_n) < \alpha\}$$

In order to prove the theorem we prove for every integer k and $\varepsilon > 0$.

$$P\{\max.(R_1,R_2,\ldots,R_k) < (\alpha-\varepsilon)\sqrt{k}\} - \frac{1}{k\varepsilon^2}$$

$$\leq \underline{\lim}_{n\to\infty} P_n(\alpha) \leq \overline{\lim}_{n\to\infty} P_n(\alpha) \leq P\{\max(R_1,\ldots,R_k) < \alpha\sqrt{k}\}.$$

Let $n_j = [\,j\frac{n}{k}]$, $j = 0,1,\ldots,k.$

and

$$P_{n_k}(\alpha) = P\{\max(S_{n_1},S_{n_2},\ldots,S_{n_k}) < \alpha\sqrt{n}\}.$$

Using central limit theorem for multivariate random variables we get

$$\lim_{n\to\infty} P_{n_k}(\alpha) = P\{\max(R_1,R_2,\ldots,R_k) < \alpha\sqrt{k}\}. \quad (3.2.23)$$

Denote

$$E_\ell = P\{S_\ell \geq \alpha\sqrt{n},\ S_1 \leq \alpha\sqrt{n},\ldots,S_{\ell-1} \leq \alpha\sqrt{n}\}.$$

Clearly

$$\sum_{\ell=1}^{n} E_\ell = 1-P_n(\alpha) \leq 1. \quad (3.2.24)$$

For $n_i < \ell \leq n_{i+1}$ we write

$$E_\ell = P\{S_\ell \geq \alpha\sqrt{n}, S_1 < \alpha\sqrt{n},\ldots,S_{\ell-1} < \alpha\sqrt{n}, |S_{n_{i+1}} - S_\ell| \geq \varepsilon\sqrt{n}\}$$

$$+P\{S_\ell \geq \alpha\sqrt{n}, S_1 < \alpha\sqrt{n}\ldots,S_{\ell-1} < \alpha\sqrt{n}, |S_{n_{i+1}} - S_\ell| < \varepsilon\sqrt{n}\}.$$

(3.2.25)

The first term of the right hand side of (3.2.25) is

$$E_{\ell}\, P\{|S_{n_{i+1}} - S_{\ell}| \geq \varepsilon\sqrt{n}\}$$

follows from the fact that S_{ℓ} is sum of independent random variables, further by Chebyshev's inequality it is less than or equal to

$$E_{\ell}\, \frac{1}{k\varepsilon^2}$$

Therefore,

$$1 - P_n(\alpha) \leq \frac{1}{k\varepsilon^2} + \sum_{i=1}^{k} \sum_{n_i < \ell \leq n_{i+1}} P\{S_{\ell} \geq \alpha\sqrt{n},\ S_1 < \alpha\sqrt{n}, \ldots,$$

$$\ldots, S_{\ell-1} < \sqrt{n}, |S_{n_{i+1}} - S_{\ell}| < \varepsilon\sqrt{n}\}.$$

The double sum is less than $1 - P_{n_k}(\alpha - \varepsilon)$, therefore

$$1 - P_n(\alpha) \leq \frac{1}{k\varepsilon^2} + 1 - P_{n_k}(\alpha - \varepsilon).$$

Since $P_n(\alpha) \leq P_{n_k}(\alpha)$ we get

$$P_{n_k}(\alpha - \varepsilon) - \frac{1}{k\varepsilon^2} \leq P_n(\alpha) \leq P_{n_k}(\alpha).$$

We hold k and ε fixed and let $n \to \infty$, in view of (3.2.23) we write

$$P\{\max(R_1,R_2,\dots,R_k) < (\alpha-\varepsilon)\sqrt{k}\} - \frac{1}{k\varepsilon^2}$$

$$\leq \underline{\lim}_{n\to\infty} P_n(\alpha) \leq \overline{\lim}_{n\to\infty} P_n(\alpha) \leq P\{\max(R_1,R_2,\dots,R_k) < \alpha\sqrt{k}\}. \tag{3.2.26}$$

Let $k\to\infty$ for ε fixed in (3.2.26), and we get

$$P\{\max(R_1,R_2,\dots,R_k) < (\alpha-\varepsilon)\sqrt{k}\}$$

$$\leq \underline{\lim}_{n\to\infty} P_n(\alpha) \leq \overline{\lim}_{n\to\infty} P_n(\alpha)$$

$$\leq P\{\max(R_1,R_2,\dots,R_k) < \alpha\sqrt{k}\}.$$

Finally taking $\varepsilon\to 0$ we get

$$\lim_{n\to\infty} P_n(\alpha) = P\{\max(R_1,R_2,\dots,R_k) < \alpha\sqrt{k}\}. \tag{3.2.27}$$

Using (3.2.21) and (3.2.27) we get

$$\lim_{\lambda\to\infty} P\{\sqrt{\lambda}K_\lambda^-(t) \leq \alpha\} = \lim_{\lambda\to\infty}\lim_{r\to\infty} P\{\sup_{1\leq k\leq 2^r} X(\tfrac{k}{2^r}) \leq \alpha\}$$

$$= P\{\sup_{0\leq t\leq 1} X(t) \leq \alpha\}.$$

Hence

$$\lim_{\lambda\to\infty} P\{\sqrt{\lambda}\, K_\lambda^-(t) \leq \alpha\} = P\{K^-(t) \leq \alpha\}.$$

3.3 Sequential Estimation

Statistical inference about the processes with independent increments is concerned to the problem of estimation or testing of unknown parameters. For this purpose a sample of appropriate size is taken. In fixed sample size procedure, a sample size is fixed and it does not depend on the data which are available. A sequential estimation referes to a technique in which sample size is not fixed in advance but it depends by some rule on the data already collected and observed, hence it is a random variable. Sequential procedure requires, less number of observations on an average as compared to the fixed sample procedure to achieve the same goal.

We define an exponential class of stochastic processes below and under some conditions we obtain maximum likelihood equation, sufficient estimator, efficient estimator and Cramer-Rao type inequality.

Let $\underset{\sim}{X}(t) = [X_1(t),\ldots,X_m(t)]'$ be a m-dimensional stochastic process defined on $(\Omega, \mathbb{F}, P)$ with the values in (E,ε), where $E \subset \mathbb{R}^m$ is a state space, ε is a σ-field of all Borel sets in E and $t \in T \subset (0,\infty)$. P represents the probability measure which depends on an unknown parameter $\theta = [\theta_1,\ldots, \theta_k]' \in \Theta$, Θ is an open interval of $\mathbb{R}^k$. Further we assume that $\mathbb{F}_t$ represents the σ-field generated by the random vectors $\{\underset{\sim}{X}(s),\ s \leq t\}$.

Definition 4. The stochastic process $\underset{\sim}{X}(t)$ belong to the exponential class, if the following conditions are fulfilled

(i) $\underset{\sim}{X}(t)$ is a SIIP with $P(\underset{\sim}{X}(0)=0) = 1$ for all $\theta \in \Theta$ and continuous in probability.

(ii) The probability distributions at time t are dominated by a σ-finite measure ν and the densities with respect to ν may be represented in the form

$$f(x,t,\theta) = g(x,t)\exp\{a'(\theta)\underset{\sim}{X} + b(\theta)t\}, \qquad (3.3.1)$$

where $\underset{\sim}{X} = (x_1,x_2,\ldots,x_m)' \in E$, $a(\theta) = [a_1(\theta),\ldots,a_m(\theta)]'$, g is a non-negative function defined on $E \times T$ and $a_1,\ldots,a_m$, b are non-constant functions defined on Θ. (Winkler et.al.,page 130).

The multinomial process and m-dimensional Gaussian process belong to the exponential class. In the one dimensional case (m=1), the Bernoulli process, Poisson process, gamma process, negative binomial process belong to the exponential class.

We assume that the functions $a_1(\theta),\ldots,a_m(\theta),b(\theta)$ in (3.3.1) are differentiable with respect to the components of the parameter $\theta = [\theta_1,\ldots,\theta_k]'$. We denote

$$B = \text{grad}_\theta\, b(\theta) = [\frac{\partial}{\partial\theta_1} b(\theta), \ldots, \frac{\partial}{\partial\theta_k} b(\theta)]' \quad \text{and}$$

$$A = \text{grad}_\theta\, a(\theta) = (\text{grad}_\theta\, a_1(\theta), \ldots, \text{grad}_\theta\, a_m(\theta))$$
$$= ((\frac{\partial}{\partial\theta_i} a_j(\theta)))$$

Clearly A is of order $k \times m$. Further we assume that the differentiation is allowed under the sign of integration and the components $E_\theta X_i(t)$ of $E_\theta(\underset{\sim}{X}(t))$ are differentiable with respect to θ_j, $j=1,2,\ldots,k$. Differentiating both the sides of the relation

$$\int_E f(x,t,\theta)\, d\nu = 1$$

with respect to θ_j, we get

$$E_\theta(\sum_{i=1}^{m} \frac{\partial}{\partial\theta_j} a_i(\theta)\, X_i(t) + \frac{\partial b(\theta)}{\partial\theta_j} t) = 0$$

for $j=1,2,\ldots,k$. Therefore

$$A\, E_\theta(\underset{\sim}{X}(t)) + B.t. = 0. \qquad (3.3.2)$$

Clearly if $k=m$ and A^{-1} exists we get

$$E_\theta(\underset{\sim}{X}(t)) = -A^{-1} B\, t. \qquad (3.3.3)$$

Similarly differentiating $E_\theta(X_i(t))$ with respect to θ_j for $j=1,2,\ldots,k$ we get

$$\frac{\partial}{\partial \theta_j} E_\theta(X_i(t)) = \frac{\partial}{\partial \theta_j} \int_E x_i f(x, t, \theta) d\nu$$

$$= E\left[X_i \sum_{\ell=1}^{m} \frac{\partial}{\partial \theta_j} a_\ell(\theta) x_\ell + x_i \frac{\partial}{\partial \theta_j} b(\theta) t\right]$$

$$\text{for } i=1,2,\ldots, m. \qquad (3.3.4)$$

Hence,

$$G_t = \text{grad}_\theta\, E_\theta(\underset{\sim}{X}'(t))$$

$$= A\, E_\theta(\underset{\sim}{X}(t)\, \underset{\sim}{X}'(t)) + t.\ B\, E_\theta(\underset{\sim}{X}'(t)). \quad (3.3.5)$$

Using (3.3.2) we get

$$G_t = A\, E_\theta(\underset{\sim}{X}(t)\, \underset{\sim}{X}'(t)) - A\, E_\theta(\underset{\sim}{X}(t))\, E_\theta(\underset{\sim}{X}'(t))$$

$$= A\, K_t, \qquad (3.3.6)$$

K_t represents variance-covariance matrix of $\underset{\sim}{X}(t)$.

If k=m and A^{-1} exists then the covariance matrix K_t becomes

$$K_t = A^{-1}\, \text{grad}_\theta\, (A^{-1} B)'\, t. \qquad (3.3.7)$$

We obtain expectation and covariance matrix for some processes.

<u>Definition 5</u>. A stochastic process

$\underset{\sim}{X}(t) = [X_1(t),\ldots,X_m(t)]'$ for $t = 0,1,2,\ldots,$ with stationary and independent increments such that $X_i(t)$ takes values in the set $\{0,1,\ldots,\}$ for $i=1,2,\ldots,m$; is called a multinomial process if the probability distribution of $\underset{\sim}{X}(t)$ is given by

$$P(\underset{\sim}{X}(t)=\underset{\sim}{x}) = \frac{t!}{x_1!x_2!\ldots x_m!(t-\sum_{i=1}^{m}x_i)!} p_1^{x_1},\ldots p_m^{x_m} q^{t-\sum_{i=1}^{m}x_i} \quad (3.3.8)$$

where $\underset{\sim}{x}=(x_1,\ldots,x_m)'$, $x_i \in \{0,1,\ldots,t\}$, $\sum_{i=1}^{m} x_i \leq t$ and $0 < p_i < 1$

$i=1,\ldots,m$ and $q = 1 - \sum_{i=1}^{m} p_i > 0$. (Winkler et.al., page 131).

Clearly a multinomial process belongs to the exponential class and the proper identification of the functions $a_i(p)$ and $b(p)$ which occur in (3.3.1) will be

$$a_i(p) = \log \frac{p_i}{q} \text{ and } b(p)=\log q \quad i=1,2,\ldots,m,$$

where $p=(p_1,p_2,\ldots,p_m)'$. It can be seen easily that

$$\begin{aligned} \frac{\partial}{\partial p_j} a_i(p) &= \frac{1}{q} && \text{if } i\neq j \\ &= \frac{1}{q} + \frac{1}{p_i} && \text{if } i=j \end{aligned}$$

and

$$\frac{\partial}{\partial p_j} b(p) = -\frac{1}{q} .$$

Hence we get

$$A = \frac{1}{q} E + \text{diag} \left[\frac{1}{p_1}, \ldots, \frac{1}{p_m} \right]$$

and $B = [-\frac{1}{q},\ldots,-\frac{1}{q}]'$. E is a square matrix of order m with each of its element equal to unity. A^{-1} exists and it is given by

$$\begin{bmatrix} p_1(1-p_1) & -p_1p_2 & \cdots & -p_1p_m \\ -p_2p_1 & p_2(1-p_2) & \cdots & -p_2p_m \\ \cdots & \cdots & \cdots & \cdots \\ -p_mp_1 & -p_mp_2 & \cdots & p_m(1-p_m) \end{bmatrix}$$

Therefore from (3.3.3)

$$\begin{aligned} E_p\, \underset{\sim}{X}(t) &= -A^{-1} B.\ t \\ &= [p_1p_2,\ldots,p_m]'\ t\ . \end{aligned}$$

We can also note that

$$\begin{aligned} G_t &= \text{grad}_p\ E_p(\underset{\sim}{X}'(t)) \\ &= \text{grad}_p\ (p.t) \\ &= I\ .\ t. \end{aligned}$$

Where I denotes an identity matrix of order m. Thus we get from (3.3.7)

$$\begin{aligned} K_t &= A^{-1}\, G_t \\ &= A^{-1}\, t. \end{aligned}$$

Definition 6. A stochastic process

$\underset{\sim}{X}(t)=[X_1(t),\ldots,X_m(t)]'$ for $t \in [0,\infty)$ with stationary and independent increments taking values in $\mathbb{R}^m$ is called a m-dimensional Gaussian process if the probability density function of $\underset{\sim}{X}(t)$ is given by

$$f(x,t,\theta)=\frac{1}{\sqrt{(2\pi)^m|\Sigma|t}}\exp\{-\frac{1}{2t}(\underset{\sim}{x}-\theta t)'\,\Sigma^{-1}(\underset{\sim}{x}-\theta t)\},\quad(3.3.9)$$

where $\theta=(\theta_1,\ldots,\theta_m)'$ is the unknown expectation vector and Σ is a given non-singular covariance matrix, $|\Sigma|$ denotes the determinant of matrix Σ. (Winkler et al., page 131).

Comparing the expressions (3.3.9) and (3.3.1) one can note that the m-dimensional Gaussian process belongs to the exponential class and

$$a'(\theta) = \theta'\Sigma^{-1} \text{ and } b(\theta)= -\frac{1}{2}\,\theta'\,\Sigma^{-1}\theta\ .$$

Therefore,

$$A = \Sigma^{-1} \text{ and } B = -\Sigma^{-1}\theta\ .$$

Using (3.3.3) and (3.3.7) we get

$$E_\theta(X(t)) = \theta . t,$$

$$G_t = I.t,$$

$$K_t = \Sigma.\ t.$$

For sequential estimation, sample size is not fixed and we need stopping times to stop the sampling. Therefore let τ be a stopping time defined on Ω with values in $T \cup \{\infty\}$ such that

$$\{\omega \varepsilon \Omega \mid \tau(\omega) \leq t\} \varepsilon \mathbb{F}_t \text{ for } t \varepsilon T ;$$

where $\mathbb{F}_t$ is a σ-field generated by the random vectors $\{\underline{X}(s),\ s \leq t\}$. The next theorem provides the joint distribution of τ and $\underline{X}(\tau)$ which is useful for further inference.

Theorem 7. Let $\underline{X}(t)$ be a process which belongs to the exponential class and let τ be any finite stopping time. Then a probability measure P_{θ_0} not depending on the unknown parameter θ exists for every fixed $\theta_0 \varepsilon \Theta$ such that

$$P((\tau,\underline{X}(\tau)) \varepsilon S) = \int_{\{\omega \mid (\tau,\underline{X}(\tau)) \varepsilon S\}} \exp\{\alpha'(\theta)\underline{X}(\tau)+\beta(\theta)\tau\} dP_{\theta_0}$$

$$= Q_\theta(S) \qquad (3.3.10)$$

where $S \subset T \times E$ is an $\mathcal{T} \times \mathcal{E}$ measurable and

$$\alpha(\theta) = a(\theta)-a(\theta_0),\ \beta(\theta) = b(\theta)-b(\theta_0).$$

Proof. See Winkler et.al., page 131.

Define $U = T \times E$ and $\mathcal{U}$ as σ-field of Borel sets of U. Therefore $Q_\theta(S)$, $S \varepsilon \mathcal{U}$ will be dominated by the measure $Q^* = Q_{\theta_0}$. We write (3.3.10) as follows.

$$Q_\theta(S) = {}_S\int f^*(u,\theta) . Q^*(du).$$

and the density function or likelihood function will be

$$f^*(u,\theta) = \exp\{\alpha'(\theta)\underset{\sim}{x}(u) + \beta(\theta)t(u)\}.$$

$$= \exp\{\sum_{i=1}^{m}\alpha_i(\theta)x_i(u) + \beta(\theta)t(u)\}. \qquad (3.3.11)$$

Assuming $f^*(u, \theta)$ differentiable with respect to θ_j and treating $f^*(u, \theta)$ as a likelihood function we get the maximum likelihood equations as follows

$$\frac{\partial}{\partial\theta_j} \log f^*(u,\theta) = 0 \text{ for } j=1,2,\ldots,m.$$

$$\sum_{i=1}^{m}\left(\frac{\partial}{\partial\theta_j}\alpha_i(\theta)\right)X_i(u) + \left(\frac{\partial}{\partial\theta_j}\beta(\theta)\right)t(u) = 0$$

$$\sum_{i=1}^{m}\left(\frac{\partial}{\partial\theta_j}a_i(\theta)\right)X_i(u) + \frac{\partial}{\partial\theta_j}b(\theta)\ t(u) = 0\quad j=1,2,\ldots,m.$$

Hence,

$$\text{grad}_\theta \log f^*(u, \theta) = A\,\underline{X}(u) + B\ t(u) = 0 \qquad (3.3.12)$$

We obtain maximum likelihood estimators for a multinomial process and m-dimensional Gaussian process.

If $\underset{\sim}{X}(t)$ is a multinomial process then clearly

$$f^*(u,p) = \exp\{\sum_{i=1}^{m}x_i(u).\log\frac{p_i q_o}{q\ q_{i.o}} + t(u)\log\frac{q}{q_o}\}.$$

We get the following system of linear equations

$$\frac{\partial}{\partial p_j} \log f^*(u,p) = \frac{x_j(u)}{p_j} + \frac{\sum_{i=1}^{m} x_i(u)}{q} - \frac{t(u)}{q}$$

$$= 0 \qquad j=1,2,\ldots,m. \qquad (3.3.13)$$

The relation (3.3.13) implies that

$$\frac{x_1(u)}{p_1} = \frac{x_2(u)}{p_2} = \cdots\cdots = \frac{x_m(u)}{p_m} = \frac{t(u)-\sum_{i=1}^{m} x_i(u)}{q}$$

and we get $p_j = \dfrac{x_j(u)}{t(u)}$ for $j= 1,2,\ldots,m$.

Hence the maximum likelihood estimator of p will be

$$\hat{p} = \tau^{-1}\underset{\sim}{X}(\tau) .$$

Clearly $E(\hat{p}) = EE(\tau^{-1}\underset{\sim}{x}(\tau) \mid \tau) = p$,

which implies that the maximum likelihood estimator of p is unbiased.

$$Var(\hat{p}) = E(\tau^{-2}\underset{\sim}{X}(\tau)) - [E(\tau^{-1}\underset{\sim}{X}(\tau))]^2$$

$$= E\,E(\tau^{-2}\underset{\sim}{X}(\tau) \mid \tau) - p^2$$

$$= E(\tau^{-1}A^{-1}) + p^2 - p^2$$

$$= A^{-1}E(\tau^{-1}).$$

If $\underline{X}(t)$ is a m-dimensional Gaussian process then

$$f^*(u,\theta) = \exp\{(\theta-\theta_o)'\Sigma^{-1}\underline{X}(u) - \tfrac{1}{2}(\theta'\Sigma^{-1}\theta - \theta_o'\Sigma^{-1}\theta_o)t(u)\}.$$

Therefore likelihood equations become

$$\text{grad}_\theta \ \log f^*(u,\theta) = \Sigma^{-1}\underline{X}(u) - \Sigma^{-1}\theta t(u) = 0.$$

Hence maximum likelihood estimator of θ will be.

$$\hat{\theta} = \tau^{-1}\underline{X}(\tau)$$

Note that

$$E\ \hat{\theta} = E\ E(\tau^{-1}\underline{X}(\tau) \mid \tau)$$
$$= \theta .$$

Thus maximum likelihood estimator $\hat{\theta}$ turns out to be unbiased.

$$\begin{aligned} Var(\hat{\theta}) &= E(\tau^{-1}\underline{X}(\tau))^2 - [E(\tau^{-1}\underline{X}(\tau))]^2 \\ &= E\ E(\tau^{-1}\underline{X}^2(\tau) \mid \tau) - \theta^2 \\ &= E\ \tau^{-1}\Sigma + \theta^2 - \theta^2 \\ &= \Sigma .\ E(\tau^{-1}). \end{aligned}$$

Further we show that the $E(\underline{X}(t))$ and covariance matrix of $\underline{X}(t)$ depends only upon $E(\tau)$ and $Var(\tau)$ under certain assumptions.

Let $\Psi = (\Psi_1, \ldots, \Psi_p)'$ be a function defined on $U \times \Theta$. The components Ψ_j, $j=1,\ldots,p$ are assumed to be measurable, Q^*

integrable and differentiable with respect to $\theta_i, i=1,2,\dots,k$. Further we assume that for $j=1,\dots,k$ the function $H_{1j}^{(\ell)}$ and $H_{2j}^{(\ell)}$ are independent of θ. Moreover

$$\int_U H_{rj}^{(\ell)} Q^*(du) < \infty, \quad r=1,2, \qquad (3.3.14)$$

and

$$\left|\Psi_\ell(u,\theta)\ \frac{f^*(u,\theta'_{(j)})-f^*(u,\theta)}{\theta'_j-\theta_j}\right| \le H_{1j}^{(\ell)}, \qquad (3.3.15)$$

$$\left|\frac{\Psi_\ell(u,\theta'_{(j)})-\Psi_\ell(u,\theta)}{\theta'_j-\theta_j}\right| f^*(u,\theta) \le H_{2j}^{(\ell)} \qquad (3.3.16)$$

where $\theta'_{(j)} = (\theta_1,\dots,\theta_{j-1}, \theta'_j, \theta_{j+1},\dots,\theta_k)'$.

We include below a theorem which helps in expressing the $E_\theta(\underset{\sim}{X}(t))$ and covariance matrix of $\underset{\sim}{X}(t)$ in terms of $E_\theta(\tau)$ and $Var(\tau)$.

<u>Theorem 8</u>. Let $\underset{\sim}{X}(t)$ be a process which belongs to the exponential class, τ be a stopping time and $\Psi=(\Psi_1,\dots,\Psi_p)'$ a vector function with the properties (3.3.14),(3.3.15) and (3.3.16). Then

$$E_\theta[(A\underset{\sim}{X}(\tau)+B\tau)\ \Psi] = \text{grad}_\theta\ (E_\theta\ h') - E_\theta\ (\text{grad}_\theta\ h'). \qquad (3.3.17)$$

<u>Proof</u>. We note that

$$E_\theta\ \Psi(\tau,X(\tau),\theta) = \int_U \Psi_\ell f^*(u,\theta)Q^*(du) \text{ for } \ell=1,2,\dots,p.$$

Due to the assumptions (3.3.14),(3.3.15) and (3.3.16), differentiation under the sign of integration is valid and we get

$$\frac{\partial}{\partial\theta_j} E_\theta \Psi_\ell (\tau, X(\tau), \theta)$$

$$= \int_U \Big[\frac{\partial}{\partial\theta_j} + \Psi_\ell \frac{\partial}{\partial\theta_j}\Big\{\sum_{i=1}^{m} \alpha_i(\theta)x_i(u)+\beta(\theta)t(u)\Big\}\Big]$$

$$\times\, f^*(u; \theta)Q^*\ (du)$$

$$= E\,\Big[\frac{\partial}{\partial\theta_j}\Psi_\ell + \frac{\partial}{\partial\theta_j}\Big\{\sum_{i=1}^{m}\alpha_i(\theta)\ x_i(u)+\beta(\theta)t(u)\Big\}\Psi_\ell\ \Big].$$

$$j=1,2,\ldots,k;\ell=1,2,\ldots,p.$$

Therefore,

$$\mathrm{grad}_\theta(E_\theta\ \Psi') = E_\theta(\mathrm{grad}_\theta\ \Psi')+ E_\theta[A\ \underline{X}(\tau)+B\ \tau)\Psi'].$$

Hence the theorem.

This theorem provides a generalisation of Wald's first equation which we discuss below

(a) If p=1 and $\Psi\ (\tau,X(\tau),\theta)$ =1, then the relation (3.3.17) gives

$$A E_\theta \underline{X}(\tau)+B E_\theta(\tau) = 0 \qquad (3.3.18)$$

If A^{-1} exists then

$$E_\theta \underline{X}(\tau) = -A^{-1}B\ E_\theta\ \tau \qquad (3.3.19)$$

(b) If P=1 and $\Psi(\tau, \underset{\sim}{X}(\tau), \theta) = \tau$ then we get from (3.3.17)

$$A\, E_\theta\, (\tau \underline{X}(\tau)) + B\, E_\theta(\tau^2) = \text{grad}_\theta (E_\theta \tau) \qquad (3.3.20)$$

(c) If P=m and $\Psi(\tau, \underset{\sim}{X}(\tau), \theta) = \underset{\sim}{X}(\tau)$ using (3.3.17) we write

$$\begin{aligned}
\text{grad}_\theta\, (E_\theta\, \underset{\sim}{X}'(\tau)) &= A\, E_\theta(\underset{\sim}{X}(\tau)\underset{\sim}{X}'(\tau)) + B\, E_\theta(\tau\, \underset{\sim}{X}'(\tau)) \\
&= A[E_\theta\, (\underset{\sim}{X}(\tau)\underset{\sim}{X}'(\tau)) - E_\theta \underset{\sim}{X}(\tau)\, E\, \underset{\sim}{X}(\tau)] + B[E_\theta(\underset{\sim}{X}'(\tau)) \\
&\qquad - E_\theta\, \tau\, E\, \underset{\sim}{X}(\tau)] + (A\, E_\theta \underset{\sim}{X}(\tau) + B\, E_\theta\ )E\, \underset{\sim}{X}'(\tau) \\
&= AK_\tau + B\, k_\tau
\end{aligned}$$

follows due to (3.3.18). K_τ represnets

$E_0(\underset{\sim}{X}(\tau)\, \underset{\sim}{X}'(\tau)) - E_\theta \underset{\sim}{X}(\tau)\, E\, \underset{\sim}{X}'(\tau)$ and k_τ represents

$E_\theta(\tau\, \underset{\sim}{X}(\tau)) - E_\theta \tau\, E_\theta\, \underset{\sim}{X}'(\tau)$. If we write $\text{grad}_\theta\, (E_\theta \underset{\sim}{X}'_\tau(\tau)) = G_\tau$ we get

$$G_\tau = A\, K_\tau + B\, k_\tau \quad . \qquad (3.3.21)$$

In order to express K_τ in terms of $E_\theta(\tau)$ and $Var(\tau)$ we assume A^{-1} exists. Therefore

$$\begin{aligned}
K_\tau &= A^{-1}\, G_\tau - A^{-1}\, Bk_\tau \\
&= A^{-1}[-\text{grad}_\theta (A^{-1}B)'\, E_\theta\, \tau - \text{grad}_\theta\, (E_\theta \tau)\ (A^{-1}B))'] \\
&\qquad - A^{-1}B\ [E_\theta \tau\ \underset{\sim}{X}'(\tau) - E_\theta\ \tau E_\theta \underset{\sim}{X}'(\tau)] \\
&= A^{-1}[-\text{grad}_\theta\ (A^{-1}B)'\, E_\theta \tau - \text{grad}_\theta (E_\theta \tau)(A^{-1}B)' \\
&\qquad - A^{-1}B[\text{grad}'_\theta\ (E_\theta \tau)(A^{-1})' - (A^{-1}B)'\ E_\theta \tau^2 + (E_\theta \tau)^2 (A^{-1}B)']
\end{aligned}$$

$$=(A^{-1}B)(A^{-1}B)'[E_\theta\tau^2-(E_\theta\tau)^2]-A^{-1}\,grad_\theta\,(A^{-1}B)'E_\theta\tau$$

$$-A^{-1}[B\,grad_\theta'\,(E_\theta\tau)+grad_\theta\,(E_\theta\tau)B'](A^{-1})'$$

$$=(A^{-1}B)(A^{-1}B)'Var(\tau)-A^{-1}grad_\theta\,(A^{-1}B)'E_\theta\tau$$

$$-A^{-1}[B\,grad_\theta'\,(E_\theta\tau)+grad_\theta\,(E_\theta\tau)B'](A^{-1})' \quad (3.3.22)$$

The relation (3.3.22) depends only on $E_\theta(\tau)$ and $Var(X(\tau))$.

(d) If $p=k$ and $\Psi(\tau,X(\tau),\theta)=AX(\tau)+B\tau$, then (3.3.17) yields

$$\Gamma=E_\theta(AX(\tau)+B\tau)(AX(\tau)+B\tau)'$$

$$=grad_\theta\,E_\theta(A\underline{X}(\tau)+B\tau)'-E[grad_\theta\,(A\underline{X}(\tau)+B\tau)'].$$

Using (3.3.18) we get

$$\Gamma=-E[grad_\theta\,(A\underline{X}(\tau)+B\tau)']$$

$$=grad_\theta(E_\theta\,\underline{X}'(\tau))A'-grad_\theta\,(E_\theta\,X'(\tau)\,A')$$

$$+grad_\theta\,(E_\theta\tau)B'-grad_\theta\,(E_\theta\tau\,B')$$

$$=grad_\theta\,(E_\theta\,\underline{X}'(\tau)A')+grad_\theta(E_\theta\tau)B'-grad_\theta\,[E_\theta(A\underline{X}(\tau)+B\tau)']$$

Using (3.3.18) we get

$$=G_\tau A'+grad_\theta\,(E_\theta\tau)\,B'.$$

If A^{-1} exists then,we have

$$\Gamma=[-grad_\theta\,(A^{-1}B)'E_\theta\tau-grad_\theta\,(E_\theta\tau)(A^{-1}B)']A'$$

$$+grad_\theta\,(E_\theta\tau)\,B'$$

$$=-grad_\theta\,(A^{-1}B)'A'E_\theta\tau \quad (3.3.23)$$

If $\{\underline{X}(t), t \geq 0\}$ is a SIIP which belongs to the exponential class such that $P(\underline{X}(0)=0)=1$ then for every fixed observation interval $[0,t]$ the last observation $\underline{X}(t)$ is itself sufficient statistic for unknown parameter θ, follows due to the factorability of probability density functions of $\underline{X}(t)$ (Ghosh, page 175). Moreover if we consider in general $\underset{\sim}{X}(t)$ an m-dimensional process with independent and stationary increments such that $P(\underset{\sim}{X}(0)=\underset{\sim}{0})=1$ then also $\underset{\sim}{X}(t)$ is a sufficient statistic in order to estimate the parameter θ (Franz et al, 1976). Now we include a theorem which provides a sufficient estimator in case of random observation time.

<u>Theorem 9</u>. Let $\{\underline{X}(t), t \in T\}$ be a process of the exponential class, τ any finite stopping time and denote by $\mathcal{A}$ the smallest σ-field in $\mathfrak{F}_\tau$ with respect to which the pair $(\tau, X(\tau))$ is measurable. Then $\mathcal{A}$ is a sufficient σ-field to estimate θ. In other words $(\tau, \underline{X}(\tau))$ is a sufficient statistic.

<u>Proof</u>. Let us obtain an expression for $P_\theta(F \cap A)$ in order to prove the theorem where $F \in \mathfrak{F}_\tau$ and $A \in \mathcal{A} \subset \mathfrak{F}_\tau$. Define a bounded stopping time $\tau_s = \min(s, \tau)$, $s \in T$. $\mathfrak{F}_{\tau_s}$ represents σ-field having elements of the type

$$\{F \in \mathfrak{F} : F \cap \{w \mid \tau_s(w) \leq t\} \in \mathfrak{F}_t \text{ for every } t \in T\}$$

Using (3.3.10) we write

$$P_\theta(F\cap A) = \int_{F\cap A} \exp\{\alpha'(\theta)\underline{X}(\tau)+\beta(\theta)\tau\}dp_{\theta_0}$$

$$= \int_A I_F(\underline{X}(\tau))\exp\{\alpha'(\theta)\underline{X}(\tau)+\beta(\theta)\tau\}dP_{\theta_0}$$

$$= \int_\Omega I_A(\underline{X}(\tau)).I_F(\underline{X}(\tau))\exp\{\alpha'(\theta)\underline{X}(\tau)+\beta(\theta)\tau\}dp_{\theta_0}$$

$$= E[I_A(\underline{X}(\tau))\; I_F(\underline{X}(\tau))\exp\{\alpha'(\theta)\underline{X}(\tau)+\beta(\theta)\tau\}]$$

$$= \int_\Omega E_{\theta_0}[I_A(\underline{X}(\tau))\; I_F(\underline{X}(\tau))$$

$$\exp\{\alpha'(\theta)\underline{X}(\tau)+\beta(\theta)\tau\}|A]dp_{\theta_0}$$

$$= \int_\Omega I_A(\underline{X}(\tau)\exp\{\alpha'(\theta)\underline{X}(\tau)+\beta(\theta)\tau\}$$

$$E_{\theta_0}(I_F(\underline{X}(\tau))|A)dp_{\theta_0}$$

$$= \int_A P_{\theta_0}(F|A)\exp\{\alpha'(\theta)\underline{X}(\tau)+\beta(\theta)\tau\}dp_{\theta_0} \quad (3.3.24)$$

We can also write

$$P_\theta(F\cap A) = \int_A P_\theta(F|A)dp_\theta$$

$$= \int_A P_\theta(F|A)\exp\{\alpha'(\theta)\underline{X}(\tau)+\beta(\theta)\tau\}dP_{\theta_0} \quad (3.3.25)$$

With the help of (3.3.24) and (3.3.25) we write

$$P_\theta(F|A) = P_{\theta_0}(F|A)$$

For every $\theta\varepsilon\Theta$ with probability one Hence $(\tau, \underline{X}\tau))$ is a sufficient statistic for θ.

Since $(\tau, \underset{\sim}{X}(\tau))$ is a sufficient statistic one may restrict to $(\tau, \underset{\sim}{X}(\tau))$ for further inference about the parameter θ. Moreover the family of distributions of $(\tau, \underset{\sim}{X}(\tau))$ is complete (Franz, 1979). Furtheremore we discuss a generalisation of the Cramer-Rao type inequality for the exponential class of multi-dimensional processes. Let $h(\theta)=(h_1(\theta),\ldots,h_p(\theta))'$ be a given parameter function to be estimated. In order to estimate $h(\theta)$ one can use an estimator $\Psi(\tau, \underset{\sim}{X}(\tau))$, based on the sufficient statistic $(\tau, \underset{\sim}{X}(\tau))$ such that $E_\theta \Psi = h(\theta)$. For a given $\underset{\sim}{X}(\tau)$ and $h(\theta)$ we need to determine a sequential procedure (τ, Ψ) in which Ψ is an unbiased estimator of $h(\theta)$ and τ is a finite stopping time. Further we assume that the components $h_j(\theta), j=1,2,\ldots,p$ of $h(\theta)$ are non-constant and differentiable with respect to θ. We denote $H = \text{grad}_\theta (h'(\theta))$. In view of theorem 8 it can be deduced that

$$E_\theta [(A \underset{\sim}{X}(\tau) + B\tau) \Psi'] = H. \qquad (3.3.26)$$

We discuss below a theorem which enables us to get Cramer-Rao type inequality.

<u>Theorem 10</u>. Let $\{\underset{\sim}{X}(t), t \in T\}$ be a process of the exponential class and τ be a stopping time, each of them having finite second order moment. Suppose Γ^{-1} exists where

$$\Gamma = E_{\theta}[(A\underline{X}(\tau)+B\tau)(A\underline{X}(\tau)+B\tau)'].$$

If Ψ is an unbiased estimator for $h(\theta)$ with finite second order moment, then the inequality

$$Z'E_{\theta}(\Psi-h)(\Psi-h)'Z \geq Z'H'\Gamma^{-1}HZ \qquad (3.3.27)$$

holds for every vector $z = (z_1, z_2, \ldots, z_p)'$.

In (3.3.27) equality holds at the point $\theta = \theta^*$ if and only if the estimator Ψ can be represented almost everywhere as

$$\Psi = H'\Gamma^{-1}(A\underline{X}(\tau)+B\tau)+h \text{ at } \theta = \theta^*. \qquad (3.3.28)$$

<u>Proof</u>. Let us denote

$$Y' = (A\underline{X}(\tau)+B\tau)\,\Gamma^{-1}H-(\Psi-h)'.$$

In view of (3.3.18) and $E_{\theta}\Psi = h$ we get $E_{\theta}Y' = 0$. Define $V = Y'z$. Clearly $E_0 V = 0$. Further we obtain an expression for $Var(V)$ to establish (3.3.27).

$$Var(V) = Z'E_{\theta}(YY')Z$$
$$= Z'E_0[H'(\Gamma^{-1})(A\underline{X}(\tau)+B\tau)-(\Psi-h)]$$
$$[(AX(\tau)+B\tau)'(\Gamma^{-1})H-(\Psi-h)']Z.$$

Using (3.3.26) we simplify and get

$$Var(V) = Z'E_0(\Psi-h)(\Psi-h)'Z-Z'H'\Gamma^{-1}HZ \geq 0.$$

Hence

$$Z' E_\theta(\Psi - h)(\Psi - h)' Z \geq Z' H' \Gamma^{-1} HZ.$$

Equality holds at $\theta = \theta^*$ if and only if var (V)=0 for all non-zero Z. Since $E_0 V = 0$, Y = 0 almost everywhere and we get

$$\Psi = H' \Gamma^{-1}(A\underset{\sim}{X}(\tau) + B\tau) + h \quad \text{at} \quad \theta = \theta^*,$$

almost everywhere.

We discuss below illustrations of Cramer-Rao type inequality.

Example 1. Let $\underset{\sim}{X}(t)$ be a multinomial process having stationary and independent increments and the probability distribution of $\underset{\sim}{X}(t)$ is given by (3.3.8). We use (3.3.23) to obtain Γ^{-1}.

$$\Gamma = -\text{grad}_p\ (A^{-1}B)'\ A' E_p \tau$$

$$= -\text{grad}_p\ (-P)' A'\ E_p \tau$$

where

$$p = (p_1\ p_2\ \dots\ p_m)'.$$

Therefore

$$\Gamma^{-1} = \frac{1}{E_p \tau}\ A^{-1}.$$

Hence from (3.3.27) we get

$$Z' E_p(\Psi - h)(\Psi - h)' Z \geq \frac{1}{E_p \tau} Z' H' A^{-1} HZ.$$

Equality holds at $p=p^*$ if and only if Ψ is of the form

$$\Psi = \frac{1}{E_p\tau} H'(X(\tau)-P\tau) + h(p) .$$

Example 2. Let $X(t)$ be a m-dimensional Gaussian process (Definition 5). The relation (3.3.23) yields

$$\Gamma = -\text{grad}_\theta (A^{-1}\theta)'A'E_0\tau$$

$$= A'E_\theta\tau .$$

Hence

$$\Gamma^{-1} = A^{-1} \frac{1}{E_\theta\tau}$$

$$= \frac{1}{E_\theta\tau} \Sigma .$$

Using (3.3.27) one can write

$$Z'E_0(\Psi-h)(\Psi-h)'Z \geq \frac{1}{E_0\tau} Z'H'\Sigma H Z.$$

Equality holds at $\theta = 0^*$ if and only if

$$\Psi = \frac{1}{E_0\tau} H'(X(\tau) - \theta\tau) + h(\theta).$$

Following is the discussion about efficient estimator in the sense that in (3.3.27) equality holds for all 0 or for some $\theta^* \varepsilon \Theta$. We define below an efficient sequential procedure.

Definition 11. A sequential procedure (τ, Ψ) is said to be efficient at $\theta = 0^*$, if for the considered stopping time τ, the parameter function $h(\theta)$ and the unbiased estimator Ψ in

(3.3.27) equality holds at the point $\theta = \theta^*$. (Winkler et.al., page 136).

<u>Definition 12</u>. A sequential procedure (τ, Ψ) is called efficient if it is efficient for all $\theta \varepsilon \Theta$. (Winkler et. al., page 136).

A parameter function $h(\theta)$ is considered to be an estimable if there exists an estimator Ψ with $E_\theta \Psi = h$, accordingly if $h(\theta)$ is an estimate parameter function and Ψ is an unbiased, efficient estimator of $h(\theta)$ for $\theta \varepsilon \Theta$ one can say that $h(\theta)$ is efficiently estimable.

A theorem which we include below gives a necessary condition for a sequential procedure (τ, Ψ) to be an efficient.

<u>Theorem 13</u>. Let $\{\underset{\sim}{X}(t), t \varepsilon T\}$ be a process which belongs to the exponential class, τ be a finite stopping time and let Ψ be an unbiased estimator for $h(\theta)$ with finite second order moment. We assume that the components $h_j(\theta)$, $j=1,\ldots,p$ of $h(\theta)$ are non-constant and differentiable with respect to θ, Γ^{-1} exists where

$$\Gamma = E_\theta[(A\underset{\sim}{X}(\tau) + B\tau)(A\underset{\sim}{X}(\tau)+B\tau)'].$$

If the sequential procedure (τ, Ψ) is efficient, then there exist coefficients $c_i, i=0,1,\ldots,m$ with $\sum_{i=0}^{m} c_i^2 > 0$ and $d \neq 0$ such that

$$c_o \tau + c' \underset{\sim}{X}(\tau) = d \qquad (3.3.29)$$

holds almost surely, where $c = (c_1, c_2, \ldots, c_m)'$.

<u>Proof</u>. (See Winkler et.al., page 136).

We obtain below a general expression for efficiently estimable function $h(\theta)$ under the assumptions of the theorem 10. If (τ, Ψ) is an efficient sequential procedure using (3.3.29) we write

$$c_o E_\theta \tau + c' E_\theta \underset{\sim}{X}(\tau) = d.$$

Assuming A^{-1} exists(3.3.19) yields

$$c_o E_\theta \tau - c' A^{-1} B \, E_\theta \tau = d.$$

Hence

$$E_\theta \tau = \frac{d}{c_o - c'(A^{-1}B)} . \qquad (3.3.30)$$

Since ψ is efficient, using (3.3.28) we get

$$\Psi = H' \Gamma^{-1}(A \underset{\sim}{X}(\tau) + B\tau) + h.$$

The relation (3.3.23) yields

$$\Psi = -H' [\text{grad}_\theta (A^{-1}B)' A' E_\theta \tau]^{-1} (A \underset{\sim}{X}(\tau) + B\tau) + h .$$

Using (3.3.30) we get

$$\Psi = -\frac{c_o - c'(A^{-1}B)}{d} H.[\text{grad}_\theta (A^{-1}B)' A']^{-1} (A \underset{\sim}{X}(\tau) + B\tau) + h$$

$$\psi = -\frac{c_o - c'(A^{-1}B)}{d} M(A \underset{\sim}{X}(\tau) + B\tau) + h \qquad (3.3.31)$$

where $M = H'[-\text{grad}_\theta\ (A^{-1}B)'\ A']^{-1}$.

In order to simplify (3.3.31) further we denote

$$\bar{C} = (\ c_o, c_1, \ldots, c_m)'$$

$$C^{(i)} = (c_o, c_1, \ldots, c_{i-1}, c_{i+1}, \ldots,\ c_m)'$$

$$\bar{X}\ (\tau) = (\tau, X_1(\tau), \ldots, X_m(\tau))'.$$

$$X^{(i)}(\tau) = (\tau, X_1(\tau), \ldots, X_{i-1}(\tau), X_{i+1}(\tau), \ldots, X_m(\tau))', \quad i=0,1,\ldots m.$$

Let the columns of the matrix A be denoted by the vectors $a_1, a_2, \ldots, a_m$ and we write $A = (a_1, a_2, \ldots, a_m)$.

Define

$$= (B_o, a_1, a_2, \ldots, a_m)$$

$$= (a_o, a_1, \ldots, a_m) \quad \text{where} \quad B = a_o.$$

$$A^{(i)} = (a_o, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m).$$

Clearly $\quad A^{(o)} = A,\ C^{(o)} = C,\ X^{(o)}(\tau) = X(\tau)$.

Now we can write the relation (3.3.29) as

$$\bar{C}'\bar{X}(\tau) = d\ . \tag{3.3.32}$$

The relation (3.3.31) can be written as

$$\psi = \frac{c_o - c'(A^{-1}B)}{d}\ M\ \bar{A}\ \underline{X}(\tau) + h. \tag{3.3.33}$$

If $c_i \neq 0$ for any i, then (3.3.29) can be written as

$$c_i X_i(\tau) + c^{(i)'} \underset{\sim}{X}^{(i)}(\tau) = d.$$

Hence

$$X_i(\tau) = \frac{1}{c_i}\left(d - c^{(i)'} \underset{\sim}{X}^{(i)}(\tau)\right)$$

and (3.3.31) simplifies to

$$\psi = \frac{c_o - c'(A^{-1}B)}{d} M[A^{(i)} \underset{\sim}{X}^{(i)}(\tau) + \frac{a_i}{c_i}(d - c^{(i)'} \underset{\sim}{X}^{(i)}(\tau))] + h$$

Setting $C^{(i)} = \frac{1}{c_i} a_i c^{(i)'}$ we write

$$= \frac{c_o - c'(A^{-1}B)}{d} M[(A^{(i)} - C^{(i)}) \underset{\sim}{X}^{(i)}(\tau) + \frac{d}{c_i} a_i] + h \quad (3.3.34)$$

Denote

$$\frac{c_o - c'(A^{-1}B)}{d} M \frac{d}{c_i} a_i + h = k$$

and

$$\frac{c_o - c'(A^{-1}B)}{d} M(A^{(i)} - C^{(i)}) = K,$$

where k is a vector and K is a matrix with constant elements follows due to the fact that ψ is an estimator which must not depend upon θ. If $A^{(i)^{-1}}$ exists then with the help of an identity

$$a_i = \frac{c_i}{c_i - c^{(i)'} \mathcal{A}^{(i)^{-1}} a_i} (\mathcal{A}^{(i)} - C^{(i)}) \mathcal{A}^{(i)^{-1}} a_i$$

we get

$$h = k - \frac{d}{c_i - c^{(i)'} \mathcal{A}^{(i)^{-1}} a_i} K \mathcal{A}^{(i)^{-1}} a_i .$$

$$= \frac{c_i k - k\, c^{(i)'} c^{(i)^{-1}} a_i - dK \mathcal{A}^{(i)^{-1}} a_i}{c_i - c^{(i)'} \mathcal{A}^{(i)^{-1}} a_i} .$$

Taking $k^* = c_i k$ and $K^* = k\, c^{(i)'} + dK$, we get the following representation for efficiently estimable functions

$$h = \frac{k^* - K^* (\mathcal{A}^{(i)^{-1}} a_i)}{c_i - c^{(i)'} (\mathcal{A}^{(i)^{-1}} a_i)} \qquad (3.3.35)$$

We can write the relation (3.3.34) as follows

$$\psi = K \underset{\sim}{X}^{(i)}(\tau) + k$$

$$= \frac{c_i K^* - k^*\, c^{(i)'}}{dc_i} \underset{\sim}{X}^{(i)}(\tau) + \frac{k^*}{c_i} \qquad (3.3.36)$$

which is the corresponding efficient estimator.

Note that ψ is an efficient estimator, therefore using (3.3.28) we get at $\theta = \theta^*$

$$\psi = H' \Gamma^{-1}(A\underset{\sim}{X}(\tau)+B\tau\ ) + h(\theta^*).$$

Hence

$$E_\theta\ \psi = (H' \Gamma^{-1}A)\ E_\theta \underset{\sim}{X}(\tau)+(H' \Gamma^{-1}B)E_\theta \tau + h(\theta^*).$$

So

$$h(\theta) = DE_\theta \underset{\sim}{X}(\tau) + d_1 E_\theta\ \tau + d_2 \quad . \qquad (3.3.37)$$

where $D = (H'\Gamma^{-1}\ A),\ d_1=(H'\ \Gamma^{-1}B),\ \ d_2 = h(\theta^*).$

Thus an efficiently estimable parameter vector $h(\theta)$ posses representation given by (3.3.37).

We now discuss some efficiently estimable parameter functions and corresponding efficient estimator in case of multinomial process and m-dimensional Gaussian process.

Example 1. Let $\underset{\sim}{X}(t)$ be a multinomial process. We deal with a fixed time procedure. Let $\tau = d$ almost surely, $d \in \{1,2,...\}$ and $c_o=1,\ c_1=c_2,...=c_m=0$. Thus we get from (3.3.35)

$$h = \frac{k^* - K^*(A^{(o)^{-1}}a_o)}{c_o - c^{(0)'}(A^{(o)^{-1}}a_o)}$$

$$= k^* - K^*\ P$$

and (3.3.36) gives

$$\Psi = \frac{c_o K^* - k^* \; c^{(o)'}}{dc_o} \underset{\sim}{X}^{(o)}(\tau) + \frac{k^*}{c_o}$$

$$= \frac{K^*}{d} \underline{X}(d) + k^*.$$

If we have $X_1(\tau)=d$, $d \in \{0,1,2,\ldots\}$ then we have

$$c_o =, \; c_1=1, c_2=c_3 \ldots = c_m = 0.$$

Therefore from (3.3.30).

$$E_p(\tau) = \frac{d}{c_o - c^{1'}(A^{-1}B)} = \frac{d}{p_1} \quad ,$$

from (3.3.35) we get

$$h = \frac{k^* - K^*(A^{(1)^{-1}} a_1)}{c_1 - c^{(1)'}(A^{(1)^{-1}} a_1)}$$

$$= k^* - K^* A^{(1)^{-1}} a_1$$

and from (3.3.36) we get

$$\Psi = \frac{c_1 K^* - k^* \; c^{(1)'}}{dc_1} X^{(1)'}(\tau) + \frac{k^*}{c_1} \; .$$

$$= k^* + \frac{1}{d} K^*(\tau, X_2(\tau), \ldots, X_m(\tau))'$$

Clearly

$$A^{(1)} = \begin{bmatrix} -1/q & 1/q_1 & 1/q & \cdots & 1/q \\ -1/q & 1/p_2+1/q & 1/q & \cdots & 1/q \\ -1/q & 1/q & 1/p_3+1/q & \cdots & 1/q \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -1/q & 1/q & 1/q & \cdots & 1/p_m+1/q \end{bmatrix}$$

and $a_1 = (1/p_1+1/q, 1/p_2, \ldots, 1/p_m)'$.

Therefore

$$A^{(1)^{-1}} = \begin{bmatrix} -(p_2+p_3+\ldots+p_m+q) & p_2 & p_3 & \cdots & p_m \\ -p_2 & p_2 & 0 & \cdots & 0 \\ -p_3 & 0 & p_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ -p_m & 0 & 0 & & p_m \end{bmatrix}$$

and

$$A^{(1)^{-1}} a_1 = (-1/p_1, -p_2/p_1, \ldots -p_m/p_1)' .$$

Thus we get that the efficiently estimable functions are

$$h(p) = k^* + \frac{1}{p_1} K^*(1, p_2, \ldots, p_m)'$$ and the corresponding estimators are

$$\Psi(\tau, \underset{\sim}{X}(\tau)) = k^* + \frac{1}{d} K^*(\tau, X_2(\tau), \ldots, X_m(\tau))'$$

Example 2. Let $\underset{\sim}{X}(t)$ be a m-dimensional , Gaussian process. We take $\tau = d$ almost surely, d being fixed. We have $c_o=1$, $c_1=c_2,\ldots=c_m=0$ and efficiently estimable functions are of the type

$$h(\theta) = k^* - K^*(A^{-1}B)$$
$$= k^* + K^*\theta$$

and the corresponding estimator will be

$$\Psi(d, \underline{X}(d)) = k^* + \frac{1}{d} K^* \underset{\sim}{X}(d) \ .$$

In general if $c_o \neq 0$ then from (3.3.30) we get

$$E_\theta \tau = \frac{d}{c_o + c'\theta} \ .$$

Using (3.3.35) and (3.3.36) we get an efficiently estimable function

$$h(\theta) = \frac{k^* + K^*\theta}{c_o + c'\theta}$$

with corresponding estimator

$$\Psi(\tau, \underline{X}(\tau)) = \frac{k^*}{c_o} + \frac{c_o K^* - k^* c'}{d c_o} \underline{X}(\tau).$$

3.4 Estimation of the canonical measure G of SIIP

Let $\{X(t), t \geq 0\}$ be a SIIP satisfying a condition $P(X(0)=0)=1$. We also assume $EX(t)$ exists and is finite. In view of theorem 6 of chapter 1, X(t) is infinitely divisible. Hence with the help of (1.3.7) we express the characteristic function $\phi_t(u)$ of X(t) as follows :

$$\log \phi_t(u) = iuat + t\int_{-\infty}^{\infty} \left(e^{iux}-1-\frac{iux}{1+x^2}\right)\frac{1+x^2}{x^2}\,dG(x)$$

with G(•) bounded, monotone non-decreasing, right continuous function in x having $\lim_{x\to-\infty} G(x) = 0$ and a is a constant. We assume $\phi_t(u)$ is continuous at t= 0. Finite dimensional distributions of a SIIP can be determined if the parameters G and a are known. We include a method of estimation of G below. Another method of estimation of G and a is suggested in the paper by Rubin and Tucker (page, 648) which requires theory of stochastic integrals and so it is not discussed here.

We define for an integer $n \geq 1$

$$X_{n.k} = X\left(\frac{k}{n}\right)-X\left(\frac{k-1}{n}\right), \quad k=1,2,\ldots,n.$$

Therefore for every n we write

$$X(1) = X_{n.1} + \dots + X_{n.k}. \tag{3.4.1}$$

Since $X(t)$ is a SIIP $\{X_{n.k}\ 1 \le k \le n\}$ is a sequence of independent and identically distributed random variables. We denote distribution function of $X_{n.k}$ by $F_n(x)$ for fixed n and $\alpha_n = \int_{-y}^{y} x\, dF_n(x)$ for arbitrary $y > 0$. The relation (3.4.1) yields, $X(1)$ is the limit law of distribution of $\sum_{k=1}^{n} X_{n.k}$ as $n \to \infty$. Therefore

$$G_n(y) = n \int_{-\infty}^{y} \frac{x^2}{1+x^2}\, dF_n(x+\alpha_n) \to G(y) \tag{3.4.2}$$

as $n \to \infty$ for all $y \in C(G)$, where $C(G)$ is a set of points of discontinuities of G (Rubin et. al., page 644). $G_n(y)$ involves α_n therefore we define

$$\bar{G}_n(y) = n \int_{-\infty}^{y} \frac{x^2}{1+x^2}\, dF_n(x) \tag{3.4.3}$$

and establish that $\bar{G}_n(y) \to G(y)$ as $n \to \infty$ for all $y \in C(G)$. We include below the necessary results.

Lemma 14 : If $G_n^{**}(y) = n \int_{-\infty}^{y} \frac{(x-\alpha_n)^2}{1+(x-\alpha_n)^2}\, dF_n(x)$

then $G_n^{**}(y) \to G(y)$ as $n \to \infty$ for all $y \in C(G)$.

Proof : Let y be a fixed point which belongs to $C(G)$. Clearly for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$y \pm \delta \in C(G),$$

$$|G(y+\delta)-G(y-\delta)| < \frac{\varepsilon}{2} \qquad (3.4.4)$$

The relation (3.4.4) and the fact that $G_n(y) \to G(y)$ as $n \to \infty$ enables us to write for every $\varepsilon > 0$

$$|\{G_n(y+\delta)-G_n(y-\delta)\}-\{G(y+\delta)-G(y-\delta)\}| < \frac{\varepsilon}{2}, \text{for all } n > N. \qquad (3.4.5)$$

Now

$$|G_n^{**}(y+\alpha_n)-G_n^{**}(y)| = \left| n\int_y^{y+\alpha_n} \frac{(x-\alpha_n)^2}{1+(x-\alpha_n)^2} dF_n(x) \right|$$

$$= |G_n(y-\alpha_n)-G_n(y)|$$

$$\leq |G_n(y+\alpha_n)-G_n(y-\alpha_n)|.$$

Since $|\alpha_n| < \delta$ for all $n > N$ we write

$$|G_n^{**}(y+\alpha_n) - G_n^{**}(y)| \leq |G_n(y+\delta)-G_n(y-\delta)|$$

$$\leq |\{G_n(y+\delta)-G_n(y-\delta)\}-\{G(y+\delta)-G(y-\delta)\}|+|G(y+\delta)-G(y-\delta)|.$$

Using (3.4.4) and (3.4.5) we get

$$|G^{**}(y+\alpha_n)-G_n^{**}(y)| < \varepsilon, \quad \text{for all } n > N.$$

Hence the proof.

Lemma 15 : If $y < 0$ is a continuity point of G, then $\bar{G}_n(y) \to G(y)$ as $n \to \infty$.

Proof. Define

$$f_n(x) = \frac{x^2}{1+x^2} \times \frac{1+(x-\alpha_n)^2}{(x-\alpha_n)^2}.$$

Then

$$\bar{G}_n(y) = \int_{-\infty}^{y} f_n(x)\ dG_n^{**}(x)$$

In order to prove the lemma let us obtain

$$|\bar{G}_n(y)-G(y)| \leq |\int_{-\infty}^{y} f_n(x)dG^{**}(x)-\int_{-\infty}^{y} dG\ (x)|+|\int_{-\infty}^{y} dG^{**}(x)-G(y)|$$

$$\leq \sup_{x\leq y} |f_n(x)-1|G_n^{**}\ (y)+|G_n^{**}\ (y)-G(y)| \qquad (3.4.6)$$

Since $f_n(x)$ converges uniformly to 1 over any closed set not containing zero and in view of lemma 14, right hand side of (3.4.6) tends to zero as $n \to \infty$. Hence the proof.

Lemma 16: If $a,b \in C(G)$ such that $0 < a < b$, then

$$\bar{G}_n(b)- \bar{G}_n(a) \to G(b)-G(a) \text{ as } n \to \infty .$$

Proof: Clearly

$$|\bar{G}(b)-\bar{G}(a)-G(b)+G(a)| \leq |\int_a^b f_n(x)dG_n^{**}\ (x)-\int_a^b dG_n^{**}\ (x)|$$

$$+|\int_a^b dG_n^{**}(x)-G(b) + G(a)|$$

$$\leq \sup_{a \leq x \leq b} |f_n(x)-1| \; (G_n^{**}(b)-G_n^{**}(a))$$

$$+|G_n^{**}(b)-G_n^{**}(a)-G(b)+G(a)|.$$

Using similar argument as used in the proof of lemma 15, lemma 16 follows:-

<u>Theorem 17</u> : If $y \in C(G)$, then $\bar{G}_n(y) \to G(y)$ as $n \to \infty$.

<u>Proof</u> : If $y < 0$, theorem follows due to lemma 15. We establish below inequalities in order to prove the theorem if $y > 0$.

We write

$$\frac{1}{1+(\tau+|\alpha_n|)^2}\{n\int_{-\tau}^{\tau}x^2 dF_n - 2n\alpha_n^2 + n\alpha_n^2(F_n(\tau)-F_n(-\tau))\}$$

$$= \frac{1}{1+(\tau+|\alpha_n|)^2} \; n\int_{-\tau}^{\tau}(x-\alpha_n)^2 \, dF_n(x) \qquad (3.4.7)$$

$$\leq n\int_{-\tau}^{\tau}\frac{(x-\alpha_n)^2}{1+(x-\alpha_n)^2}\,dF_n(x)$$

$$\leq n\int_{-\tau}^{\tau}(x-\alpha_n)^2\,dF_n(x)$$

$$\leq n\int_{-\tau}^{\tau}x^2 dF_n(x) - n\alpha_n^2 \; . \qquad (3.4.8)$$

Note that

$$n\int_{-\tau}^{\tau}x^2 dF_n(x) \leq (1+\tau^2)n\int_{-\tau}^{\tau}\frac{x^2}{1+x^2}\,dF_n(x)$$

$$\leq (1+\tau^2)n\int_{-\tau}^{\tau}x^2 dF_n(x) \qquad (3.4.9)$$

It can be seen

$$n\int_{-\tau}^{\tau} \frac{x^2}{1+x^2}\, dF_n(x) \leq n\int_{-\tau}^{\tau} x^2\, dF_n(x)$$

$$\leq \{1+(\tau+|\alpha_n|)^2\} n \int_{-\tau}^{\tau} \frac{(x-\alpha_n)^2}{1+(x-\alpha_n)^2}\, dF_n(x) + \{2-F_n(\tau)+F_n(-\tau)\} n\alpha_n^2$$

$$\leq \{1+(\tau+|\alpha_n|^2\} n \int_{-\tau}^{\tau} x^2 dF_n(x)$$

$$+\{1-(\tau+|\alpha_n|)^2-F_n(\tau)+F_n(-\tau)\} n\alpha_n^2 \qquad (3.4.10)$$

$$\leq (1+\tau^2)\,[1+(\tau+|\alpha_n|)^2] n \int_{-\tau}^{\tau} \frac{x^2}{1+x^2}\, dF_n(x)$$

$$+(1+\tau^2)\{1-(\tau+|\alpha_n|)^2-F_n(\tau)+F_n(-\tau)\} n\alpha_n^2 .$$

Observe that $A_n = n\alpha_n$

$$= \sum_{k=1}^{n} \int_{-\tau}^{\tau} X_{n_k} dF_n(x) \leq E(X(1)) < \infty .$$

Hence $\{A_n\ n \geq 1\}$ is convergent and A_n is bounded. Thus we can say $\alpha_n \to 0$ as $n \to \infty$. Since $n\alpha_n$ is bounded we write

$$n\alpha_n^2 = (n\alpha_n)\,\alpha_n$$

$$= C.\,\alpha_n$$

and as $n \to \infty$, $n\alpha_n^2 \to 0$.

If $-\tau, \tau \in C(G)$, (3.4.10) yields.

$$\overline{\lim_{n\to\infty}}\ n \int_{-\tau}^{\tau} \frac{x^2}{1+x^2}\, dF_n(x) \le (1+\tau^2)\{G(\tau)-G(-\tau)\}$$

$$\le (1+\tau^2)^2 \underline{\lim_{n\to\infty}}\ n \int_{-\tau}^{\tau} \frac{x^2}{1+x^2}\, dF_n(x).$$

We can write for $o < \tau < y$

$$\underline{\lim_{n\to\infty}}\ n \int_{-\infty}^{y} \frac{x^2}{1+x^2}\, dF_n(x) \ge G(y)-G(\tau)+ \frac{1}{1+\tau^2}\{G(\tau)-G(-\tau)\}+G(-\tau),$$

and as $\tau \to 0$ we get

$$\underline{\lim_{n\to\infty}}\ n \int_{-\infty}^{y} \frac{x^2}{1+x^2}\, dF_n(x) \ge G(y).$$

Similarly

$$\lim_{n\to\infty} n \int_{-\infty}^{y} \frac{x^2}{1+x^2}\, dF_n(x) \le (1+\tau^2)\{G(\tau)-G(-\tau)\}+G(y)-G(\tau)-G(-\tau),$$

as $\tau \to 0$ we get

$$\lim_{n\to\infty} n \int_{-\infty}^{y} \frac{x^2}{1+x^2}\, dF_n(x) \le G(y).$$

Hence the proof.

We write the statement of theorem 17 as follows

$$\bar{G}_n(y) = nE\{ \frac{X_{n.k}^2}{1+X_{n.k}^2} I_{[X_{n.k} \ge y]}\} \to G(y)$$

as $n \to \infty$ for every $y \in C(G)$.

Using strong law of large numbers we get for fixed n and every y

$$G^*_{N,n}(y) = \frac{1}{N} \sum_{k=1}^{nN} \frac{X^2_{n.k}}{1+X^2_{n.k}} I_{[X_{n.k} \leq y]} \to \bar{G}_n(y)$$

as $N \to \infty$ with probability one. Using the fact that $G^*_{N,n}(y)$ and $G(y)$ are nondecreasing in y we can write

$$P\{\lim_{n\to\infty} \lim_{N\to\infty} G^*_{N,n}(y) = G(y) \text{ for all } y \in C(G)\} = 1 . \tag{3.4.11}$$

From (3.4.11) one can say $G^*_{N,n}(y)$ is a strongly consistent estimator of $G(y)$.

Estimator $G^*_{N,n}(y)$ need not be unbiased. We illustrate the same below.

Let $\{X(t), t \geq 0\}$ be Poisson process with characteristic function

$$\phi_t(u) = iuat + \lambda t\,(e^{iub}-1),$$

where $u \in R$, $a > 0$, $b > 0$, $\lambda > 0$.

Using (1.3.7) proper identification of G will be

$$\begin{aligned} G(x) &= 0 && \text{if } x < b \\ &= \lambda\, b^2(1+b^2)^{-1} && \text{if } x > b. \end{aligned}$$

If $x \leq 0$, then for every n and N

$$E\ G^*_{N,n}(x) = 0.$$

If $0 < x < b$, then for n such that $0 < \frac{a}{n} < x$ we get

$$E\ G^*_{N,n}(x) = n\ E\{ \frac{X^2_{1.n}}{1+X^2_{1.n}}\ I[X_{1.n} \le x]\}$$

$$= \frac{na^2}{n^2+a^2} \exp \{- \frac{\lambda}{n}\} > 0 \quad \text{for all } N.$$

If $x \ge b$, then

$$E\ G^*_{N,n}(x) = \sum_{j=0}^{[\frac{x-a}{b}]} \frac{(a+njb)^2}{n^2+(a+njb)^2} \exp \{- \frac{\lambda}{n}\} (\frac{\lambda}{n})^j \frac{1}{j!}$$

$$= n \exp\{- \frac{\lambda}{n}\}[\frac{a^2}{n^2+a^2} + \frac{(a+nb)^2}{n^2+(a+nb)^2} \frac{\lambda}{n}$$

$$(1+ \frac{\lambda}{n} \sum_{j=2}^{[\frac{x-a}{b}]} (\frac{\lambda}{n})^{j-2} \frac{1}{j!} \frac{(a+njb)^2}{n^2+(a+njb)^2})]. \quad (3.4.12)$$

It follows from (3.4.12) $E\ G^*_{N,n}(x)$ is greater than $\lambda b^2(1+b^2)^{-1}$.

x. o.o.x.

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